Math 131AH – Honors Real Analysis I

University of California, Los Angeles

Duc Vu

Winter 2021

This is math 131AH – Honors Real Analysis I taught by Professor Visan, and our TA is Thierry Laurens. We meet weekly on MWF from 10:00am – 10:50am for lectures. There are two textbooks used for the class, *Principles of Mathematical Analysis* by *Rudin* and *Metric Spaces* by *Copson*. Note that some of the theorems' name are not necessarily their official names. It's just a way for me to reference them without the need of searching through pages for their contents. You can find other lecture notes at my github site. Please let me know through my email if you spot any mathematical errors/typos.

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§1 Lec 1: Jan 4, 2021

§1.1 Logical Statments & Basic Set Theory

Let A and B be two statements. We write

- A if A is true.
- not A if A is false.
- A and B if both A and B are true.
- A or B if A is true or B is true or both A and B are true (inclusive "or" – it is not either A or B).
- $\underbrace{A \implies B}$: if ($A$ and B) or (not A) – We read this "A implies B" or "If A then B".

  In this case, B is at least as true as A. In particular, a false statement can imply anything.

Example 1.1
Consider the following statement: If x is a natural number (i.e., $x \in \mathbb{N} = \{1, 2, 3, \ldots\}$, then $x \geq 1$. In this case, $A =$ "x is a natural number", $B =$ "$x \geq 1$". Taking $x = 3$, we get a $T \implies T$. Taking $x = \pi$ we get $F \implies T$. If $x = 0$, we get $F \implies F$.

Example 1.2
Consider the statement: $\underbrace{\text{If a number is less than 10}}_{A}$, $\underbrace{\text{then it's less than 20}}_{B}$.

Taking

$$\begin{aligned} \text{number } &= 5, \qquad T \implies T \\ &= 15, \qquad F \implies T \\ &= 25, \qquad F \implies F \end{aligned}$$

We write $\underbrace{A \iff B}$ if A and B are true together or false together. We read this as "A is equivalent to B" or "A if and only if B". Compare these notions to similar ones from set theory. Let X is an ambient space. Let A and B be subsets of X. Then

$$\begin{aligned} A^c &= \{x \in X; x \notin A\} \\ A \cap B &= \{x \in X; x \in A \text{ and } x \in B\} \\ A \cup B &= \{x \in X; x \in A \text{ or } x \in B \text{ or } x \in A \cap B\} \\ A \subseteq B & \text{ corresponds to } A \implies B \\ A = B & \qquad A \iff B \end{aligned}$$

<u>Truth table</u>:

A	B	not A	A and B	A or B	A $\implies$ B	A $\iff$ B
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Example 1.3
Using the truth table show that $A \implies B$ is logically equivalent to (not A) or B.

A	B	A $\implies$ B	not A	(not A) or B
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Homework 1.1. Using the truth table prove De Morgan's laws:

$$\text{not (A and B)} = \text{(not A) or (not B)}$$
$$\text{not (A or B)} = \text{(not A) and (not B)}$$

Compare this to

$$(A \cap B)^c = A^c \cup B^c$$
$$(A \cup B)^c = A^c \cap B^c$$

Exercise 1.1. Negate the following statement: If A then B.
Solution:

$$\begin{aligned}\text{not}(A \implies B) &= \text{not}\,((\text{not A}) \text{ or B}) \\ &= [\text{not}(\text{not A}) \text{ and } (\text{not B})] \\ &= A \text{ and } (\text{not B})\end{aligned}$$

The negation is “A is true and B is false”.

Example 1.4
Negate the following sentence: If I speak in front of the class, I am nervous.
I speak in front of the class and I am not nervous.

Quantifiers:

- $\forall$ reads “for all” or “for any”
- $\exists$ reads “there is” or “there exists”

The negation of $\forall A, B$ is true is $\exists A$ s.t. B is false.
The negation of $\exists A, B$ is true is $\forall A, B$ is false.

Example 1.5

Negate the following: Every student had coffee or is late for class.
$\forall$ student (had coffee) or (is late for class)
$\exists$ student s.t. not[(had coffee) or (is late for class)]
$\exists$ student s.t. not (had coffee) and not (is late for class)
Ans: There is a student that did not have coffee and is not late for class.

§2 Lec 2: Jan 6, 2021

§2.1 Mathematical Induction

The natural numbers – $\mathbb{N} = \{1, 2, 3, \ldots\}$; they satisfy the Peano axioms:

N1) $1 \in \mathbb{N}$

N2) If $n \in \mathbb{N}$ then $n + 1 \in \mathbb{N}$

N3) 1 is not the successor of any natural number.

N4) If $n, m \in \mathbb{N}$ such that $n + 1 = m + 1$ then $n = m$

N5) Let $S \subseteq \mathbb{N}$. Assume that S satisfies the following two conditions:

(i) $1 \in S$

(ii) If $n \in S$ then $n + 1 \in S$

Then $S = \mathbb{N}$.

Axiom N5) forms the basis for mathematical induction. Assume we want to prove that a property $P(n)$ holds for all $n \in \mathbb{N}$. Then it suffices to verify two steps:
Step 1 (base step): $P(1)$ holds.
Step 2 (inductive step): If $P(n)$ is true for some $n \geq 1$, then $P(n+1)$ is also true, i.e., $P(n) \implies P(n+1) \forall n \geq 1$.
Indeed, if we let

$$S = \{n \in \mathbb{N} : P(n) \text{ holds}\}$$

then Step 1 implies $1 \in S$ and Step 2 implies if $n \in S$ then $n + 1 \in S$. By Axiom N5 we deduce $S = \mathbb{N}$.

Example 2.1

Prove that

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \qquad \forall n \in \mathbb{N}$$

Solution: We argue by mathematical induction. For $n \in \mathbb{N}$ let $P(n)$ denote the statement

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Step 1 (Base step): $P(1)$ is the statement

$$1^2 = \frac{1 \cdot 2 \cdot 3}{6}$$

which is true, so $P(1)$ holds.
Step 2 (Inductive step): Assume that $P(n)$ holds for some $n \in \mathbb{N}$. We want to know $P(n+1)$ holds. We know

$$1^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Let's add $(n+1)^2$ to both sides of $P(n)$

$$\begin{aligned} 1^2 + \ldots + n^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= (n+1)\left[\frac{n(2n+1)}{6} + n + 1\right] \\ &= \frac{(n+1)(n+2)(2n+3)}{6} \end{aligned}$$

So $P(n+1)$ holds.
Collecting the two steps, we conclude $P(n)$ holds $\forall n \in \mathbb{N}$. □

Example 2.2

Prove that $2^n > n^2$ for all $n \geq 5$.
Solution: We argue by mathematical induction. For $n \geq 5$ let $P(n)$ denote the statement $2^n > n^2$.
Step 1 (base step): $P(5)$ is the statement

$$32 = 2^5 > 5^2 = 25$$

which is true. So $P(5)$ holds.
Step 2 (Inductive step): Assume $P(n)$ is true for some $n \geq 5$ and we want to prove $P(n+1)$. We know

$$2^n > n^2$$

Let us manipulate the above inequality to get $P(n+1)$

$$\begin{aligned} 2^n &> n^2 \\ 2^{n+1} &> 2n^2 = (n+1)^2 + n^2 - 2n - 1 \\ 2^{n+1} &> (n+1)^2 + (n-1)^2 - 2 \end{aligned}$$

As $n \geq 5$ we have $(n-1)^2 - 2 \geq 4^2 - 2 = 14 \geq 0$. So

$$2^{n+1} > (n+1)^2$$

So $P(n+1)$ holds.
Collecting the two steps, we conclude that $P(n)$ holds $\forall n \geq 5$. □

Remark 2.3. Each of the two steps are essential when arguing by induction. Note that $P(1)$ is true. However, our proof of the second step fails if $n = 1 : (1-1)^2 - 2 = -2 < 0$.
Note that our proof of the second step is valid as soon as

$$(n-1)^2 - 2 \geq 0 \iff (n-1)^2 \geq 2 \iff n - 1 \geq 2 \iff n \geq 3$$

However, $P(3)$ fails.

Example 2.4

Prove by mathematical induction that the number $4^n + 15n - 1$ is divisible by 9 for all $n \geq 1$.

Solution: We'll argue by induction. For $n \geq 1$, let $P(n)$ denote the statement that "$4^n + 15n - 1$ is divisible by 9". We write this $9/(4^n + 15n - 1)$.

Step 1: $4^1 + 15 \cdot 1 - 1 = 18 = 9 \cdot 2$. This is divisible by 9, so $P(1)$ holds.

Step 2: Assume $P(n)$ is true for some $n \geq 1$. We want to show $P(n+1)$ holds.

$$\begin{aligned} 4^{n+1} + 15(n+1) - 1 &= 4(4^n + 15n - 1) - 60n + 4 + 15n + 14 \\ &= 4(4^n + 15n - 1) - 45n + 18 \\ &= 4(4^n + 15n - 1) - 9(5n - 2) \end{aligned}$$

By the inductive hypothesis, $9/(4^n + 15n - 1) \implies 9/4(4^n + 15n - 1)$. Also $9/9\underbrace{(5n-2)}_{\in \mathbb{N}}$.

So

$$9/\left[4(4^n + 15n - 1) - 9(5n - 2)\right]$$

So $P(n+1)$ holds. Collecting the two steps, we conclude $P(n)$ holds $\forall n \in \mathbb{N}$. $\square$

Example 2.5

Compute the following sum and then use mathematical induction to prove your answer: for $n \geq 1$

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots + \frac{1}{(2n-1)(2n+1)}$$

<u>Solution</u>: Note that $\frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\left[\frac{1}{2n-1} - \frac{1}{2n+1}\right] \forall n \geq 1$. So

$$\begin{aligned}\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \ldots + \frac{1}{(2n-1)(2n+1)} &= \frac{1}{2}\left\{\frac{1}{1} - \frac{1}{3} + \frac{1}{3} \ldots + \frac{1}{2n-1} - \frac{1}{2n+1}\right\} \\ &= \frac{1}{2}\frac{2n}{2n+1} = \frac{n}{2n+1}\end{aligned}$$

For $n \geq 1$, let $P(n)$ denote the statement

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

<u>Step 1</u>: $P(1)$ becomes $\frac{1}{1\cdot 3} = \frac{1}{3}$, which is true. So $P(1)$ holds.
<u>Step 2</u>: Assume $P(n)$ holds for some $n \geq 1$. We want to show $P(n+1)$. We know

$$\frac{1}{1\cdot 3} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

Let's add $\frac{1}{(2n+1)(2n+3)}$ to both sides

$$\begin{aligned}\frac{1}{1\cdot 3} + \ldots + \frac{1}{(2n+1)(2n+3)} &= \frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)} \\ &= \frac{2n^2+3n+1}{(2n+1)(2n+3)} \\ &= \frac{(n+1)(2n+1)}{(2n+1)(2n+3)} \\ &= \frac{n+1}{2n+3}\end{aligned}$$

So $P(n+1)$ holds.
Collecting the two steps, we conclude $P(n)$ holds for $\forall n \geq 1$. □

§3 Lec 3: Jan 8, 2021

§3.1 Equivalence Relation

The set of integers is $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n : n \in \mathbb{N}\}$.

Definition 3.1 (Equivalence Relation) — An equivalence relation $\sim$ on a non-empty set A satisfies the following three properties:

- Reflexivity: $a \sim a, \forall a \in A$
- Symmetry: If $a, b \in A$ are such that $a \sim b$, then $b \sim a$
- Transitivity: If $a, b, c \in A$ are such that $a \sim b$ and $b \sim c$, then $a \sim c$.

Example 3.2

$=$ is an equivalence relation on $\mathbb{Z}$.

Example 3.3

Let $q \in \mathbb{N}, q > 1$. For $a, b \in \mathbb{Z}$ we write $a \sim b$ if $q/(a-b)$. This is an equivalence relation on $\mathbb{Z}$. Indeed, it suffices to check 3 properties:

- Reflexivity: If $a \in \mathbb{Z}$ then $a - a = 0$, which is divisible by q. So $q/(a-a) \iff a \sim a$.
- Symmetry: Let $a, b \in \mathbb{Z}$ such that $a \sim b \iff q/(a-b)$ which means there exists $k \in \mathbb{Z}$ s.t. $a - b = kq \implies b - a = \underbrace{-k}_{\in \mathbb{Z}} \cdot q$. So $q/(b-a) \iff b \sim a$.
- Transitivity: Let $a, b, c \in \mathbb{Z}$ such that $a \sim b$ and $b \sim c$, $a \sim b \iff q/(a-b) \implies \exists n \in \mathbb{Z}$ s.t. $a - b = q \cdot n$. And $b \sim c \iff q/(b-c) \implies \exists m \in \mathbb{Z}$ s.t. $b - c = q \cdot m$. So, we must have $a - c = q\underbrace{(n+m)}_{\in \mathbb{Z}}$. So $q/(a-c) \iff a \sim c$.

§3.2 Equivalence Class

Definition 3.4 (Equivalence Class) — Let $\sim$ denote an equivalence relation on a non-empty set A. The equivalence class of an element $a \in A$ is given by

$$C(a) = \{b \in A : a \sim b\}$$

Proposition 3.5 (Properties of Equivalence Classes)

Let $\sim$ denote an equivalence relation on a non-empty set A. Then

1. $a \in C(a) \quad \forall a \in A$.
2. If $a, b \in A$ are such that $a \sim b$, then $C(a) = C(b)$.
3. If $a, b \in A$ are such that $a \not\sim b$, then $C(a) \cap C(b) = \emptyset$.
4. $A = \bigcup_{a \in A} C(a)$

Proof.

1. By reflexivity, $a \sim a \quad \forall a \in A \implies a \in C(a) \quad \forall a \in A$.
2. Assume $a, b \in A$ with $a \sim b$. Let's show $C(a) \subseteq C(b)$. Let $c \in C(a)$ be arbitrary. Then $a \sim c$ (by definition). As $a \sim b$ (by hypothesis), which implies $b \sim a$ (by symmetry). By transitivity, we obtain $b \sim c \implies c \in C(b)$. This proves that $C(a) \subseteq C(b)$.

   A similar argument shows that $C(b) \subseteq C(a)$. Putting the two together, we obtain $C(a) = C(b)$.
3. We argue by contradiction. Assume that $a, b \in A$ are such that $a \not\sim b$, but $C(a) \cap C(b) \neq \emptyset$. Let $c \in C(a) \cap C(b)$.

$$c \in C(a) \implies a \sim c$$
$$c \in C(b) \implies b \sim c \implies c \sim b \quad \text{(by symmetry)}$$

   By transitivity, $a \sim b$. This contradicts the hypothesis $a \not\sim b$. This proves that if $a \not\sim$ then $C(a) \cap C(b) = \emptyset$.
4. Clearly, $C(a) \subseteq A \quad \forall a \in A$, we get

$$\bigcup_{a \in A} C(a) \subseteq A$$

   Conversely, $A = \bigcup_{a \in A} \{a\} \subseteq \bigcup_{a \in A} C(a)$. Putting everything together, we obtain $A = \bigcup_{a \in A} C(a)$. □

Example 3.6

Take $q = 2$ in our previous example: for $a, b \in \mathbb{Z}$ we write $a \sim b$ if $2/(a-b)$. The equivalence classes are

$$C(0) = \{a \in \mathbb{Z} : 2/(a-0)\} = \{2n : n \in \mathbb{Z}\}$$
$$C(1) = \{a \in \mathbb{Z} : 2/(a-1)\} = \{2n+1 : n \in \mathbb{Z}\}$$
$$\mathbb{Z} = C(0) \cup C(1)$$

Let $F = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : b \neq 0\}$. If $(a, b), (c, d) \in F$ we write $(a, b) \sim (c, d)$ if $ad = bc$.

Example 3.7

$(1,2) \sim (2,4) \sim (3,6) \sim (-4,-8)$.

Lemma 3.8

$\sim$ is an equivalence relation on F.

Proof. We have to check 3 properties:

- Reflexivity: Fix $(a,b) \in F$. As $ab = ba$ we have $(a,b) \sim (a,b)$
- Symmetry: Let $(a,b),(c,d) \in F$ such that
$$(a,b) \sim (c,d) \iff ad = bc \iff cb = da \iff (c,d) \sim (a,b)$$
- Transitivity: Let $(a,b),(c,d),(e,f) \in F$ such that $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$.
$$(a,b) \sim (c,d) \iff ad = bc \implies adf = bcf$$
$$(c,d) \sim (e,f) \iff cf = de \implies cfb = deb$$
$$\implies adf = deb \implies \underbrace{d}_{\neq 0}(af - be) = 0, \text{ so } af = be \iff (a,b) \sim (e,f).$$

□

For $(a,b) \in F$, we denote its equivalence class by $\frac{a}{b}$. We define addition and multiplication of equivalence classes as follows:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}; \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

We have to check that these operations are well-defined. Specifically, if $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$ then

$$(ad+bc, bd) \sim (a'd' + b'c', b'd') \tag{1}$$
$$(ac, bd) \sim (a'c', b'd') \tag{2}$$

Let's check (1). We want to show

$$(ad+bc)b'd' = bd(a'd' + b'c')$$

We know

$$(a,b) \sim (a',b') \iff ab' = ba' \quad | \cdot dd'$$
$$(c,d) \sim (c',d') \iff cd' = dc' \quad | \cdot bb'$$

Adding the two (after multiplying the two terms) together, we have

$$ab'dd' + cd'bb' = ba'dd' + dc'bb'$$
$$(ad+bc)b'd' = bd(a'd' + b'c')$$

This proves addition is well defined.

Hw: Check (2)

The set of <u>rational numbers</u> is

$$\mathbb{Q} = \left\{ \frac{a}{b} : (a,b) \in F \right\}$$

§4 Lec 4: Jan 11, 2021

§4.1 Field & Ordered Field

Definition 4.1 (Field) — A field is a set F with at least two elements with two operators: addition (denoted $+$) and multiplication (denoted $\cdot$) that satisfy the following

A1) Closure: if $a, b \in F$ then $a + b \in F$

A2) Commutativity: if $a, b \in F$ then $a + b = b + a$

A3) Associativity: if $a, b, c \in F$ then $(a + b) + c = a + (b + c)$

A4) Identity: $\exists 0 \in F$ s.t. $a + 0 = 0 + a = a \ \forall a \in F$

A5) Inverse: $\forall a \in F \exists (-a) \in F$ s.t. $a + (-a) = -a + a = 0$

M1) Closure: if $a, b \in F$ then $a \cdot b \in F$

M2) Commutativity: if $a, b \in F$ then $a \cdot b = b \cdot a$

M3) Associativity: if $a, b, c \in F$ then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

M4) Identity: $\exists 1 \in F$ s.t. $a \cdot 1 = 1 \cdot a = a \ \forall a \in F$

M5) Inverse: $\forall a \in F \setminus \{0\} \, \exists a^{-1} \in F$ s.t. $a \cdot a^{-1} = a^{-1} \cdot a = 1$

D) Distributivity: if $a, b, c \in F$ then $(a + b) \cdot c = a \cdot c + b \cdot c$

Example 4.2

$(\mathbb{N}, +, \cdot)$ is not a field. A4 fails.

Example 4.3

$(\mathbb{Z}, +, \cdot)$ is not a field. M5 fails.

Example 4.4

$(\mathbb{Q}, +, \cdot)$ is a field.

Hw

Recall:

$$\mathbb{Q} = \left\{ \frac{a}{b} : (a, b) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \right\}$$

where $\frac{a}{b}$ denotes the equivalence class of $(a, b) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ with respect to the equivalence relation

$$(a, b) \sim (c, d) \iff a \cdot d = b \cdot c$$

Note $\frac{1}{2} = \frac{2}{4}$ because $(1,2) \sim (2,4)$. We defined

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \qquad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Additive identity $\frac{0}{1}$ equivalence class $(0,1)$.
Multiplicative identity $\frac{1}{1}$ equivalence class of $(1,1)$.
Additive inverse: $\frac{a}{b} \in \mathbb{Q}$ has inverse $-\frac{a}{b}$
Multiplicative inverse: $\frac{a}{b} \in \mathbb{Q} \setminus \left\{\frac{0}{1}\right\}$ has inverse $\frac{b}{a}$.

Proposition 4.5

Let $(F, +, \cdot)$ be a field. Then

1. The additive and multiplicative identities are unique.
2. The additive and multiplicative inverses are unique.
3. If $a, b, c \in F$ s.t. $a + b = a + c$ then $b = c$. In particular, if $a + b = a$ then $b = 0$.

3'. If $a, b, c \in F$ s.t. $a \neq 0$ and $a \cdot b = a \cdot c$ then $b = c$. In particular, $a \neq 0$ and $a \cdot b = a$ then $b = 1$.

4. $a \cdot 0 = 0 \cdot a = 0 \ \forall a \in F$.
5. If $a, b \in F$ then $(-a) \cdot b = a \cdot (-b) = -(a \cdot b)$
6. If $a, b \in F$ then $(-a) \cdot (-b) = a \cdot b$
7. If $a \cdot b = 0$ then $a = 0$ or $b = 0$.

Proof. 1. We'll show the additive identity is unique. Assume

$$\exists 0, 0' \in F \text{ s.t. } \forall a \in F, \begin{cases} a + 0 = 0 + a = a & (i) \\ a + 0' = 0' + a = a & (ii) \end{cases}$$

Take $a = 0'$ in (i) and $a = 0$ in (ii) to get

$$\left.\begin{aligned} 0' + 0 &= 0' \\ 0' + 0 &= 0 \end{aligned}\right\} \implies 0 = 0'$$

2. We'll show that the additive inverse is unique. Let $a \in F$. Assume $\exists (-a), a' \in F$ s.t.

$$\begin{cases} -a + a = a + (-a) = 0 \\ a' + a = a + a' = 0 \end{cases}$$

We have

$$a' + a = 0 \qquad | + (-a)$$

$$(a' + a) + (-a) = 0 + (-a) \overset{A3,A4}{\implies} a' + (a + (-a)) = -a$$
$$\overset{A5}{\implies} a' + 0 = -a \overset{A4}{\implies} a' = -a$$

3. Assume $a+b=a+c \quad |+(-a)$ to the left

$$\begin{aligned} -a+(a+b) &= -a+(a+c) \\ \overset{A3}{\Longrightarrow} (-a+a)+b &= (-a+a)+c \\ \overset{A5}{\Longrightarrow} 0+b = 0+c &\overset{A4}{\Longrightarrow} b=c \end{aligned}$$

So if $a+b=a=a+0$, then $b=0$.

4.

$$\begin{aligned} a\cdot 0 \overset{A4}{=} a\cdot(0+0) \overset{D}{=} a\cdot 0+a\cdot 0 &\overset{(3)}{\Longrightarrow} a\cdot 0=0 \\ 0\cdot a \overset{A4}{=} (0+0)\cdot a = 0\cdot a+0\cdot a &\overset{(3)}{\Longrightarrow} 0\cdot a=0 \end{aligned}$$

5. $(-a)\cdot b+a\cdot b \overset{D}{=} (-a+a)\cdot \overset{A5}{=} 0\cdot b \overset{(4)}{=} 0 \implies (-a)\cdot b=-(a\cdot b)$. Similarly, $a\cdot(-b)=-(a\cdot b)$.

6. $(-a)\cdot(-b)+[-(a\cdot b)] \overset{(5)}{=} (-a)\cdot(-b)+(-a)\cdot b \overset{D}{=} (-a)(-b+b) \overset{A5}{=} (-a)\cdot 0 \overset{(4)}{=} 0$. So $(-a)\cdot(-b)=a\cdot b$.

7. Assume $a\cdot b=0$. Assume $a\neq 0$. Want to show $b=0$. As $a\neq 0$ then $\exists a^{-1}\in F$ s.t. $a\cdot a^{-1}=a^{-1}\cdot a=1$.

$$a\cdot b=0 \quad |\cdot a^{-1} \text{ to the left}$$
$$a^{-1}\cdot(a\cdot b)=a^{-1}\cdot 0 \overset{M3,(4)}{\Longrightarrow} (a^{-1}\cdot a)\cdot b=0 \overset{M5}{\Longrightarrow} 1\cdot b=0 \overset{M4}{\Longrightarrow} b=0$$ □

Definition 4.6 (Order Relation) — An <u>order relation</u> $<$ on a non-empty set A satisfies the following properties:

- <u>Trichotomy</u>: if $a,b\in A$ then one and only one of the following statement holds: $a<b$ or $a=b$ or $b<a$.
- <u>Transitivity</u>: if $a,b,c\in A$ such that $a<b$ and $b<c$, then $a<c$.

Example 4.7

For $a,b\in\mathbb{Z}$ we write $a<b$ if $b-a\in\mathbb{N}$. This is an order relation.

<u>Notation</u>: We write

$$\begin{aligned} &a>b \text{ if } b<a \\ &a\leq b \text{ if } [a<b \text{ or } a=b] \\ &a\geq b \text{ if } b\leq a \end{aligned}$$

Definition 4.8 (Ordered Field) — Let $(F, +, \cdot)$ be a field. We say $(F, +, \cdot)$ is an ordered field if it is equipped with an order relation $<$ that satisfies the following

01) if $a, b, c \in F$ such that $a < b$ then $a + c < b + c$.

02) if $a, b, c \in F$ such that $a < b$ and $0 < c$ then $a \cdot c < b \cdot c$.

Note:

To check something is an ordered field, we have to check that it satisfies the properties of order relation and ordered field.

§5 Lec 5: Jan 13, 2021

§5.1 Ordered Field (Cont'd)

Proposition 5.1

Let $(F, +, \cdot, <)$ be an ordered field. Then,

1. $a > 0 \iff -a < 0$.
2. If $a, b, c \in F$ are such that $a < b$ and $c < 0$, then $ac > bc$.
3. If $a \in F \setminus \{0\}$ then $a^2 = a \cdot a > 0$. In particular, $1 > 0$.
4. If $a, b \in F$ are such that $0 < a < b$ then $0 < b^{-1} < a^{-1}$.

Proof. 1. Let's prove " $\implies$ ". Assume $a > 0$.

$$\overset{01}{\implies} a + (-a) > 0 + (-a) \overset{A5,A4}{\implies} 0 > -a$$

Let's prove " $\impliedby$ ". Assume $-a < 0$

$$\overset{01}{\implies} -a + a < 0 + a \overset{A5,A4}{\implies} 0 < a$$

2. Assume $a < b$ and $c < 0$

$$\left.\begin{aligned} &a < b \\ &c < 0 \overset{01}{\implies} -c > 0 \end{aligned}\right\} \overset{02}{\implies} a \cdot (-c) < b \cdot (-c)$$

$$\begin{aligned} &\overset{01}{\implies} -ac + (ac + bc) < -bc + (ac + bc) \\ &\overset{A3,A2}{\implies} (-ac + ac) + bc < -bc + (bc + ac) \\ &\overset{A5,A3}{\implies} 0 + bc < (-bc + bc) + ac \\ &\overset{A4,A5}{\implies} bc < 0 + ac \\ &\overset{A4}{\implies} bc < ac \end{aligned}$$

3. By trichotomy, exactly one of the following hold:

$$a > 0 \overset{02}{\implies} a \cdot a > 0 \cdot a \implies a^2 > 0$$

or

$$a < 0 \overset{2)}{\implies} a \cdot a > 0 \cdot a \implies a^2 > 0$$

4. First we show that if $a > 0$ then $a^{-1} > 0$. Let's argue by contradiction. Assume $\exists a \in F$ s.t. $a > 0$ but $a^{-1} < 0$. Then

$$\left.\begin{aligned} a &> 0 \\ a^{-1} &< 0 \end{aligned}\right\} \overset{(2)}{\implies} a \cdot a^{-1} < 0 \overset{M5}{\implies} 1 < 0$$

This contradicts (3). So if $a > 0$ then $a^{-1} > 0$.

Say

$$
\begin{aligned}
0 < a < b \quad | \cdot a^{-1} \cdot b^{-1} \\
&\overset{02}{\implies} 0 \cdot (a^{-1} \cdot b^{-1}) < a \cdot (a^{-1} \cdot b^{-1}) < b \cdot (a^{-1} \cdot b^{-1}) \\
&\overset{M3,M2}{\implies} 0 < (a \cdot a^{-1}) \cdot b^{-1} < b \cdot (b^{-1} \cdot a^{-1}) \\
&\overset{M5,M3}{\implies} 0 < 1 \cdot b^{-1} < (b \cdot b^{-1}) \cdot a^{-1} \\
&\overset{M4,M5}{\implies} 0 < b^{-1} < 1 \cdot a^{-1} \\
&\overset{M4}{\implies} 0 < b^{-1} < a^{-1}
\end{aligned}
$$

□

Theorem 5.2 (Ordered Field)

Let $(F, +, \cdot)$ be a field. The following are equivalent

1) F is an ordered field.

2) There exists $P \subseteq F$ that satisfies the following properties

   01') For every $a \in F$ one and only one of the following statements holds: $a \in P$ or $a = 0$ or $-a \in P$.

   02') If $a, b \in P$ then $a + b \in P$ and $a \cdot b \in P$.

Proof. Let's show 1) $\implies$ 2). Define $P = \{a \in F : a > 0\}$. Let's check (01'). Fix $a \in F$. By trichotomy for the order relation on F we get that exactly one of the following statements is true:

- $a > 0 \implies a \in P$.
- $a = 0$.
- $a < 0 \implies -a > 0 \implies -a \in P$.

Let's check (02'). Fix $a, b \in P$.

$$
\left.\begin{aligned} a \in P \implies a > 0 \\ b \in P \implies b > 0 \end{aligned}\right\} \overset{01}{\implies} a + b > 0 + b \overset{A4}{=} b > 0 \implies a + b \in P
$$

And

$$
\left.\begin{aligned} a \in P \implies a > 0 \quad | \cdot b \\ b \in P \implies b > 0 \qquad \end{aligned}\right\} \overset{02}{\implies} a \cdot b > 0 \cdot b = 0 \implies a \cdot b \in P
$$

Let's check that 2) $\implies$ 1).
For $a, b \in F$ we write $a < b$ if $b - a \in P$. Let's check this is an order relation.

- Trichotomy: Fix $a, b \in F$. By 01') exactly one of the following hold:

$$\begin{aligned} b-a \in P &\implies a<b \\ b-a=0 &\implies a=b \\ -(b-a) \in P &\implies a-b \in P \implies b<a \end{aligned}$$

- Transitivity Assume $a, b, c \in F$ s.t. $a<b$ and $b<c$

$$\left.\begin{aligned} a<b &\implies b-a \in P \\ b<c &\implies c-b \in P \end{aligned}\right\} \overset{02'}{\implies} (b-a)+(c-b) \in P \implies c-a \in P \implies a<c$$

Now let's check that with this order relation, F is an ordered field. We have to check 01 and 02.

01) Fix $a, b, c \in F$ s.t. $a<b \implies b-a \in P \implies b-a \in P \implies (b+c)-(a+c) \in P \implies a+c<b+c$.

02) Fix $a, b, c \in F$ s.t. $a<b$ and $0<c$

$$\left.\begin{aligned} a<b &\implies b-a \in P \\ 0<c &\implies c-0=c \in P \end{aligned}\right\} \overset{02'}{\implies} (b-a) \cdot c \in P \overset{D}{\implies} b \cdot c-a \cdot c \in P \implies a \cdot c<b \cdot c \quad \square$$

We extend the order relation $<$ from $\mathbb{Z}$ to the field $(\mathbb{Q},+,\cdot)$ by writing $\frac{a}{b}>0$ if $a \cdot b>0$. Let's see this is well defined. Specifically, we need to show that if $\frac{a}{b}=\frac{c}{d}$, i.e., $(a, b) \sim (c, d)$ and $a \cdot b>0$ then $c \cdot d>0$.

$$\begin{aligned} (a, b) \sim (c, d) &\implies a \cdot d=b \cdot c \quad \mid \cdot (ad) \\ &\implies 0<(ad)^2=(ab) \cdot (cd) \text{ where } a \cdot d \neq 0 \end{aligned}$$

So

$$\left.\begin{aligned} &0<(ab) \cdot (cd) \\ &0<ab \end{aligned}\right\} \implies cd>0 \implies \frac{c}{d}>0$$

Let $P=\left\{\frac{a}{b} \in \mathbb{Q}: \frac{a}{b}>0\right\}$. By the theorem, to prove that $\mathbb{Q}$ is an ordered field, it suffices to show that P satisfies (01') and (02').

Hw: check (01') and (02')

§6 Lec 6: Jan 15, 2021

§6.1 Least Upper Bound & Greatest Lower Bound

Definition 6.1 (Boundedness – Maximum and Minimum) — Let $(F,+,\cdot,<)$ be an ordered field. Let $\emptyset \neq A \subseteq F$. We say that A is bounded above if $\exists M \in F$ s.t. $a \leq M \forall a \in A$. Then M is called an upper bound for A. If moreover, $M \in A$ then we say that M is the maximum of A.
We say that A is bounded below if $\exists m \in F$ s.t. $m \leq a \forall a \in A$. Then m is called a lower bound for A. If moreover, $m \in A$ then we say that m is the minimum of A.
We say that A is bounded if A is bounded both above and below.

Example 6.2
$A = \left\{1 + \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$ bounded.

- 3 is an upper bound for A.
- $\frac{3}{2}$ is the maximum of A.
- 0 is a lower bound for A ; 0 is the minimum of A.

Example 6.3
$A = \left\{x \in \mathbb{Q} : 0 < x^4 \leq 16\right\}$ bounded.

- 2 is the maximum of A.
- -2 is the minimum of A.

Example 6.4

$A = \left\{x \in \mathbb{Q} : x^2 < 2\right\}$ bounded.

- 2 is an upper bound for A.
- -2 is lower bound for A.
- A does not have a maximum. Indeed, let $x \in A$. We'll construct $y \in A$ s.t. $y > x$. Define $y = x + \frac{2-x^2}{2+x}$.

$$\left.\begin{array}{l} x \in A \implies x \in \mathbb{Q} \implies 2 - x^2, 2 + x \in \mathbb{Q} \\ x \in A \implies 2 + x > 0 \implies \frac{1}{2+x} \in \mathbb{Q} \end{array}\right\} \implies \frac{2-x^2}{2+x} \in \mathbb{Q} \implies y \in \mathbb{Q}(i)$$

Also note

$$\left.\begin{array}{l} 2 - x^2 > 0 (\text{as } x \in A) \\ 2 + x > 0 \implies \frac{1}{2+x} > 0 \end{array}\right\} \implies \frac{2-x^2}{2+x} > 0$$

So $y = x + \frac{2-x^2}{2+x} > x$ (ii). Let's compute $y^2 = \left(\frac{2x+x^2+2-x^2}{2+x}\right)^2 = \frac{2(x^2+4x+4)+2x^2-4}{x^2+4x+4} = 2 + \underbrace{\frac{2(x^2-2)}{(x+2)^2}}_{<0}$. So $y^2 < 2$. (iii)

So collecting (i) – (iii) we get $y \in A$ and $y > x$.

Homework 6.1. Show that the maximum and minimum of a set are unique, if they exist.

Definition 6.5 (Least Upper Bound) — Let $(F, +, \cdot, <)$ be an ordered field. Let $\emptyset \neq A \subseteq F$ and assume A is bounded above. We say that L is the least upper bound of A if it satisfies:

1. L is an upper bound of A.
2. If M is an upper bound of A then $L \leq M$.

We write $L = \sup A$ and we say L is the supremum of A.

Lemma 6.6

The least upper bound of a set is unique, if it exists.

Proof. Say that a set $\emptyset \neq A \subseteq F$, A bounded above, admits two least upper bounds L, M.
L is a least upper bound $\overset{(1)}{\implies}$ L is an upper bound for A.
M is a least upper bound $\overset{(2)}{\implies}$ $M \leq L$.
M is a least upper bound for A $\overset{(1)}{\implies}$ M is an upper bound for A $\implies$ L is a least upper bound for A $\overset{(2)}{\implies}$ $L \leq m$. So $L = M$. ☐

Definition 6.7 (Greatest Lower Bound) — Let $(F, +, \cdot, <)$ be an ordered field. Let $\emptyset \neq A \subseteq F$ and assume A is bounded below. We say that l is the greatest lower bound of Aif it satisfies

1. l is a lower bound of A.
2. If m is a lower bound of A then $m \leq l$.

We write $l = \inf A$ and we say l is the infimum of A.

Homework 6.2. Show that the greatest lower bound of a set is unique if it exists.

Definition 6.8 (Bound Property) — Let $(F, +, \cdot, <)$ be an ordered field. Let $\emptyset \neq S \subseteq F$. We say that S has the the least upper bound property if it satisfies the following: For any non-empty subset A of S is bounded above, there exists a least upper bound of A and $\sup A \in S$.
We say that S has the greatest lower bound property if it satisfies the following: $\forall \emptyset \neq A \subseteq S$ with A bounded below, $\exists \inf A \in S$.

Example 6.9

$(\mathbb{Q}, +, \cdot, <)$ is an ordered field.
$\emptyset \neq \mathbb{N} \subseteq \mathbb{Q}$, $\mathbb{N}$ has the least upper bound property. Indeed if $\emptyset \neq A \subseteq \mathbb{N}$, A bounded above, then the largest elements in A is the least upper bound of A and $\sup A \in \mathbb{N}$. $\mathbb{N}$ also has the greatest lower bound property.

Example 6.10

$(\mathbb{Q}, +, \cdot, <)$ is an ordered field.
$\emptyset \neq \mathbb{Q} \subseteq \mathbb{Q}$, $\mathbb{Q}$ does not have the least upper bound property.
Indeed, $\emptyset \neq A = \left\{x \in \mathbb{Q} : x \geq 0 \text{ and } x^2 < 2\right\} \subseteq \mathbb{Q}$. A is bounded above by 2. However, $\sup A = \sqrt{2} \notin \mathbb{Q}$.

Proposition 6.11

Let $(F, +, \cdot, <)$ be an ordered field. Then F has the least upper bound property if and only if it has the greatest lower bound property.

Proof. ($\implies$) Assume F has the least upper bound property. Let $\emptyset \neq A \subseteq F$ bounded below. WTS $\exists \inf A \in F$. A is bounded below $\implies$ $\exists m \in F$ s.t. $m \leq a \forall a \in A$. Let $B = \{b \in F : b \text{ is a lower bound for } A\}$. Note $B \neq \emptyset$ (as $m \in B$), $B \subseteq F$, B is bounded above (every element in A is an upper bound for B) and F has the least upper bound property $\implies$ $\sup B \in F$.

Claim 6.1. $\sup B = \inf A$ (to be proven in Lec 7). □

§7 Lec 7: Jan 20, 2021

§7.1 Least Upper & Greatest Lower Bound (Cont'd)

Proof. (Cont'd of proposition 6.11)

Claim 7.1. $\sup B = \inf A$.

Method 1:

- $\sup B$ is a lower bound for A. Indeed, let $a \in A$. We know that $a \geq b \quad \forall b \in B$. $\sup B$ is the least upper bound for $B \implies a \geq \sup B$. As $a \in A$ was arbitrary, we conclude that $\sup B \leq a \quad \forall a \in A$ and so $\sup B$ is a lower bound for A.

- If l is a lower bound for A then $l \leq \sup B$. Well, l is a lower bound for $A \implies l \in B$ and $\sup B$ is an upper bound for B. So $l \leq \sup B$.

Collecting the two bullet points above, we find that $\inf A = \sup B$.
Method 2: Let $\emptyset \neq A \subseteq F$ s.t. A is bounded below. Let $B = \{-a : a \in A\}$. Note $B \subseteq F$ by A5. $B \neq \emptyset$ because $A \neq \emptyset$. B is bounded above: indeed if m is a lower bound for A then $-m$ is an upper bound for B.

$$m \leq a \quad \forall a \in A \implies -m \geq -a \quad \forall a \in A$$

F has the least upper bound property. Altogether, it implies that $\sup B \in F$. In Hw3, you show $-\sup B = \inf A \in F$ (by A5). □

Homework 7.1. Prove the "$\Longleftarrow$" direction.

Theorem 7.1 (Existence of $\mathbb{R}$)

There exists an ordered field with the least upper bound property. We denote it $\mathbb{R}$ and we call it the set of real numbers. $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield. Moreover, we have the following uniqueness property: If $(F, +, \cdot, <)$ is an ordered field with the least upper bound property, then F is order isomorphic with $\mathbb{R}$, that is, there exists a bijection $\phi : \mathbb{R} \to F$ such that

i) $\phi(x \underbrace{+}_{\mathbb{R}} y) = \phi(x) \underbrace{+}_{F} \phi(y)$

ii) $\phi(x \underbrace{\cdot}_{\mathbb{R}} y) = \phi(x) \underbrace{\cdot}_{F} \phi(y)$

iii) If $x \underbrace{<}_{\mathbb{R}} y$ then $\phi(x) \underbrace{<}_{F} \phi(y)$

Theorem 7.2 (Archimedean Property)

$\mathbb{R}$ has the Archimedean property, that is, $\forall x \in \mathbb{R} \quad \exists n \in \mathbb{N}$ s.t. $x < n$.

Proof. We argue by contradiction. Assume

$$\exists x_0 \in \mathbb{R} \text{ s.t. } x_0 \geq n \quad \forall n \in \mathbb{N}$$

Then $\emptyset \neq \mathbb{N} \subseteq \mathbb{R}$. $\mathbb{N}$ is bounded above by x_0. $\mathbb{R}$ has the least upper bound property $\implies \exists L = \sup \mathbb{N} \in \mathbb{R}$.

$$\left.\begin{array}{r} L = \sup \mathbb{N} \\ L - 1 < L \end{array}\right\} \implies L - 1 \text{ is not an upper bound for } \mathbb{N}$$

$\implies \exists n_0 \in \mathbb{N}$ s.t. $n_0 > L - 1$. So $\sup \mathbb{N} = L < n_0 + 1 \in \mathbb{N}$, which is a contradiction. □

Remark 7.3. $\mathbb{Q}$ has the Archimedean property.

If $r \in \mathbb{Q}$ is s.t. then choose $n = 1$. For $r \in \mathbb{Q}$ is s.t. $r > 0$, then write $r = \frac{p}{q}$ with $p, q \in \mathbb{N}$. Choose $n = p + 1$ since $\frac{p}{q} < p + 1$.

Corollary 7.4

If $a, b \in \mathbb{R}$ such that $a > 0, b > 0$ then there exists $n \in \mathbb{N}$ s.t. $n \cdot a > b$.

Proof. Apply the Archimedean Property to $x = \frac{b}{a}$. □

Corollary 7.5

If $\epsilon > 0$ there exists $n \in \mathbb{N}$ s.t. $\frac{1}{n} < \epsilon$.

Proof. Apply the Archimedean property to $x = \frac{1}{\epsilon}$. □

Lemma 7.6

For any $a \in \mathbb{R}$ there exists $N \in \mathbb{Z}$ s.t. $N \leq a \leq N + 1$.

Proof. Case 1: $a = 0$. Take $N = 0$.
Case 2: $a > 0$. Consider $A = \{n \in \mathbb{Z} : n \leq a\} \subseteq \mathbb{R}$, $A \neq \emptyset (0 \in A)$. A is bounded above by a. $\mathbb{R}$ has the least upper bound property. So $\exists L = \sup A \in \mathbb{R}$.

$$L - 1 < L = \sup A \implies L - 1 \text{ is not an upper bound for } A$$

$\implies \exists N \in A$ s.t. $L - 1 < N \implies L < N + 1$ but $L = \sup A$, so $N + 1 \notin A$. So

$$\left.\begin{array}{r} N \in A \implies N \leq a \\ N + 1 \notin A \implies N + 1 > a \end{array}\right\} \implies N \leq a < N + 1$$

Case 3: $a < 0 \implies -a > 0$. By case 2, $\exists n \in \mathbb{Z}$ s.t. $n \leq -a < n + 1$. So $-n - 1 < a \leq -n$. If $a = -n$, let $N = -n$ and so $N \leq a < N + 1$. If $a < -n$ let $N = -n - 1$ and so $N \leq a < N + 1$. □

Definition 7.7 (Dense Set) — We say that a subset A of $\mathbb{R}$ is dense in $\mathbb{R}$ if for every $x, y \in \mathbb{R}$ such that $x < y$ there exists $a \in A$ such that $x < a < y$.

Lemma 7.8

$\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof. Let $x, y \in \mathbb{R}$ such that $x < y$. Since $y - x > 0$ by corollary 7.5, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < y - x \implies \frac{1}{n} + x < y$.
Consider $nx \in \mathbb{R}$. By the lemma 7.6, $\exists m \in \mathbb{Z}$ s.t.

$$m \leq nx < m + 1 \implies \frac{m}{n} \leq x < \frac{m+1}{n}$$

Then

$$x < \frac{m+1}{n} = \frac{m}{n} + \frac{1}{n} \leq x + \frac{1}{n} < y$$

w where $\frac{m+1}{n} \in \mathbb{Q}$. □

Lemma 7.9

$\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

§8 Lec 8: Jan 22, 2021

§8.1 Construction of the Reals

Recall that we say a set $A \subseteq \mathbb{R}$ is dense if for every $x, y \in \mathbb{R}$ s.t. $x < y$, there exists $a \in A$ s.t. $x < a < y$. Last time we proved

Lemma 8.1

$\mathbb{Q}$ is dense in $\mathbb{R}$.

Remark 8.2. For any two rational numbers $r_1, r_2 \in \mathbb{Q}$ s.t. $r_1 < r_2$, there exists $s \in \mathbb{Q}$ s.t. $r_1 < s < r_2$.

Indeed if $r_1 < 0 < r_2$ then we may take $s = 0$.
Assume $0 < r_1 < r_2$. Write $r_1 = \frac{a}{b}, a_2 = \frac{c}{d}$ with $a, b, c, d \in \mathbb{N}$. Take $s = \frac{ad+bc}{2bd} \in \mathbb{Q}$. Note $r_1 < s < r_2$.

$$r_1 < s \iff \frac{a}{b} < \frac{ad+bc}{2bd} \iff 2ad < ad + bc \iff ad < bc \iff \frac{a}{b} < \frac{c}{d} \iff r_1 < r_2$$

Homework 8.1. Construct s in the remaining cases.

Lemma 8.3

$\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

Proof. Let $x, y \in \mathbb{R}$ s.t. $x < y \implies x + \sqrt{2} < y + \sqrt{2}$. $\mathbb{Q}$ is dense in $\mathbb{R}$. So $\exists q \in \mathbb{Q}$ s.t. (since $\mathbb{Q}$ is dense in $\mathbb{R}$)

$$x + \sqrt{2} < q < y + \sqrt{2} \implies x < q - \sqrt{2} < y$$

Claim 8.1. $q - \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$.

Otherwise, $\exists r \in \mathbb{Q}$ s.t. $q - \sqrt{2} = r \implies \sqrt{2} = q - r \in \mathbb{Q}$, contradiction. $\square$

Theorem 8.4 (Construction of $\mathbb{R}$(Existence))

There exists an ordered field with the least upper bound property. We denote it $\mathbb{R}$ and call it the set of real numbers. $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.

Proof. We will construct an ordered field with the least upper bound property using Dedekind cuts. The elements of the field are certain subsets of $\mathbb{Q}$ called cuts.

Definition 8.5 ((Dedekind) Cuts) — A cut is a set $\alpha \subseteq \mathbb{Q}$ that satisfies:

a) $\emptyset \neq \alpha \neq \mathbb{Q}$

b) If $q \in \alpha$ and $p \in \mathbb{Q}$ s.t. $p < q$ then $p \in \alpha$.

c) For every $q \in \alpha$ there exists $r \in \alpha$ s.t. $r > q$ (α has no maximum)

Intuitively, we think of a cut as $\mathbb{Q} \cap (\infty, a)$. Of course, at this point we haven't yet constructed $\mathbb{R}$...
Note that if $\mathbb{Q} \ni q \notin \alpha$ then $q > p \forall p \in \alpha$. Indeed, otherwise, if $\exists p_0 \in \alpha$ s.t. $q \leq p_0$ then by ii) we would have $q \in \alpha$. Contradiction.
We define

$$F = \{\alpha : \alpha \text{ is a cut}\}$$

We will show F is an ordered field with the least upper bound property.
Order: For $\alpha, \beta \in F$ we write $\alpha < \beta$ if α is a proper subset of β, that is, $\alpha \subsetneq \beta$

- Transitivity: If $\alpha, \beta, \gamma \in F$ s.t. $\alpha < \beta$ and $\beta < \gamma$ then $\alpha \subsetneq \beta \subsetneq \gamma \implies \alpha \subsetneq \gamma \implies \alpha < \gamma$.

- Trichotomy: First note that at most one of the following hold

$$\alpha < \beta, \quad \alpha = \beta, \quad \beta < \alpha$$

To prove trichotomy, it thus suffices to show that at least one of the following holds: $\alpha < \beta, \alpha = \beta, \beta < \alpha$. We show this by contradiction: Assume $\alpha < \beta, \alpha = \beta, \beta < \alpha$ all fail. Then we have

$$\left.\begin{matrix} \alpha \nsubseteq \beta \\ \alpha \neq \beta \\ \beta \nsubseteq \alpha \end{matrix}\right\} \implies \begin{cases} \exists p \in \alpha \setminus \beta \\ \exists q \in \beta \setminus \alpha \end{cases}$$

Now

$$p \notin \beta \implies p > r \quad \forall r \in \beta \tag{1}$$
$$q \notin \alpha \implies q > s \quad \forall s \in \alpha \tag{2}$$

Take $r = q$ in (1) and $s = p$ in (2) to get $p > q > p$. Contradiction!

So $<$ defines an order relation on F.
Let's show that F has the least upper bound property. Let $\emptyset \neq A \subseteq F$ bounded above by $\beta \in F$. Define

$$\gamma = \bigcup_{\alpha \in A} \alpha$$

Claim 8.2. $\gamma \in F$.

- $\gamma \neq \emptyset$ because $A \neq \emptyset$ and $\emptyset \neq \alpha \in A$.

- $\gamma \neq \mathbb{Q}$ because β being an upper bound for A

$$\implies \beta \geq \alpha \forall \alpha \in A \implies \beta \supseteq \alpha \forall \alpha \in A \implies \beta \supseteq \bigcup_{\alpha \in A} \alpha = \gamma$$

  As $\beta \neq \mathbb{Q} \implies \gamma \neq \mathbb{Q}$.

- Let $q \in \gamma$ and let $p \in \mathbb{Q}$ s.t. $p < q$. As $q \in \gamma \implies \exists \alpha \in A$ s.t. $q \in \alpha$ and $\mathbb{Q} \ni p < q$. So $p \in \alpha \implies p \in \gamma$.

- Let $q \in \gamma \implies \exists \alpha \in A$ s.t. $q \in \alpha \implies \exists r \in \alpha$ s.t. $q < r$. Then $r \in \gamma$ and $q < r$.

Collecting all these properties, we deduce $\gamma \in F$.

Claim 8.3. $\gamma = \sup A$.

- Note $\alpha \subseteq \gamma \forall \alpha \in A \implies \alpha \leq \gamma \forall \alpha \in A$. So γ is an upper bound for A.

- Let δ be an upper bound for $A \implies \delta \geq \alpha \forall \alpha \in A \implies \delta \supseteq \alpha \forall \alpha \in A$. So $\delta \supseteq \bigcup_{\alpha \in A} \alpha = \gamma \implies \delta \geq \gamma$.

Addition: If $\alpha, \beta \in F$ we define

$$\alpha + \beta = \{p + q : p \in \alpha \text{ and } q \in \beta\}$$

Let's check A1, namely, $\alpha + \beta \in F$.

- Note $\alpha + \beta = \emptyset$ because $\alpha \neq \emptyset \implies \exists p \in \alpha$ and $\beta \neq \emptyset \implies \exists q \in \beta$ which implies $p + q \in \alpha + \beta$.

- Note $\alpha + \beta \neq \mathbb{Q}$. Indeed $\alpha \mathbb{Q} \implies \exists r \in \mathbb{Q} \setminus \alpha \implies r > p \forall p \in \alpha$ and $\beta \neq \mathbb{Q} \implies \exists s \in \mathbb{Q} \setminus \beta \implies s > q \forall q \in \beta$ which implies $r + s > p + q \forall p \in \alpha$ and $\forall q \in \beta \implies r + s \notin \alpha + \beta$

- Let $r \in \alpha + \beta$ and $s \in \mathbb{Q}$ s.t. $s < r$

$$\begin{aligned} r \in \alpha + \beta &\implies r = p + q \text{ for some } p \in \alpha \text{ and some } q \in \beta \\ s < r &\implies s < p + q \implies \underbrace{s - p}_{\in \mathbb{Q}} < \underbrace{q}_{\in \beta} \implies s - p \in \beta \end{aligned}$$

  So $s = p + (s - p) \in \alpha + \beta$.

- Let $r \in \alpha + \beta \implies r = p + q$ for some $p \in \alpha$ and some $q \in \beta$

$$\left. \begin{aligned} \alpha \in F &\implies \exists p' \in \alpha \ni p' > p \\ \beta \in F &\implies \exists q' \in \beta \ni q' > q \end{aligned} \right\} \implies \alpha \ni p' + q' \in \beta > p + q = r$$

  So $p' + q' \in \alpha + \beta$ s.t. $p' + q' > r$.

So collecting all these properties, we see that $\alpha + \beta \in F$. $\square$

§9 Lec 9: Jan 25, 2021

§9.1 Construction of the Reals (Cont'd)

Recall: A cut is set $\alpha \subseteq \mathbb{Q}$ such that

i) $\emptyset \neq \alpha \neq \mathbb{Q}$

ii) If $q \in \alpha$ and $p \in \mathbb{Q}$ with $p < q$ then $p \in \alpha$

iii) $\forall q \in \alpha \quad \exists r \in \alpha$ s.t. $r > q$.

We defined

$$F = \{\alpha : \alpha \text{ is a cut}\}$$

We defined an order relation on F : for $\alpha, \beta \in F$ we write $\alpha < \beta \iff \alpha \subsetneq \beta$. We showed that F has the least upper bound property with respect to this order relation.
We defined an addition operation on F : for $\alpha, \beta \in F$

$$\alpha + \beta = \{p + q : p \in \alpha \text{ and } q \in \beta\}$$

We checked A1. Let's check A2: for $\alpha, \beta \in F$

$$\begin{aligned} \alpha + \beta &= \{p + q : p \in \alpha, q \in \beta\} \\ &= \{q + p : q \in \beta, p \in \alpha\} \text{ (since addition in } \mathbb{Q} \text{ satisfies A2)} \\ &= \beta + \alpha \end{aligned}$$

Let's check A3: for $\alpha, \beta, \gamma \in F$

$$\begin{aligned} (\alpha + \beta) + \gamma &= \{s + r : s \in \alpha + \beta, r \in \gamma\} \\ &= \{(p + q) + r : p \in \alpha, q \in \beta, r \in \gamma\} \\ &= \{p + (q + r) : p \in \alpha, q \in \beta, r \in \gamma\} \text{ (since addition in } \mathbb{Q} \text{ satisfies A3} \\ &= \{p + t : p \in \alpha, t \in \beta + \gamma\} \\ &= \alpha + (\beta + \gamma) \end{aligned}$$

Let's check A4: Let $0^* = \{q \in \mathbb{Q} : q < 0\}$.

Claim 9.1. $0^* \in F$

- Note $0^* \neq \emptyset$ since $-1 \in 0^*$
- Note $0^* = \mathbb{Q}$ since $2 \notin 0^*$
- Let $q \in 0^*$ and let $p \in \mathbb{Q}$ and $p < q$

$$\left.\begin{aligned} q \in 0^* \implies q < 0 \\ p < q \end{aligned}\right\} \implies p < 0$$

  So $p \in 0^*$.

- Let $q \in 0^* \implies q < 0 \implies \exists r \in \mathbb{Q}$ s.t. $q < r < 0$. So $r \in 0^*$ and $r > q$.

Collecting all these properties we got $0^* \in F$.

Claim 9.2. $\alpha + 0^* = \alpha \qquad \forall \alpha \in F$.

- Let's check $\alpha + 0^* \subseteq \alpha$.

  Let $r \in \alpha + 0^* \implies r = p + q$ for some $p \in \alpha$ and some $q \in 0^*$. $q \in 0^* \implies q < 0$. So

  $$\left.\begin{array}{l} \mathbb{Q} \ni r = p + q < p \\ p \in \alpha \in F \end{array}\right\} \implies r \in \alpha$$

  As r was arbitrary in $\alpha + 0^*$ we find $\alpha + 0^* \subseteq \alpha$.

- Let's check $\alpha \subseteq \alpha + 0^*$. Let $p \in \alpha \implies \exists r \in \alpha$ s.t. $r > p$. We write

  $$p = \underbrace{r}_{\in \alpha} + \underbrace{(p - r)}_{\in 0^*} \in \alpha + 0^*$$

  As $p \in \alpha$ was arbitrary, this shows $\alpha \subseteq \alpha + 0^*$

Collecting everything, we get $\alpha + 0^* = \alpha$.
Let's check A5: Fix $\alpha \in F$. Define

$$\beta = \{q \in \mathbb{Q} : \exists r \in \mathbb{Q} \text{ with } r > 0 \ni -q - r \notin \alpha\}$$

Claim 9.3. $\beta \in F$.

- Note that $\beta \neq \emptyset$.

  As $\alpha \neq \mathbb{Q} \implies \exists p \in \mathbb{Q} \setminus \alpha$. Then $-(p+1) \in \beta$ because $-[-(p+1)] - 1 = (p+1) - 1 = p \notin \alpha$.

- Note that $\beta \neq \mathbb{Q}$.

  As $\alpha \neq \emptyset \implies \exists p \in \alpha$. Then $-p \notin \beta$ because $\forall r \in \mathbb{Q}, r > 0$ we have

  $$\left.\begin{array}{l} -(-p) - r = p - r < p \\ p \in \alpha \in F \end{array}\right\} \implies p - r \in \alpha$$

  So $-p \notin \beta$.

- Let $q \in \beta$ and let $p \in \mathbb{Q}$ s.t. $p < q$

  $$q \in \beta \implies \exists r \in \mathbb{Q}, r > 0 \ni -q - r \notin \alpha \implies -q - r > s \forall s \in \alpha$$

  So $-p - r > -q - r > s \forall s \in \alpha \implies -p - r \notin \alpha \implies p \in \beta$.

- Let $q \in \beta$. Want to find $s \in \beta$ s.t. $s > q$.

  $$\begin{aligned} q \in \beta &\implies \exists r \in \mathbb{Q} \ni r > 0 \text{ and } -q - r \notin \alpha \\ &\implies -\left(2 + \frac{r}{2}\right) - \frac{r}{2} = -q - r \notin \alpha \\ &\implies q + \frac{r}{2} \in \beta \end{aligned}$$

  Let $s = q + \frac{r}{2}$.

Collecting all the properties, we get $\beta \in F$.

Claim 9.4. $\alpha + \beta = 0^*$.

- Let's check that $\alpha + \beta \subseteq 0^*$.

  Let $s \in \alpha + \beta \implies s = p + q$ with $p \in \alpha$ and $q \in \beta$. Since $q \in \beta \implies \exists r \in \mathbb{Q}, r > 0 \ni -q - r \notin \alpha \implies -q - r > p$. So $\underbrace{p+q}_{\in \mathbb{Q}} < -r < 0$. So $s = p + q \in 0^*$. Thus $\alpha + \beta \subseteq 0^*$.

- Let's check $0^* \subseteq \alpha + \beta$. Let $r \in 0^* \implies r \in \mathbb{Q}, r < 0$.

  Claim 9.5. $\exists N \in \mathbb{N}$ s.t. $N \cdot \left(-\frac{r}{2}\right) \in \alpha$ but $(N+1)\left(-\frac{r}{2}\right) \notin \alpha$.

  Let's prove this by contradiction. Assume

  $$\left\{n\left(-\frac{r}{2}\right) : n \in \mathbb{N}\right\} \subseteq \alpha$$

  We will show that in this case $\mathbb{Q} \subseteq \alpha$ thus reaching a contradiction.

  Fix $q \in \mathbb{Q}$. By the Archimedean property for $\mathbb{Q}$, $\exists n \in \mathbb{N}$ s.t. $n > \underbrace{q \cdot \left(-\frac{2}{r}\right)}_{\in \mathbb{Q}}$. So

  $$\left.\begin{aligned} &n \cdot \left(-\tfrac{r}{2}\right) > q \\ &n \cdot \left(-\tfrac{r}{2}\right) \in \alpha \in F \end{aligned}\right\} \implies q \in \alpha$$

  As $q \in \mathbb{Q}$ was arbitrary, this shows $\mathbb{Q} \subseteq \alpha$. Contradiction!

  Write $r = \underbrace{N\left(-\frac{r}{2}\right)}_{\in \alpha} + (N+2) \cdot \frac{r}{2}$ and note that $(N+2)\frac{r}{2} \in \beta$ since

  $$-(N+2) \cdot \frac{r}{2} - \frac{r}{2} = (N+1) \cdot \left(-\frac{r}{2}\right) \notin \alpha$$

  As $r \in 0^*$ was arbitrary, this shows $0^* \subseteq \alpha + \beta$. Thus, $\alpha + \beta = 0^*$.

Let's check 01: if $\alpha, \beta, \gamma \in F$ s.t. $\alpha < \beta \implies \alpha \subsetneq \beta$ then $\alpha + \gamma \subsetneq \beta + \gamma \implies \alpha + \gamma < \beta + \gamma$. WE define multiplication on F as follows: for $\alpha < \beta \in F$ with $\alpha > 0$, $\beta > 0$ we define

$$\alpha \cdot \beta = \{q \in \mathbb{Q} : q < r \cdot s \text{ for some } 0 < r \in \alpha \text{ and some } 0 < s \in \beta\}$$

For $\alpha \in F$ we define $\alpha \cdot 0^* = 0^*$. We define

$$\alpha \cdot \beta = \begin{cases} (-\alpha) \cdot (-\beta), & \text{if } \alpha < 0, \beta < 0 \\ -\left[(-\alpha) \cdot \beta\right], & \text{if } \alpha < 0, \beta > 0 \\ -\left[\alpha \cdot (-\beta)\right], & \text{if } \alpha > 0, \beta < 0 \end{cases}$$

You checked M1 through M5 for positive cuts. This extends readily to all cuts.

Homework 9.1. Check (D) and (02).

We identify a rational number $r \in \mathbb{Q}$ with the cut

$$r^* = \{q \in \mathbb{Q} : q < r\}$$

One can check that

$$\begin{aligned} r^* + s^* &= (r+s)^* \\ r^* \cdot s^* &= (r \cdot s)^* \\ r < s &\iff r^* < s^* \end{aligned}$$

§10 Lec 10: Jan 27, 2021

§10.1 Sequences

Definition 10.1 (Sequence) — A sequence of real number is a function $f : \{n \in \mathbb{Z} : n \geq m\} \to \mathbb{R}$ where m is a fixed integer (m is usually 0 or 1). We write the sequence as $f(m), f(m+1), f(m+2), \ldots$ or as $\{f(n)\}_{n \geq m}$ or as $\{f_n\}_{n \geq m}$.

Example 10.2

1. $\{a_n\}_{n \geq 1}$ with $a_n = 3 - \frac{1}{n}$ bounded, strictly increasing.
2. $\{a_n\}_{n \geq 1}$ with $a_n = (-1)^n$ bounded, not monotone.
3. $\{a_n\}_{n \geq 0}$ with $a_n = n^2$ bounded below, strictly increasing.
4. $\{a_n\}_{n \geq 0}$ with $a_n = \cos\left(\frac{n\pi}{3}\right)$ bounded, not monotone.

Definition 10.3 (Boundedness of Sequence) — We say that a sequence $\{a_n\}_{n \geq 1}$ of real numbers is bounded below/bounded above/bounded if the set $\{a_n : n \geq 1\}$ is bounded below/bounded above/bounded.

We say that the sequence $\{a_n\}_{n \geq 1}$ is

- increasing if $a_n \leq a_{n+1} \quad \forall n \geq 1$
- strictly increasing if $a_n < a_{n+1} \quad \forall n \geq 1$
- decreasing if $a_n \geq a_{n+1} \quad \forall n \geq 1$
- strictly decreasing if $a_n > a_{n+1} \quad \forall n \geq 1$.
- monotone if it's either increasing or decreasing

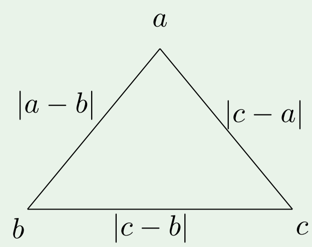
To define the notion of convergence of a sequence, we need a notion of distance between two real numbers.

Definition 10.4 (Absolute Value) — For $x \in \mathbb{R}$, the absolute value of x is

$$|x| = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$$

This function satisfies the following:

1. $|x| \geq 0 \quad \forall x \in \mathbb{R}$
2. $|x| = 0 \iff x = 0$
3. $|x+y| < |x| + |y| \quad \forall x, y \in \mathbb{R}$ (the triangle inequality)



$$\underbrace{|c-b|}_{x+y} \leq \underbrace{|c-a|}_{x} + \underbrace{|a-b|}_{y}$$

4. $|x \cdot y| = |x| \cdot |y| \quad \forall x, y \in \mathbb{R}$

Homework 10.1. $||x| - |y|| \leq |x - y| \quad \forall x, y \in \mathbb{R}$.

We think of $|x - y|$ as the distance between $x, y \in \mathbb{R}$.

Definition 10.5 (Convergent Sequence) — We say that a sequence $\{a_n\}_{n\geq 1}$ of real numbers <u>converges</u> if

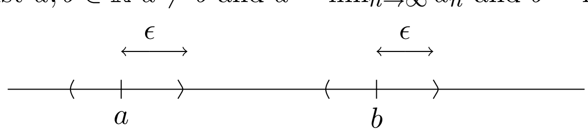
$$\exists a \in \mathbb{R} \ni \forall \epsilon > 0 \exists n_\epsilon \in \mathbb{N} \ni |a_n - a| < \epsilon \quad \forall n \geq n_\epsilon$$

We say that a is the <u>limit</u> of $\{a_n\}_{n\geq 1}$ and we write $a = \lim_{n\to\infty} a_n$ or $a_n \overset{n\to\infty}{\longrightarrow} a$

Lemma 10.6

The limit of a convergent sequence is unique.

Proof. We argue by contradiction. Assume that $\{a_n\}_{n\geq 1}$ is a convergent sequence and assume that there exist $a, b \in \mathbb{R}$ $a \neq b$ and $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} a_n$.



Let $0 < \epsilon < \frac{|b-a|}{2}$ (we can choose such an ϵ because $\mathbb{Q}$ is dense in $\mathbb{R}$)

$$a = \lim_{n\to\infty} a_n \implies \exists n_1(\epsilon) \in \mathbb{N} \ni |a_n - a| < \epsilon \forall n \geq n_1(\epsilon)$$
$$b = \lim_{n\to\infty} a_n \implies \exists n_2(\epsilon) \in \mathbb{N} \ni |a_n - b| < \epsilon \forall n \geq n_2(\epsilon)$$

Set $n_\epsilon = \max\{n_1(\epsilon), n_2(\epsilon)\}$. Then for $n \geq n_\epsilon$ we have

$$|b-a| = |b - a_n + a_n - a| \leq \underbrace{|b-a_n|}_{<\epsilon} + \underbrace{|a_n - a|}_{<\epsilon} < 2\epsilon < |b-a|$$

Contradiction! □

Exercise 10.1. Show that the sequence given by $a_n = \frac{1}{n} \forall n \geq 1$ converges to 0.

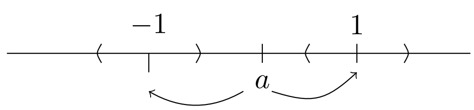
Proof. Let $\epsilon > 0$. By the Archemedean Property, $\exists n_\epsilon \in \mathbb{N} \ni n_\epsilon > \frac{1}{\epsilon}$. Then for $n \geq n_\epsilon$ we have

$$\left|0 - \frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{n_\epsilon} < \epsilon$$

By definition, $\lim_{n\to\infty} \frac{1}{n} = 0$. □

Exercise 10.2. Show that the sequence given by $a_n = (-1)^n \forall n \geq 1$ does not converge.

Proof. We argue by contradiction.



Assume $\exists a \in \mathbb{R}$ s.t. $a = \lim_{n\to\infty}(-1)^n$.
Let $0 < \epsilon < 1$. Then $\exists n_\epsilon \in \mathbb{N}$ s.t.

$$|a - (-1)^n| < \epsilon \quad \forall n \geq n_\epsilon$$

Taking $n = 2n_\epsilon$ we get $|a-1| < \epsilon$ and $n = 2n_\epsilon + 1$ we get $|a+1| < \epsilon$. By the triangle inequality,

$$2 = |1+1| = |1 - a + a + 1| \leq |1-a| + |a+1| < 2\epsilon < 2$$

Contradiction! □

Lemma 10.7

A convergent sequence is bounded.

Proof. Let $\{a_n\}_{n\geq 1}$ be a convergent sequence and let $a = \lim_{n\to\infty} a_n$.

$$\exists n_1 \in \mathbb{N} \ni |a - a_n| < 1 \quad \forall n \geq n_1$$

So $|a_n| \leq |a_n - a| + |a| < 1 + |a| \quad \forall n \geq n_1$. Let

$$M = \max\{1 + |a|, |a_1|, |a_2|, \ldots, |a_{n_1} - 1|\}$$

Clearly, $|a_n| \leq M \quad \forall n \geq 1$ so $\{a_n\}_{n\geq 1}$ is bounded. □

Theorem 10.8

Let $\{a_n\}_{n\geq 1}$ be a convergent sequence and let $a = \lim_{n\to\infty} a_n$. Then for any $k \in \mathbb{R}$, the sequence $\{ka_n\}_{n\geq 1}$ converges and $\lim_{n\to\infty} ka_n = ka$.

Proof. If $k = 0$ then $ka_n = 0 \quad \forall n \geq 1$. So $\lim_{n\to\infty} ka_n = 0 = k \cdot a$
Assume $k \neq 0$. Let $\epsilon > 0$.
Aside: want to find $n_\epsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\epsilon$

$$|ka_n - ka| < \epsilon \iff |a_n - a| < \frac{\epsilon}{|k|}$$

As $a = \lim_{n\to\infty} a_n$, $\exists n_{\epsilon,k} \in \mathbb{N}$ s.t.

$$|a_n - a| < \frac{\epsilon}{|k|} \quad \forall n \geq n_{\epsilon,k}$$

So $|ka_n - ka| = |k| \cdot |a_n - a| < |k| \cdot \frac{\epsilon}{|k|} = \epsilon$. □

§11 Lec 11: Jan 29, 2021

§11.1 Convergent and Divergent Sequences

Theorem 11.1 (Properties of Convergent Sequences)
Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two convergent sequences of real numbers and let $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$. Then

1. the sequence $\{a_n + b_n\}_{n\geq 1}$ converges and $\lim_{n\to\infty}(a_n + b_n) = a + b$,
2. the sequence $\{a_n \cdot b_n\}$ converges and $\lim_{n\to\infty}(a_n b_n) = a \cdot b$,
3. if $a \neq 0$ and $a_n \neq 0 \forall n \geq 1$ then $\left\{\frac{1}{a_n}\right\}_{n\geq 1}$ converges and $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$,
4. if $a \neq 0$ and $a_n \neq 0 \forall n \geq 1$, then $\left\{\frac{b_n}{a_n}\right\}_{n\geq 1}$ converges and $\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{b}{a}$.
5. for any $k \in \mathbb{R}$, $\{ka_n\}_{n\geq 1}$ converges and $\lim_{n\to\infty} ka_n = ka$ (from theorem 10.8)

Proof. 1. Let $\epsilon > 0$.

Aside(Goal): Want to find $n_\epsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\epsilon$

$$|(a+b) - (a_n + b_n)| < \epsilon$$
$$|(a+b) - (a_n + b_n)| \leq \underbrace{|a - a_n|}_{<\frac{\epsilon}{2}} + \underbrace{|b - b_n|}_{<\frac{\epsilon}{2}} < \epsilon$$

Now back to the main proof, as $\lim_{n\to\infty} a_n = a, \exists n_1(\epsilon) \in \mathbb{N}$ s.t.

$$|a - a_n| < \frac{\epsilon}{2} \qquad \forall n \geq n_1(\epsilon)$$

As $\lim_{n\to\infty} b_n = b, \exists n_2(\epsilon) \in \mathbb{N}$ s.t.

$$|b - b_n| < \frac{\epsilon}{2} \qquad \forall n \geq n_2(\epsilon)$$

Let $n_\epsilon = \max\{n_1(\epsilon), n_2(\epsilon)\}$. Then for $n \geq n_\epsilon$ we have $|(a+b) - (a_b + b_n)| \leq |a - a_n| + |b - b_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. By definition, $\lim_{n\to\infty}(a_b + b_n) = a + b$.

2. Let $\epsilon > 0$.

Aside(Goal): Want to find $n_\epsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\epsilon$

$$|ab - a_n b_n| < \epsilon$$
$$|ab - a_n b_n| = |(a - a_n)b + a_n(b - b_n)|$$
$$\leq \underbrace{|a - a_n| \cdot |b|}_{<\frac{\epsilon}{2}} + \underbrace{|a_n|\,|b - b_n|}_{<\frac{\epsilon}{2}} < \epsilon$$

Take $|a - a_n| < \frac{\epsilon}{2(|b|+1)}$. Take $M > 0$ s.t. $|a_n| \leq M \forall n \geq 1$

$$|b - b_n| < \frac{\epsilon}{2M}$$

Now, back to the main proof, as $\{a_n\}_{n\geq 1}$ converges, it is bounded. Let $M > 0$ such that $|a_n| \leq M \ \forall n \geq 1$. As $\lim_{n\to\infty} a_n = a, \exists n_1(\epsilon) \in \mathbb{N}$ s.t.

$$|a - a_n| < \frac{\epsilon}{2(|b|+1)} \qquad \forall n \geq n_1(\epsilon)$$

As $\lim_{n\to\infty} b_n = b, \exists n_2(\epsilon) \in \mathbb{N}$ s.t.

$$|b - b_n| < \frac{\epsilon}{2M} \qquad \forall n \geq n_2(\epsilon)$$

Set $n_\epsilon = \max\{n_1(\epsilon), n_2(\epsilon)\}$. For $n \geq n_\epsilon$ we have

$$\begin{aligned}|ab - a_n b_n| &= |(a - a_n)b + a_n(b - b_n)| \\ &\leq |a - a_n|\,|b| + |a_n|\,|b - b_n| \\ &< \frac{\epsilon}{2(|b|+1)} \cdot |b| + M \cdot \frac{\epsilon}{2M} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

By definition, $\lim_{n\to\infty}(a_n b_n) = ab$.

3. Let $\epsilon > 0$.

Aside(Goal): Want to find $n_\epsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\epsilon$

$$\begin{aligned}&\left|\frac{1}{a} - \frac{1}{a_n}\right| < \epsilon \\ &\left|\frac{1}{a} - \frac{1}{a_n}\right| = \frac{|a_n - a|}{|a|\cdot|a_n|} < \epsilon \\ &|a_n - a| < \epsilon|a|\cdot|a_n| \qquad (\text{!!! − NONONO})\end{aligned}$$

Now, back to the proof, as $a = \lim_{n\to\infty} a_n, \exists n_1(a) \in \mathbb{N}$ s.t.

$$|a - a_n| < \frac{|a|}{2} \qquad \forall n \geq n_1$$

Then, for all $n \geq n_1$ we have

$$|a_n| \geq |a| - |a - a_n| > |a| - \frac{|a|}{2} = \frac{|a|}{2}$$

As $a = \lim_{n\to\infty} a_n, \exists n_2(\epsilon, a)$ s.t.

$$|a - a_n| < \frac{\epsilon|a|^2}{2} \qquad \forall n \geq n_2(\epsilon, a)$$

Let $n_\epsilon = \max\{n_1(a), n_2(\epsilon, a)\}$. For $n \geq n_\epsilon$ we have

$$\left|\frac{1}{a} - \frac{1}{a_n}\right| = \frac{|a - a_n|}{|a|\cdot|a_n|} < \frac{\epsilon|a|^2}{2|a|} \cdot \frac{2}{|a|} = \epsilon$$

By definition, $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$.

□

Example 11.2

Find the limit of

$$\lim_{n\to\infty} \frac{n^3+5n+8}{3n^3+2n^2+7}$$

which can rewritten as

$$\lim_{n\to\infty} \frac{1+\frac{5}{n^2}+\frac{8}{n^3}}{3+\frac{2}{n}+\frac{7}{n^3}} = \frac{1+5\lim\frac{1}{n^2}+8\lim\frac{1}{n^3}}{3+2\lim\frac{1}{n}+7\lim\frac{1}{n^3}}$$

which is equivalent to

$$= \frac{1+5\cdot 0+8\cdot 0}{3+2\cdot 0+7\cdot 0} = \frac{1}{3}$$

Theorem 11.3 (Monotone Convergence)

Every bounded monotone sequence converges.

Proof. We'll show that an increasing sequence bounded above converges. A similar argument can be used to show that a decreasing sequence bounded below converges. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers that is bounded above and $a_{n+1} \geq a_n \quad \forall n \geq 1$.
As $\emptyset \neq \{a_n : n \geq 1\} \subseteq \mathbb{R}$ is bounded above and $\mathbb{R}$ has the least upper bound property, $\exists a \in \mathbb{R}$ s.t. $a = \sup\{a_n : n \geq 1\}$.

Claim 11.1. $a = \lim_{n\to\infty} a_n$.

Let $\epsilon > 0$. Then $a - \epsilon$ is not an upper bound for $\{a_n : n \geq 1\} \implies \exists n_\epsilon \in \mathbb{N}$ s.t. $a - \epsilon < a_{n_\epsilon}$. Then for $n \geq n_\epsilon$ we have

$$a - \epsilon < a_{n_\epsilon} \leq a_n \leq a < a + \epsilon \iff |a_n - a| < \epsilon$$

This proves the claim. □

Homework 11.1. Prove for the decreasing sequence.

Definition 11.4 (Divergent Sequence) — Let $\{a_n\}$ be a sequence of real numbers. We write $\lim_{n\to\infty} a_n = \infty$ and say that a_n diverges to $+\infty$ if $\forall M > 0, \quad \exists n_M \in \mathbb{N}$ s.t. $a_n > M \quad \forall n \geq n_M$.
We write $\lim_{n\to\infty} a_n = -\infty$ and say that a_n diverges to $-\infty$ if $\forall M < 0 \quad \exists n_M \in \mathbb{N}$ s.t. $a_n < M \quad \forall n \geq n_M$.

Homework 11.2.

1. Show that $\lim_{n\to\infty}(\sqrt[3]{n}+1) = \infty$.
2. Show that the sequence given by $a_n = (-1)^n n \quad \forall n \geq 1$ does not diverge to ∞ or to $-\infty$.
3. Let $\{a_n\}_{n\geq 1}$ be a sequence of positive real numbers. Show that

$$\lim_{n\to\infty} a_n = 0 \iff \lim_{n\to\infty} \frac{1}{a_n} = \infty$$

§12 Lec 12: Feb 1, 2021

Example 12.1

Show that $\lim_{n\to\infty} \frac{n^2+1}{n+3} = \infty$.
Aside: Want to find $n_M \in \mathbb{N}$ s.t. $\forall n \geq n_M$ we have

$$\frac{n^2+1}{n+3} > M$$

So

$$\frac{n^2+1}{n+3} > \frac{n^2}{n+3} > \frac{n^2}{4n} = \frac{n}{4} > M$$

Now, back to the main proof, let $M > 0$. By the Archimedean property there exists $n_M \in \mathbb{N}$ s.t.

$$n_M > 4M$$

Then for $n \geq n_M$ we have

$$\frac{n^2+1}{n+3} > \frac{n^2}{n+3} > \frac{n^2}{4n} = \frac{n}{4} \geq \frac{n_M}{4} > M$$

By the definition, $\lim_{n\to\infty} \frac{n^2+1}{n+3} = \infty$.

§12.1 Cauchy Sequences

Definition 12.2 (Cauchy Sequence) — We say that a sequence of real numbers $\{a_n\}_{n\geq 1}$ is a Cauchy sequence if

$$\forall \epsilon > 0 \quad \exists n_\epsilon \in \mathbb{N} \quad \text{s.t. } |a_n - a_m| < \epsilon \quad \forall n, m \geq n_\epsilon$$

Theorem 12.3 (Cauchy Criterion - Sequence)

A sequence of real numbers is Cauchy if and only if it converges.

We will split the proof of this theorem into various lemmas and propositions.

Proposition 12.4

Any convergent sequence is a Cauchy sequence.

Proof. Let $\{a_n\}_{n\geq 1}$ be a convergent sequence and let $a = \lim_{n\to\infty} a_n$. Let $\epsilon > 0$. As $a_n \overset{n\to\infty}{\longrightarrow} a$, $\exists n_\epsilon \in \mathbb{N}$ s.t.

$$|a - a_n| < \frac{\epsilon}{2} \quad \forall n \geq n_\epsilon$$

Then for $n, m \geq n_\epsilon$, we have

$$|a_n - a_m| \leq |a_n - a| + |a - a_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$ □

Lemma 12.5

A Cauchy sequence is bounded.

Proof. Let $\{a_n\}_{n\geq 1}$ be a Cauchy sequence. Then $\exists n_1 \in \mathbb{N}$ s.t. $|a_n - a_m| < 1 \quad \forall n, m \geq n_1$. So, taking $m = n_1$, we get

$$|a_n| \leq |a_{n_1}| + |a_n - a_{n_1}| < |a_{n_1}| + 1 \quad \forall n \geq n_1$$

Let $M = \max\{|a_1|, |a_2|, \dots, |a_{n_1 - 1}|, |a_{n_1} + 1|\}$. Clearly, $|a_n| \leq M \quad \forall n \geq 1$. □

Definition 12.6 (Subsequence) — Let $\{k_n\}_{n\geq 1}$ be a sequence of natural numbers s.t. $k_1 \geq 1$ and $k_{n+1} > k_n \quad \forall n \geq 1$. Using induction, it's easy to see that $k_n \geq n \quad \forall n \geq 1$. If $\{a_n\}_{n\geq 1}$ is a sequence, we say that $\{a_{k_n}\}_{n\geq 1}$ is a subsequence of $\{a_n\}_{n\geq 1}$.

Example 12.7

The following are subsequences of $\{a_n\}_{n\geq 1}$:

$$\{a_{2n}\}_{n\geq 1}, \{a_{2n-1}\}_{n\geq 1}, \{a_{n^2}\}_{n\geq 1}, \{a_{p_n}\}_{n\geq 1}$$

where p_n denotes the n^{th} prime.

Theorem 12.8

Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. Then $\lim_{n\to\infty} a_n = a \in \mathbb{R} \cup \{\pm\infty\}$ if and only if every subsequence $\{a_{k_n}\}_{n\geq 1}$ of $\{a_n\}_{n\geq 1}$ satisfies $\lim_{n\to\infty} a_{k_n} = a$.

Proof. We will consider $a \in \mathbb{R}$. The cases $a \in \{\pm\infty\}$ can be handled by analogous arguments.

" $\Longleftarrow$ " Take $k_n = n \quad \forall n \geq 1$

" $\Longrightarrow$ " Assume $\lim_{n\to\infty} a_n = a$ and let $\{a_{k_n}\}_{n\geq 1}$ be a subsequence of $\{a_n\}_{n\geq 1}$. Let $\epsilon > 0$. As $a_n \overset{n\to\infty}{\longrightarrow} a, \quad \exists n_\epsilon \in \mathbb{N}$ s.t.

$$|a - a_n| < \epsilon \quad \forall n \geq n_\epsilon$$

Recall that $k_n \geq n \, \forall n \geq 1$. So for $n \geq n_\epsilon$ we have $k_n \geq n \geq n_\epsilon$ and so

$$|a - a_{k_n}| < \epsilon \quad \forall n \geq n_\epsilon$$

By definition,

$$\lim_{n\to\infty} a_{k_n} = a$$

□

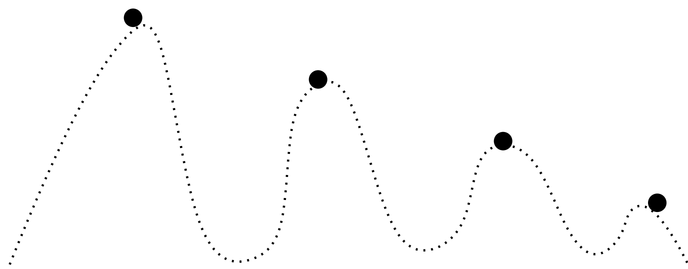
Proposition 12.9

Every sequence of real numbers has a monotone subsequence.

Proof. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. We say that the n^{th} term is <u>dominant</u> if

$$a_n > a_m \quad \forall m > n$$

We distinguished 2 cases:
<u>Case 1</u>: There are infinitely many dominant terms:



Then a subsequence formed by these dominant terms is strictly decreasing.
<u>Case 2</u>: There are none or finitely many dominant terms. Let N be larger than the largest index of the dominant terms. So $\forall n \geq N$ a_n is not dominant. Set $k_1 = N$, $a_{k_1} = a_N$. a_{k_1} is not dominant $\implies \exists k_2 > k_1$ s.t. $a_{k_2} \geq a_{k_1}$, $k_2 > k_1 = N \implies a_{k_2}$ is not dominant $\implies \exists k_3 > k_2$ s.t. $a_{k_3} \geq a_{k_2}$. Proceeding inductively we construct a subsequence $\{a_{k_n}\}_{n\geq 1}$ s.t.

$$a_{k_{n+1}} \geq a_{k_n} \quad \forall n \geq 1$$ □

Theorem 12.10 (Bolzano – Weierstrass)
Any bounded sequence has a convergent subsequence.

Proof. Let $\{a_n\}_{n\geq 1}$ be a bounded sequence. By the previous proposition, there exists $\{a_{k_n}\}_{n\geq 1}$ monotone subsequence of $\{a_n\}_{n\geq 1}$. As $\{a_n\}_{n\geq 1}$ is bounded, so is $\{a_{k_n}\}_{n\geq 1}$. As bounded monotone sequences converge, $\{a_{k_n}\}_{n\geq 1}$ converges. □

Corollary 12.11
Every Cauchy sequence has a convergent subsequence.

Lemma 12.12
A Cauchy sequence with a convergent subsequence converges.

Proof. Let $\{a_n\}_{n\geq 1}$ be a Cauchy sequence s.t. $\{a_{k_n}\}_{n\geq 1}$ is a convergent subsequence. Let $a = \lim_{n\to\infty} a_{k_n}$. Let $\epsilon > 0$. As $a_{k_n} \xrightarrow{n\to\infty} a$, $\exists n_1(\epsilon)$ s.t. $|a - a_{k_n}| < \frac{\epsilon}{2}\, \forall n \geq n_1(\epsilon)$. As $\{a_n\}_{n\geq 1}$ is Cauchy, $\exists n_2(\epsilon)$ s.t. $|a_n - a_m| < \frac{\epsilon}{2}\, \forall n, m \geq n_2(\epsilon)$. Let $n_\epsilon = \max\{n_1(\epsilon), n_2(\epsilon)\}$. Then for $n \geq n_\epsilon$ we have

$$|a - a_n| \leq |a - a_{k_n}| + |a_{k_n} - a_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for $k_n \geq n \geq n_\epsilon$. By definition,

$$\lim_{n\to\infty} a_n = a$$

Combining the last two results, we see that a Cauchy sequence of real numbers converges. $\square$

§13 Lec 13: Feb 3, 2021

§13.1 Limsup and Liminf

Let $\{a_n\}_{n\geq 1}$ be a bounded sequence of real numbers (convergent or not). The asymptotic behavior of $\{a_n\}_{n\geq 1}$ depends on sets of the form $\{a_n : n \geq N\}$ for $N \in \mathbb{N}$.

As $\{a_n\}_{n\geq 1}$, the set $\{a_n : n \geq N\}$ (where $N \in \mathbb{N}$ is fixed) is a non-empty bounded subset of $\mathbb{R}$.

As $\mathbb{R}$ has the least upper bound property (and so also the greatest lower bound property), the set $\{a_n : n \geq N\}$ has an infimum and a supremum in $\mathbb{R}$.

For $N \geq 1$, let $u_N = \inf\{a_n : n \geq N\}$ and $v_N = \sup\{a_n : n \geq N\}$. Clearly, $u_N \leq v_N \quad \forall N \geq 1$. For $N \geq 1$, $\{a_n : n \geq N\} \supseteq \{a_n : n \geq N+1\}$

$$\implies \begin{cases} \inf\{a_n : n \geq N\} \leq \inf\{a_n : n \geq N+1\} \\ \sup\{a_n : n \geq N\} \geq \sup\{a_n : n \geq N+1\} \end{cases}$$

So $u_N \leq u_{N+1}$ and $v_{N+1} \leq v_N \quad \forall N \geq 1$. Thus $\{u_N\}_{N\geq 1}$ is increasing and $\{v_N\}_{N\geq 1}$ is decreasing. Moreover, $\forall N \geq 1$ we have

$$u_1 \leq u_2 \leq \ldots \leq u_N \leq v_N \leq \ldots \leq v_2 \leq v_1$$

So the sequences $\{u_N\}_{N\geq 1}$ and $\{v_N\}_{N\geq 1}$ are bounded. As monotone bounded sequence converges, $\{u_N\}_{N\geq 1}$ and $\{v_N\}_{N\geq 1}$ must converge.

Let

$$\begin{aligned} u &= \lim_{N\to\infty} u_N = \sup\{u_N : N \geq 1\} \coloneqq \sup_N u_N \\ v &= \lim_{N\to\infty} v_N = \inf\{v_N : N \geq 1\} \coloneqq \inf_N v_N \end{aligned}$$

From (*), we see that

$$\begin{aligned} & u_M \leq v_N \qquad \forall M, N \geq 1 \\ \implies & \lim_{M\to\infty} u_M \leq v_N \qquad \forall N \geq 1 \\ \implies & u \leq v_N \qquad \forall N \geq 1 \\ \implies & u \leq \lim_{N\to\infty} v_N \\ \implies & u \leq v \end{aligned}$$

Moreover, if $\lim_{n\to\infty} a_n$ exists, then for all $N \geq 1$, we have

$$u_N = \inf\{a_n : n \geq N\} \leq a_n \leq \sup\{a_n : n \geq N\} = v_N \quad \forall n \geq N$$

So

$$\begin{aligned} \implies & u_N \leq \lim_{n\to\infty} a_n \leq v_N \\ \implies & u = \lim_{N\to\infty} u_N \leq \lim_{n\to\infty} a_n \leq \lim_{N\to\infty} v_N = v \end{aligned}$$

Definition 13.1 (lim sup and lim inf) — Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. We define

$$\limsup_{n\to\infty} a_n = \lim_{N\to\infty} \sup\{a_n : n \geq N\} = \lim_{N\to\infty} v_N = \inf_N v_N = \inf_N \sup_{n\geq N} a_n$$
$$\liminf_{n\to\infty} a_n = \lim_{N\to\infty} \inf\{a_n : n \geq N\} = \lim_{N\to\infty} u_N = \sup_N u_N = \sup_N \inf_{n\geq N} a_n$$

with the convention that if $\{a_n\}_{n\geq 1}$ is unbounded above then

$$\limsup_{n\to\infty} a_n = \infty$$

and if $\{a_n\}_{n\geq 1}$ is unbounded below then

$$\liminf_{n\to\infty} a_n = -\infty$$

Remark 13.2.

$$\inf\{a_n : n \geq 1\} \leq \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n \leq \sup\{a_n : n \geq 1\}$$

where $\liminf_{n\to\infty} a_n$ is the smallest value that infinitely many a_n get close to and $\limsup_{n\to\infty} a_n$ is the largest value that infinitely many a_n get close to.

Example 13.3

$a_n = 3 + \frac{(-1)^n}{n} \implies \lim_{n\to\infty} a_n = 3 \implies \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = 3$

$$\inf\{a_n : n \geq 1\} = 2 \neq 3$$
$$\sup\{a_n : n \geq 1\} = \frac{7}{2} \neq 3$$

Theorem 13.4 (lim, lim sup, and lim inf)

Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers.

1. If $\lim_{n\to\infty} a_n$ exists in $\mathbb{R} \cup \{\pm\infty\}$, then $\liminf a_n = \limsup a_n = \lim_{n\to\infty} a_n$.
2. If $\liminf a_n = \limsup a_n \in \mathbb{R} \cup \{\pm\infty\}$, then $\lim_{n\to\infty} a_n$ exists and

$$\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$$

Proof. 1. We distinguish three cases.

<u>**Case i)**</u> $\lim_{n\to\infty} a_n = -\infty$. It's enough to show $\limsup a_n = -\infty$ since $\liminf a_n \leq \limsup a_n$. Fix $M < 0$. As $\lim_{n\to\infty} a_n = -\infty$, $\exists n_M \in \mathbb{N}$ s.t. $a_n < M \quad \forall n \geq n_M$. Then for $N \geq n_M$, we have $v_N = \sup\{a_n : n \geq N\} \leq M$. Note that when taking sup(inf), $<$ can become $\leq$; e.g. $a_n = 3 - \frac{1}{n}$ where $a_n < 3 \quad \forall n \geq 1$ but $\sup_{n\geq 1} a_n = 3$.

By definition, $\limsup_{n\to\infty} a_n = \lim_{N\to\infty} v_N = -\infty$.

Case ii) $\lim_{n\to\infty} a_n = \infty$ Exercise

Case iii) $\lim_{n\to\infty} a_n = a \in \mathbb{R}$.

Fix $\epsilon > 0$. Then $\exists n_\epsilon \in \mathbb{N}$ s.t. $|a - a_n| < \epsilon \quad \forall n \geq n_\epsilon$. So

$$a - \epsilon < a_n < a + \epsilon \qquad \forall n \geq n_\epsilon$$

Thus for $N \geq n_\epsilon$ we have

$$a - \epsilon \leq \inf\{a_n : n \geq N\} \leq \sup\{a_n : n \geq N\} \leq a + \epsilon$$
$$a - \epsilon \leq u_N \leq v_N \leq a + \epsilon$$

So

$$\forall N \geq n_\epsilon \begin{cases} |u_N - a| \leq \frac{\epsilon}{2} < \epsilon \\ |v_N - a| \leq \frac{\epsilon}{2} < \epsilon \end{cases}$$

By definition,

$$\begin{cases} \liminf a_n = \lim_{N\to\infty} u_N = a \\ \limsup a_n = \lim_{N\to\infty} v_N = a \end{cases}$$

2. We distinguish three cases.

Case i) $\liminf a_n = \limsup a_n = -\infty$.

We will use $\limsup a_n = -\infty$. Fix $M < 0$. Then since $\limsup a_n = \lim_{N\to\infty} v_N = -\infty$, $\exists N_M \in \mathbb{N}$ s.t. $v_N < M \quad \forall N \geq N_M$. In particular, $v_{N_M} = \sup\{a_n : n \geq N_M\} < M$

$$\implies a_n < M \qquad \forall n \geq N_M$$

By definition, $\lim_{n\to\infty} a_n = -\infty$.

Case ii) $\liminf a_n = \limsup a_n = \infty$ exercise

Case iii) $\liminf a_n = \limsup a_n = a \in \mathbb{R}$.

Fix $\epsilon > 0$.

$$a = \liminf a_n = \lim_{N\to\infty} u_N \implies \exists N_1(\epsilon) \in \mathbb{N} \ni |u_N - a| < \epsilon \quad \forall N \geq N_1$$

So $a - \epsilon < u_{N_1} = \inf\{a_n : n \geq N_1\} < a + \epsilon$

$$\implies a - \epsilon < a_n \qquad \forall n \geq N_1$$

And

$$a = \limsup a_n = \lim_{N\to\infty} v_N \implies \exists N_2(\epsilon) \in \mathbb{N} \ni |v_N - a| < \epsilon \quad \forall N \geq N_2$$

So $a - \epsilon < v_{N_2} = \sup\{a_n : n \geq N_2\} < a + \epsilon$.

$$\implies a_n < a + \epsilon \qquad \forall n \geq N_2$$

Thus for $n \geq \max\{N_1, N_2\}$ we have

$$a - \epsilon < a_n < a + \epsilon \iff |a_n - a| < \epsilon$$

By definition, $\lim_{n\to\infty} a_n = a$.

□

§14 Lec 14: Feb 5, 2021

§14.1 Limsup and Liminf (Cont'd)

Recall: For a sequence $\{a_n\}_{n\geq 1}$ of real numbers, we define

$$\liminf a_n = \sup_N \inf_{n\geq N} a_n = \lim_{N\to\infty} u_N \text{ where } u_N = \inf\{a_n : n \geq N\}$$
$$\limsup a_n = \inf_N \sup_{n\geq N} a_n = \lim_{N\to\infty} v_N \text{ where } v_N = \sup\{a_n : n \geq N\}$$

Last time, we proved that

$$\lim_{n\to\infty} a_n \text{ exists in } \mathbb{R}\cup\{\pm\infty\} \iff \liminf a_n = \limsup a_n$$

Theorem 14.1 (Existence of Monotonic Subsequence)
Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. Then there exists a monotonic subsequence of $\{a_n\}_{n\geq 1}$ whose limit is $\limsup a_n$. Also, there exists a monotonic subsequence of $\{a_n\}_{n\geq 1}$ whose limit is $\liminf a_n$.

Proof. We will prove the statement about $\limsup a_n$. Similar arguments can be used to prove the statement about $\liminf a_n$. HW!

Note that it suffices to find a subsequence of $\{a_{k_n}\}_{n\geq 1}$ of $\{a_n\}_{n\geq 1}$ s.t.

$$\lim_{n\to\infty} a_{k_n} = \limsup a_n$$

As every sequence has a monotone subsequence, $\{a_{k_n}\}_{n\geq 1}$ has a monotone subsequence $\{a_{p_{k_n}}\}_{n\geq 1}$. Then as $\lim a_{k_n}$ exists, $\lim_{n\to\infty} a_{p_{k_n}}$ exists and

$$\lim_{n\to\infty} a_{p_{k_n}} = \lim a_{k_n} = \limsup a_n$$

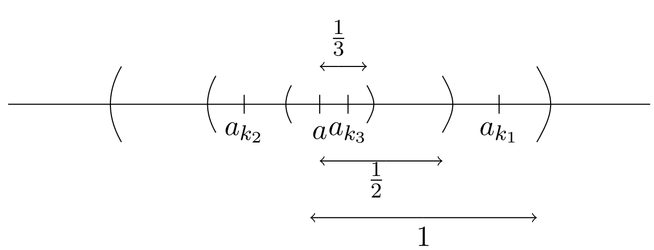
Finally, note that $\{a_{p_{k_n}}\}_{n\geq 1}$ is a subsequence of $\{a_n\}_{n\geq 1}$.

Let's find a subsequence of $\{a_n\}_{n\geq 1}$ whose limit is $\limsup a_n$.

Case 1: $\limsup a_n = -\infty$.

We showed that in this case, $\lim_{n\to\infty} a_n = -\infty$. Choose $\{a_{k_n}\}_{n\geq 1}$ to be $\{a_n\}_{n\geq 1}$.

Case 2: $\limsup a_n = a \in \mathbb{R}$.



$$a = \limsup a_n = \lim_{N\to\infty} v_N$$

Then $\exists N_1 \in \mathbb{N}$ s.t. $|a - v_N| < 1 \quad \forall N \geq N_1$. In particular,

$$\begin{aligned} &a - 1 < v_{N_1} < a + 1 \\ \implies &a - 1 < \sup\{a_n : n \geq N_1\} \\ \implies &\exists k_1 \geq N_1 \quad \ni \quad a - 1 < a_{k_1} \\ \implies &a - 1 < a_{k_1} < v_{N_1} < a + 1 \end{aligned}$$

So $|a - a_{k_1}| < 1$.
As $a = \lim_{N\to\infty} v_N$, $\exists N_2 \in \mathbb{N}$ s.t. $|a - v_N| < \frac{1}{2} \quad \forall N \geq N_2$.
Let $\tilde{N}_2 = \max\{N_2, k_1 + 1\}$
In particular,

$$\left.\begin{aligned} &a - \tfrac{1}{2} < v_{\tilde{N}_2} < a + \tfrac{1}{2} \\ &a - \tfrac{1}{2} < \sup\left\{a_n : n \geq \tilde{N}_2\right\} \\ &\exists k_2 \geq \tilde{N}_2 \text{ s.t. } a - \tfrac{1}{2} < a_{k_2} \end{aligned}\right\} \implies a - \frac{1}{2} < a_{k_2} \leq v_{N_2} < a + \frac{1}{2}$$

So, $|a - a_{k_2}| < \frac{1}{2}$. To construct our subsequence we proceed inductively. Assume we have found $k_1 < k_2 < \ldots < k_n$ and $a_{k_1}, \ldots, a_{k_n}$ s.t.

$$\left|a - a_{k_j}\right| < \frac{1}{j} \quad \forall 1 \leq j \leq n$$

As $a = \lim_{N\to\infty} v_N \implies \exists N_{n+1} \in \mathbb{N}$ s.t. $|a - v_N| < \frac{1}{n+1} \quad \forall N \geq N_{n+1}$. Let $\tilde{N}_{n+1} = \max\{N_{n+1}, k_n + 1\}$. Then

$$\begin{aligned} &a - \frac{1}{n+1} < v_{\tilde{N}_{n+1}} < a + \frac{1}{n+1} \\ &\implies a - \frac{1}{n+1} < \sup\left\{a_n : n \geq \tilde{N}_{n+1}\right\} \\ &\implies \exists k_{n+1} \geq \tilde{N}_{n+1} > k_n \text{ s.t. } a - \frac{1}{n+1} < a_{k_{n+1}} \\ &\implies a - \frac{1}{n+1} < a_{k_{n+1}} \leq v_{\tilde{N}_{n+1}} < a + \frac{1}{n+1} \\ &\implies |a_{k_{n+1}} - a| < \frac{1}{n+1} \end{aligned}$$

Case 3: $\limsup a_n = \infty$. □ HW!

Definition 14.2 (Subsequential Limit) — Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. A subsequential limit of $\{a_n\}_{n\geq 1}$ is any $a \in \mathbb{R} \cup \{\pm\infty\}$ that is the limit of a subsequence of $\{a_n\}_{n\geq 1}$.

Example 14.3 1. $a_n = n(1 + (-1)^n)$

The subsequential limits are

$$0 = \lim_{n\to\infty} a_{2n+1}$$
$$\infty = \lim_{n\to\infty} a_{2n}$$

2. $a_n = \cos\left(\frac{n\pi}{3}\right)$

The subsequential limits are

$$1 = \lim_{n\to\infty} a_{6n}$$
$$\frac{1}{2} = \lim_{n\to\infty} a_{6n+1} = \lim_{n\to\infty} a_{6n+5}$$
$$-\frac{1}{2} = \lim_{n\to\infty} a_{6n+2} = \lim_{n\to\infty} a_{6n+4}$$
$$-1 = \lim_{n\to\infty} a_{6n+3}$$

Theorem 14.4 (Properties of the Set of Subsequential Limit)
Let $\{a_n\}_{n\ge1}$ be a sequence of real numbers and let A denote its set of subsequential limits:

$$A = \left\{a \in \mathbb{R} \cup \{\pm\infty\} : \exists \{a_{k_n}\}_{n\ge1} \text{ subsequence of } \{a_n\}_{n\ge1} \text{ s.t. } \lim_{n\to\infty} a_{k_n} = a\right\}$$

Then:

1. $A \ne \emptyset$.
2. $\lim_{n\to\infty} a_n$ exists (in $\mathbb{R} \cup \{\pm\infty\}$) $\iff$ A has exactly one element.
3. $\inf A = \liminf a_n$ and $\sup A = \limsup a_n$.

Proof. 1. By the previous theorem, $\liminf a_n, \limsup a_n \in A$. So $A \ne \emptyset$.

2. "$\implies$" Assume $\lim_{n\to\infty} a_n$ exists. Then if $\{a_{k_n}\}_{n\ge1}$ is a subsequence of $\{a_n\}_{n\ge1}$, we have
$$\lim_{n\to\infty} a_{k_n} = \lim_{n\to\infty} a_n$$
So $A = \{\lim_{n\to\infty} a_n\}$.
"$\impliedby$" If A has a single element, $\liminf a_n = \limsup a_n$ and so $\lim_{n\to\infty} a_n$ exists.

3. We will prove

Claim 14.1. $\liminf a_n \le a \le \limsup a_n \quad \forall a \in A$.

Assuming the claim, let's see how to finish the proof. The claim implies

- $\liminf a_n$ is a lower bound for $A \implies \liminf a_n \leq inf A$. On the other hand, $\liminf a_n \in A \implies \liminf a_n \geq \inf A$. Thus, $\liminf a_n = \inf A$.
- $\limsup a_n$ is an upper bound for $A \implies \limsup a_n \geq \sup A$. But $\limsup a_n \in A \implies \limsup a_n \leq \sup A$. Thus, $\limsup a_n = \sup A$.

Let's prove the claim. Fix $a \in A \implies \exists \{a_{k_n}\}_{n\geq 1}$ subsequence of $\{a_n\}_{n\geq 1}$ s.t. $\lim_{n\to\infty} a_{k_n} = a$.

$$
\begin{aligned}
&\{a_n : n \geq N\} \supset \{a_{k_n} : n \geq N\} \\
&\implies \underbrace{\inf\{a_n : n \geq N\}}_{\text{increasing seq}} \leq \underbrace{\inf\{a_{k_n} : n \geq N\}}_{\text{increasing seq}} \leq \underbrace{\sup\{a_{k_n} : n \geq N\}}_{\text{deceasing seq}} \leq \underbrace{\sup\{a_n : n \geq N\}}_{\text{decreasing seq}} \\
&\implies \lim_{N\to\infty} \inf\{a_n : n \geq N\} \leq \lim_{N\to\infty} \inf\{a_{k_n} : n \geq N\} \leq \lim_{N\to\infty} \sup\{a_{k_n} : n \geq N\} \\
&\qquad\qquad \leq \lim_{N\to\infty} \sup\{a_n : n \geq N\} \\
&\implies \liminf a_n \leq \underbrace{\liminf a_{k_n}}_{=\lim a_{k_n} = a} \leq \underbrace{\limsup a_{k_n}}_{=\lim a_{k_n} = a} \leq \limsup a_n
\end{aligned}
$$

□

§15 Lec 15: Feb 8, 2021

§15.1 Limsup and Liminf (Cont'd)

Theorem 15.1 (Cesaro – Stolz)
Let $\{a_n\}_{n\geq 1}$ be a sequence of non-zero real numbers. Then

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \overset{1)}{\leq} \liminf |a_n|^{\frac{1}{n}} \overset{2)}{\leq} \limsup |a_n|^{\frac{1}{n}} \overset{3)}{\leq} \limsup \left|\frac{a_{n+1}}{a_n}\right|$$

In particular, if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists then $\lim_{n\to\infty} |a_n|^{\frac{1}{n}}$ exists and

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

Example 15.2
Find $\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{\frac{1}{n}}$.
If we let $a_n = n$ then $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{n} \overset{n\to\infty}{\longrightarrow} 1$. By Cesaro – Stolz, we get $\{\sqrt[n]{n}\}_{n\geq 1}$ converges and

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

Proof. We will prove inequality 3). Analogous arguments yield inequality 1). Let

$$l = \limsup |a_n|^{\frac{1}{n}} \geq 0$$
$$L = \limsup \left|\frac{a_{n+1}}{a_n}\right| \geq 0$$

We want to show $l \leq L$. If $L = \infty$, then it's clear. Henceforth we assume $L \in \mathbb{R}$. We will prove

Claim 15.1. l is a lower bound for the set

$$(L, \infty) = \{M \in \mathbb{R} : M > L\}$$

Assuming the claim for now, let's see how to finish the proof. Note (L,∞) is a non-empty subset of $\mathbb{R}$ which is bounded below (by L). As $\mathbb{R}$ has the least upper bound property, $\inf(L,\infty)$ exists in $\mathbb{R}$. In fact,

$$\inf(L,\infty) = L$$

As l is a lower bound for (L,∞), we must have $l \leq L$.
Let's prove the claim. Fix $M \in (L,\infty)$. We will show

$$l \leq M$$

We have $M > L = \limsup \left|\frac{a_{n+1}}{a_n}\right| = \inf_N \sup_{n\geq N} \left|\frac{a_{n+1}}{a_n}\right|$.

$$\begin{aligned}
&\implies \exists N_0 \in \mathbb{N} \ni \sup_{n\geq N_0} \left|\frac{a_{n+1}}{a_n}\right| < M \\
&\implies \left|\frac{a_{n+1}}{a_n}\right| < M \quad \forall n \geq N_0 \\
&\implies |a_{n+1}| < M \cdot |a_n| \quad \forall n \geq N_0
\end{aligned}$$

A simple inductive argument yields

$$\begin{aligned}
&|a_n| < M^{n-N_0} |a_{N_0}| \quad \forall n > N_0 \\
&\implies |a_n|^{\frac{1}{n}} < M \left(\frac{|a_{N_0}|}{M^{N_0}}\right)^{\frac{1}{n}} \quad \forall n > N_0 \\
&\implies l = \limsup |a_n|^{\frac{1}{n}} \leq \limsup M \cdot \left(\frac{|a_{N_0}|}{M^{N_0}}\right)^{\frac{1}{n}} = M \cdot \limsup \left(\frac{|a_{N_0}|}{M^{N_0}}\right)^{\frac{1}{n}} \qquad (*)
\end{aligned}$$

Claim 15.2. For $r > 0$ we have $\lim_{n\to\infty} r^{\frac{1}{n}} = 1$

Indeed, if $r \geq 1$

$$0 \leq r^{\frac{1}{n}} - 1 = \frac{r-1}{r^{n-1} + r^{n-2} + \ldots + 1} \leq \frac{r-1}{n} \xrightarrow{n\to\infty} 0$$

where we use the formula $a^n - b^n = (a-b)\left(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1}\right)$. If $r < 1$, then

$$r^{\frac{1}{n}} = \frac{1}{\left(\frac{1}{r}\right)^{\frac{1}{n}}} \xrightarrow{n\to\infty} \frac{1}{1} = 1$$

Taking $r = \frac{|a_{N_0}|}{M^{N_0}}$ in (*) we conclude that

$$l \leq M$$

□

§15.2 Series

Definition 15.3 (Convergent/Absolutely Convergent Series) — Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. For $n \geq 1$, we define the partial sum

$$s_n = a_1 + \ldots + a_n$$

The series $\sum_{n=1}^{\infty} a_n \left(\sum_{n\geq 1} a_n\right)$ is said to converge if $\{s_n\}_{n\geq 1}$ converges.
We say that the series $\sum_{n=1}^{\infty} a_n$ converges absolutely if the series $\sum_{n=1}^{\infty} |a_n|$ converges.
(Note that $\sum_{n=1}^{\infty} |a_n|$ either converges or it diverges to ∞).

Theorem 15.4 (Cauchy Criterion - Series)
A series $\sum_{n\geq 1} a_n$ converges if and only if

$$\forall \epsilon > 0 \quad \exists n_\epsilon \in \mathbb{N} \ni \left| \sum_{k=n+1}^{n+p} a_k \right| < \epsilon \quad \forall n \geq n_\epsilon \, \forall p \in \mathbb{N}$$

Proof. The series $\sum_{n\geq 1} a_n$ converges $\iff$ the sequence $\{s_n\}_{n\geq 1}$ converges $\iff$ $\{s_n\}_{n\geq 1}$ is Cauchy $\iff$ $\forall \epsilon > 0 \; \exists n_\epsilon \in \mathbb{N}$ s.t. $|s_m - s_n| < \epsilon \quad \forall m, n \geq n_\epsilon$. Without loss of generality, we may assume $m > n$ and write $m = n + p$ for $p \in \mathbb{N}$. Note

$$|s_m - s_n| = \left| \sum_{k=1}^{n+p} a_k - \sum_{k=1}^{n} a_k \right| = \left| \sum_{k=n+1}^{n+p} a_k \right|$$

So $\sum_{n\geq 1} a_n$ converges $\iff$ $\forall \epsilon > 0 \exists n_\epsilon \in \mathbb{N}$ s.t. $\left| \sum_{k=n+1}^{n+p} a_k \right| < \epsilon \; \forall n \geq n_\epsilon \, \forall p \in \mathbb{N}$. □

Corollary 15.5
If $\sum_{n\geq 1} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Proof. Taking $p = 1$, we find $\sum_{n\geq 1} a_n$ converges implies

$$\forall \epsilon > 0 \quad \exists n_\epsilon \in \mathbb{N} \text{ s.t. } |a_{n+1}| < \epsilon \quad \forall n \geq n_\epsilon$$

□

Corollary 15.6
If $\sum_{n\geq 1} a_n$ converges absolutely, then it converges.

Proof. $\sum_{n\geq 1} a_n$ converges absolutely $\implies$ $\sum_{n\geq 1} |a_n|$ converges.

$$\implies \forall \epsilon > 0 \quad \exists n_\epsilon \in \mathbb{N} \text{ s.t. } \sum_{k=n+1}^{n+p} |a_k| < \epsilon \quad \forall n \geq n_\epsilon \, \forall p \in \mathbb{N}$$

Note that by $\triangle$ inequality,

$$\left| \sum_{k=n+1}^{n+p} a_k \right| \leq \sum_{k=n+1}^{n+p} |a_k| < \epsilon \quad \forall n \geq n_\epsilon \, \forall p \in \mathbb{N}$$

So $\sum_{n\geq 1} a_n$ converges by the Cauchy criterion. □

Theorem 15.7 (Comparison Test)

Let $\sum_{n\geq 1} a_n$ be a series of real numbers with $a_n \geq 0 \quad \forall n \geq 1$.

1. If $\sum_{n\geq 1} a_n$ converges and $|b_n| \leq a_n \, \forall n \geq 1$, then $\sum_{n\geq 1} b_n$ converges.
2. If $\sum_{n\geq 1} a_n$ diverges and $b_n \geq a_n \, \forall n \geq 1$, then $\sum_{n\geq 1} b_n$ diverges.

Proof.

1. $\sum_{n\geq 1} a_n$ converges $\implies \forall \epsilon > 0 \exists n_\epsilon \in \mathbb{N}$ s.t.

$$\left| \sum_{k=n+1}^{n+p} a_k \right| < \epsilon \quad \forall n \geq n_\epsilon \, \forall p \in \mathbb{N}$$

   Then $\left|\sum_{k=n+1}^{n+p} b_k\right| \leq \sum_{k=n+1}^{n+p} |b_k| \leq \sum_{k=n+1}^{n+p} a_k < \epsilon \, \forall n \geq n_\epsilon \, \forall p \in \mathbb{N}$. So by the Cauchy criterion, $\sum_{n\geq 1} b_n$ converges.

2. $b_1 + \ldots + b_n \geq a_1 + \ldots + a_n \overset{n\to\infty}{\longrightarrow} \infty \implies \sum_{n\geq 1} b_n$ diverges.

□

Lemma 15.8

Let $r \in \mathbb{R}$. The series $\sum_{n\geq 0} r^n$ converges if and only if $|r| < 1$. If $|r| < 1$, then

$$\sum_{n\geq 0} r^n = \frac{1}{1-r}$$

Proof. First note that if $|r| \geq 1$, then

$$|r^n| = |r|^n \geq 1 \qquad \implies r^n \overset{n\to\infty}{\not\longrightarrow} 0$$

By the first corollary, $\sum_{n\geq 0} r^n$ cannot converge. Assume now that $|r| < 1$. Then

$$|r^n| = |r|^n \overset{n\to\infty}{\longrightarrow} 0$$

Also

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1-r} \overset{n\to\infty}{\longrightarrow} \frac{1}{1-r}$$

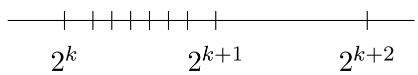
□

§16 Lec 16: Feb 10, 2021

§16.1 Series (Cont'd)

Theorem 16.1 (Dyadic Criterion)
Let $\{a_n\}_{n\geq 1}$ be a decreasing sequence of real numbers with $a_n \geq 0 \, \forall n \geq 1$. Then the series $\sum_{n\geq 1} a_n$ converges if and only if the series $\sum_{n\geq 0} 2^n a_{2^n}$ converges.

Proof. For $n \geq 1$ let $s_n = \sum_{k=1}^n a_k = a_1 + \ldots + a_n$. For $n \geq 0$ let $t_n = \sum_{k=0}^n 2^k a_{2^k} = a_1 + 2a_2 + \ldots + 2^n a_{2^n}$. Note that $\{s_n\}_{n\geq 1}$ and $\{t_n\}_{n\geq 0}$ are increasing sequences.
Thus $\sum_{n\geq 1} a_n$ converges $\iff$ $\{s_n\}_{n\geq 1}$ is bounded and $\sum_{n\geq 0} 2^n a_{2^n}$ converges $\iff$ $\{t_n\}_{n\geq 0}$ is bounded. We have to prove that $\{s_n\}_{n\geq 1}$ is bounded $\iff$ $\{t_n\}_{n\geq 0}$ is bounded.



Consider:

$$\sum_{l=2^k+1}^{2^{k+1}} a_l$$

Because $\{a_n\}_{n\geq 1}$ is decreasing, we get

$$\frac{1}{2}\left(2^{k+1} a_{2^{k+1}}\right) = 2^k a_{2^{k+1}} \leq \sum_{l=2^k+1}^{2^{k+1}} a_l \leq 2^k a_{2^k+1} \leq 2^k a_{2^k}$$

$$\frac{1}{2}\sum_{k=0}^{n} 2^{k+1} a_{2^{k+1}} \leq \sum_{k=0}^{n}\sum_{l=2^k+1}^{2^{k+1}} a_l \leq \sum_{k=0}^{n} 2^k a_{2^k}$$

$$\frac{1}{2}\sum_{l=1}^{n+1} 2^l a_{2^l} \leq \sum_{l=2}^{2^{n+1}} a_l \leq t_n$$

$$\frac{1}{2}\left(t_{n+1} - a_1\right) \leq s_{2^{n+1}} - a_1 \leq t_n$$

$$\implies \begin{cases} t_{n+1} \leq 2 s_{2^{n+1}} - a_1 \\ s_n \leq s_{2^{n+1}} \leq t_n + a_1 \text{ as } n \leq 2^{n+1} \, \forall n \geq 1 \end{cases}$$

If $\{s_n\}_{n\geq 1}$ is bounded $\implies \exists M > 0$ s.t. $|s_n| \leq M \, \forall n \geq 1$

$$\implies t_{n+1} \leq 2M + a_1 \quad \forall n \geq 1$$

If $\{t_n\}_{n\geq 0}$ is bounded $\implies \exists L > 0$ s.t. $|t_n| \leq L \, \forall n \geq 0$

$$\implies s_n \leq L + a_1 \quad \forall n \geq 1$$

□

Corollary 16.2
The series $\sum_{n\geq 1} \frac{1}{n^\alpha}$ converges if and only if $\alpha > 1$.

Proof. If $\alpha \le 0$ then $\frac{1}{n^\alpha} = n^{-\alpha} \ge 1\, \forall n \ge 1$. In particular, $\frac{1}{n^\alpha} \overset{n\to\infty}{\not\to} 0$ so $\sum_{n\ge1} \frac{1}{n^\alpha}$ cannot converge. Assume $\alpha > 0$. Then $\left\{\frac{1}{n^\alpha}\right\}_{n\ge1}$ is a decreasing sequence of positive real numbers. By the dyadic criterion,

$$\sum_{n\ge1} \frac{1}{n^\alpha} \text{ converges} \iff \sum_{n\ge0} 2^n \frac{1}{(2^n)^\alpha} \text{ converges}$$

$$\sum_{n\ge0} \frac{2^n}{(2^n)^\alpha} = \sum_{n\ge0} \left(2^{1-\alpha}\right)^n = \sum_{n\ge0} r^n \text{ where } r = 2^{1-\alpha}$$

This converges $\iff r < 1 \iff 2^{1-\alpha} < 1 \iff 1-\alpha < 0 \iff \alpha > 1$. □

Theorem 16.3 (Root Test)
Let $\sum_{n\ge1} a_n$ be a series of real numbers.

1. If $\limsup |a_n|^{\frac{1}{n}} < 1$ then $\sum_{n\ge1} a_n$ converges absolutely.
2. If $\liminf |a_n|^{\frac{1}{n}} > 1$ then $\sum_{n\ge1} a_n$ diverges.
3. The test is inconclusive if $\liminf |a_n|^{\frac{1}{n}} \le 1 \le \limsup |a_n|^{\frac{1}{n}}$.

Proof. 1. Let $L = \limsup |a_n|^{\frac{1}{n}}$.

$$L < 1 \implies 1 - L > 0 \overset{\mathbb{Q} \text{ dense in } \mathbb{R}}{\implies} \exists \epsilon \in \mathbb{R} \ni 0 < \epsilon < 1 - L \implies L < L + \epsilon < 1$$

So $L + \epsilon > L = \limsup |a_n|^{\frac{1}{n}} = \inf_N \sup_{n\ge N} |a_n|^{\frac{1}{n}}$

$$\begin{aligned} &\implies \exists N_0 \in \mathbb{N} \ni \sup_{n\ge N_0} |a_n|^{\frac{1}{n}} < L + \epsilon \\ &\implies |a_n|^{\frac{1}{n}} < L + \epsilon \quad \forall n \ge N_0 \\ &\implies |a_n| < (L+\epsilon)^n \quad \forall n \ge N_0 \end{aligned}$$

As $L + \epsilon < 1$, the series

$$\begin{aligned} \sum_{n\ge N_0} (L+\epsilon)^n &= \sum_{k\ge0} (L+\epsilon)^{N_0+k} \\ &= (L+\epsilon)^{N_0} \sum_{k\ge0} (L+\epsilon)^k \\ &= (L+\epsilon)^{N_0} \frac{1}{1-(L+\epsilon)} \end{aligned}$$

By the Comparison Test, $\sum_{n\ge N_0} a_n$ converges absolutely and note $|a_1| + \ldots + |a_{N_0-1}| \in \mathbb{R}$.

$$\implies \sum_{n\ge1} a_n \text{ converges absolutely}$$

2. Let $\{a_{k_n}\}_{n\geq 1}$ be a subsequence of $\{a_n\}_{n\geq 1}$ such that

$$\begin{aligned}
&\lim_{n\to\infty} |a_{k_n}|^{\frac{1}{k_n}} = \liminf |a_n|^{\frac{1}{n}} > 1 \\
&\implies \exists n_0 \in \mathbb{N} \ni |a_{k_n}|^{\frac{1}{k_n}} > 1 \quad \forall n \geq n_0 \\
&\implies |a_{k_n}| > 1 \qquad \forall n \geq n_0 \\
&\implies a_{k_n} \overset{n\to\infty}{\not\longrightarrow} 0 \implies a_n \overset{n\to\infty}{\not\longrightarrow} 0 \implies \sum_{n\geq 1} a_n \text{ diverges}
\end{aligned}$$

3. Consider $a_n = \frac{1}{n}\, \forall n \geq 1$. The series $\sum_{n\geq 1} a_n = \sum_{n\geq 1} \frac{1}{n}$ diverges. However,

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{n}} \overset{\text{Cesaro-Stolz}}{=} \frac{1}{\lim_{n\to\infty} \frac{n+1}{n}} = 1$$

So $\liminf \sqrt[n]{a_n} = \limsup \sqrt[n]{a_n} = 1$. Consider now $a_n = \frac{1}{n^2}\, \forall n \geq 1$. The series $\sum_{n\geq 1} a_n = \sum_{n\geq 1} \frac{1}{n^2}$ converges.

However,

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{n^2}} \overset{\text{C-S}}{=} \frac{1}{\lim_{n\to\infty} \frac{(n+1)^2}{n^2}} = 1$$

So $\liminf \sqrt[n]{a_n} = \limsup \sqrt[n]{a_n} = 1$.

□

Theorem 16.4 (Ratio Test)

Let $\sum_{n\geq 1} a_n$ be a series of non-zero real numbers.

1. If $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$ then $\sum_{n\geq 1} a_n$ converges absolutely.
2. If $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$ then $\sum_{n\geq 1} a_n$ diverges.
3. The test is conclusive if $\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$

Proof. (1) & (2) follow from the root test and the Cesaro – Stolz theorem:

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq \liminf |a_n|^{\frac{1}{n}} \leq \limsup |a_n|^{\frac{1}{n}} \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$$

For (3) consider the same examples as in the previous theorem. □

Theorem 16.5 (Abel Criterion)

Let $\{a_n\}_{n\geq 1}$ be a decreasing sequence with $\lim_{n\to\infty} a_n = 0$. Let $\{b_n\}_{n\geq 1}$ be a sequence so that $\{\sum_{k=1}^{n} b_k\}_{k\geq 1}$ is bounded. Then $\sum_{n\geq 1} a_n b_n$ converges.

Corollary 16.6 (Leibniz Criterion)
Let $\{a_n\}_{n\geq 1}$ be a decreasing sequence with $\lim_{n\to\infty} a_n = 0$. Then $\sum_{n\geq 1}(-1)^n a_n$ converges.

Proof. (Abel Criterion) Let $t_n = \sum_{k=1}^n b_k$ for $n \geq 1$. As $\{t_n\}_{n\geq 1}$ is bounded $\exists M > 0$ s.t. $|t_n| \leq M \,\forall n \geq 1$. We will use the Cauchy criterion to prove convergence of $\sum_{n\geq 1} a_n b_n$. Let $\epsilon > 0$.
As $\lim a_n = 0 \implies \exists n_\epsilon \in \mathbb{N}$ s.t. $|a_n| < \frac{\epsilon}{2M} \,\forall n \geq n_\epsilon$. For $n \geq n_\epsilon$ and $p \in \mathbb{N}$,

$$\begin{aligned}
\left|\sum_{k=n+1}^{n+p} a_k b_k\right| &= \left|\sum_{k=n+1}^{n+p} a_k (t_k - t_{k-1})\right| \\
&= \left|\sum_{k=n+1}^{n+p} a_k t_k - \sum_{k=n+1}^{n+p} a_k t_{k-1}\right| \\
&= \left|\sum_{k=n+1}^{n+p} a_k t_k - \sum_{k=n}^{n+p-1} a_{k+1} t_k\right| \\
&= \left|\sum_{k=n}^{n+p} t_k \left(a_k - a_{k+1}\right) - a_n t_n + a_{n+p+1} t_{n+p}\right| \\
&\leq \sum_{k=n}^{n+p} |t_k| \left|a_k - a_{k+1}\right| + |a_n| \cdot |t_n| + |a_{n+p+1}| \cdot |t_{n+p}| \\
&\leq \sum_{k=n}^{n+p} M(a_k - a_{k+1}) + a_n M + a_{n+p+1} M \\
&= M\left(a_n - \cancel{a_{n+p+1}}\right) + a_n M + \cancel{a_{n+p+1}} M \\
&= 2M \cdot a_n < \epsilon
\end{aligned}$$

□

§17 Lec 17: Feb 12, 2021

§17.1 Rearrangements of Series

Definition 17.1 (Rearrangement) — Let $k : \mathbb{N} \to \mathbb{N}$ be a bijective function. For a sequence $\{a_n\}_{n\geq 1}$ of real numbers, we denote

$$\tilde{a}_n = a_{k(n)} = a_{k_n}$$

Then $\sum_{n\geq 1} \tilde{a}_n$ is called a rearrangement of $\sum_{n\geq 1} a_n$

Example 17.2

Consider $a_n = \frac{(-1)^{n-1}}{n}$ $\forall n \geq 1$. The series $\sum_{n\geq 1} a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots$ Note that the sequence $\left\{\frac{1}{n}\right\}_{n\geq 1}$ is decreasing and $\lim_{n\to\infty} \frac{1}{n} = 0$. Thus, by the Leibniz criterion, $\sum_{n\geq 1} a_n$ converges. Write the series as follows:

$$\sum_{n\geq 1} a_n = 1 - \frac{1}{2} + \frac{1}{3} - \sum_{k\geq 2}\left(\frac{1}{2k} - \frac{1}{2k+1}\right)$$

Note that for $k \geq 2$

$$0 < \frac{1}{2k} - \frac{1}{2k+1} = \frac{1}{2k(2k+1)} < \frac{1}{4k^2}$$

Recall that the series $\sum_{k\geq 2} \frac{1}{4k^2}$ converges (by the dyadic criterion). By the comparison test, the series $0 < \sum_{k\geq 2}\left(\frac{1}{2k} - \frac{1}{2k+1}\right)$ converges. So $\sum_{n\geq 1} a_n < 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$. Consider next the following rearrangement:

$$\frac{1}{1} + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \dots = \sum_{k\geq 1}\left(\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k}\right)$$

Then

$$\begin{aligned}0 < \frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} &= \frac{8k^2 - 2k + 8k^2 - 6k - (16k^2 - 16k + 3)}{(4k-3)(4k-1)\cdot 2k} \\ &= \frac{8k-3}{(4k-3)(4k-1)2k} < \frac{8k}{k\cdot 3k\cdot 2k} = \frac{4}{3k^2}\end{aligned}$$

As the series $\sum_{k\geq 1} \frac{4}{3k^2}$ converges, we deduce that the series

$$\sum_{k\geq 1}\left(\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k}\right) \text{ converges}$$

Moreover,

$$\begin{aligned}\sum_{k\geq 1}\left(\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k}\right) &= 1 + \frac{1}{3} - \frac{1}{2} + \sum_{k\geq 2}\left(\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k}\right) \\ &> 1 + \frac{1}{3} - \frac{1}{2} = \frac{5}{6}\end{aligned}$$

So the two series converge to two different numbers.

Theorem 17.3 (Riemann)

Let $\sum_{n\geq 1} a_n$ be a series that converges, but it does not converge absolutely. Let $-\infty \leq \alpha \leq \beta \leq \infty$. Then there exists a rearrangement $\sum_{n\geq 1} \tilde{a}_n$ with partial sums $\tilde{s}_n = \sum_{k=1}^{n} \tilde{a}_k$ such that

$$\liminf \tilde{s}_n = \alpha \text{ and } \limsup \tilde{s}_n = \beta$$

Proof. For $n \geq 1$ let

$$b_n = \frac{|a_n| + a_n}{2} = \begin{cases} a_n, & a_n \geq 0 \\ 0, & a_n < 0 \end{cases} \implies b_n \geq 0$$

$$c_n = \frac{|a_n| - a_n}{2} = \begin{cases} 0, & a_n \geq 0 \\ -a_n, & a_n < 0 \end{cases} \implies c_n \geq 0$$

Claim 17.1. The series $\sum_{n\geq 1} b_n$ and $\sum_{n\geq 1} c_n$ both diverge.

Note $\sum_{k=1}^{n} b_k - \sum_{k=1}^{n} c_k = \sum_{k=1}^{n}(b_k - c_k) = \sum_{k=1}^{n} a_k$ which converges as $n \to \infty$.

$$\implies \sum_{k=1}^{n} b_k = \sum_{k=1}^{n} c_k + \sum_{k=1}^{n} a_k$$

So $\{\sum_{k=1}^{n} b_k\}_{n\geq 1}$ converges if and only if $\{\sum_{k=1}^{n} c_k\}_{n\geq 1}$ converges. On the other hand if $\sum_{n\geq 1} b_n$ and $\sum_{n\geq 1} c_n$ both converged, then

$$\underbrace{\sum_{k=1}^{n} b_k + \sum_{k=1}^{n} c_k}_{\text{converge as } n\to\infty} = \sum_{k=1}^{n}(b_k + c_k) = \sum_{k=1}^{n} |a_k|$$

which diverges as $n \to \infty$ – contradiction. Thus $\sum_{n\geq 1} b_n$ and $\sum_{n\geq 1} c_n$ diverge to infinity. Note also that $\sum_{n\geq 1} a_n$ converges $\implies \lim_{n\to\infty} a_n = 0$ and so $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 0$.

Let $B_1, B_2, B_3, \ldots$ denote the non-negative terms in $\{a_n\}_{n\geq 1}$ in the order which they appear.

Let $C_1, C_2, C_3, \ldots$ denote the absolute values of the negative terms in $\{a_n\}_{n\geq 1}$, in the order in which they appear.

Note $\sum_{n\geq 1} B_n$ differs $\sum_{n\geq 1} b_n$ only by terms that are zero. So $\sum_{n\geq 1} B_n = \infty$. Similarly, $\sum_{n\geq 1} C_n$ differs $\sum_{n\geq 1} c_n$ only be terms that are zero. So $\sum_{n\geq 1} C_n = \infty$.

Choose sequences $\{\alpha_n\}_{n\geq 1}$ and $\{\beta_n\}_{n\geq 1}$ so that

$$\begin{cases} \alpha_n \overset{n\to\infty}{\longrightarrow} \alpha \\ \beta_n \overset{n\to\infty}{\longrightarrow} \beta \\ \alpha_n < \beta_n \quad \forall n \geq 1 \\ \beta_1 > 0 \end{cases}$$

<u>E.g.</u>

Next we construct increasing sequences $\{k_n\}_{n\geq 1}$ and $\{j_n\}_{n\geq 1}$ as follows:

1. Choose k_1 and j_1 to be the smallest natural numbers so that

$$x_1 = B_1 + B_2 + \ldots + B_{k_1} > \beta_1 \text{ (this is possible because } \sum_{n\geq 1} B_n = \infty)$$

$$y_1 = B_1 + \ldots + B_{k_1} - C_1 - C_2 - \ldots - C_{j_1} < \alpha_1 \text{ (this is possible since } \sum_{n\geq 1} C_n = \infty)$$

2. Choose k_2 and j_2 to be the smallest natural numbers so that

$$x_2 = B_1 + \ldots + B_{k_1} - C_1 - \ldots - C_{j_1} + B_{k_1+1} + \ldots + B_{k_2} > \beta_2$$
$$y_2 = B_1 + \ldots + B_{k_1} - C_1 - C_{j_1} + B_{k_1+1} + \ldots + B_{k_2} - C_{j_1+1} - \ldots - C_{j_2} < \alpha_2$$

and so on.

Note that by definition,

$$\begin{aligned} x_n - B_{k_n} \leq \beta_n &\implies \beta_n - B_{k_n} < \beta_n < x_n \leq \beta_n + B_{k_n} \\ &\implies \left| x_n - \underbrace{B_n}_{\xrightarrow{n\to\infty} \beta} \right| \leq B_{k_n} \xrightarrow{n\to\infty} 0 \\ &\implies \lim_{n\to\infty} x_n = \beta \end{aligned}$$

Similarly,

$$\begin{aligned} y_n + C_{j_n} \geq \alpha_n &\implies \alpha_n - C_{j_n} \leq y_n < \alpha_n < \alpha_n + C_{j_n} \\ &\implies \left| y_n - \underbrace{\alpha_n}_{\xrightarrow{n\to\infty} \alpha} \right| \leq C_{j_n} \xrightarrow{n\to\infty} 0 \\ &\implies \lim_{n\to\infty} y_n = \alpha \end{aligned}$$

Finally, note that x_n and y_n are partial sums in the rearrangement

$$B_1 + B_2 + \ldots + B_{k_1} - C_1 - \ldots - C_{j_1} + B_{k_1+1} + \ldots + B_{k_2} - C_{j_1+1} - \ldots - C_{j_2} + \ldots$$

By construction, no number less than α or larger than β can occur as a subsequential limit of the partial sums. □

Theorem 17.4 (Absolute Convergence and Convergence of Rearrangement)
If a series $\sum_{n\geq 1} a_n$ converges absolutely, then any rearrangement $\sum_{n\geq 1} \tilde{a}_n$ converges to $\sum_{n\geq 1} a_n$.

Proof. For $n \geq 1$ let $s_n = \sum_{k=1}^n a_k$, $\tilde{s}_n = \sum_{k=1}^n \tilde{a}_k$. As $\sum_{n\geq 1} a_n$ converges absolutely, $\forall \epsilon > 0 \, \exists n_\epsilon \in \mathbb{N}$ s.t.

$$\sum_{k=n+1}^{n+p} |a_k| < \epsilon \quad \forall n \geq n_\epsilon \, \forall p \in \mathbb{N}$$

Choose N_ϵ sufficiently large so that $a_1, \dots, a_{n_\epsilon}$ belong to the set $\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n\}$. Then for $n > N_\epsilon$ the terms $a_1, \dots, a_{n_\epsilon}$ cancel in $s_n - \tilde{s}_n$

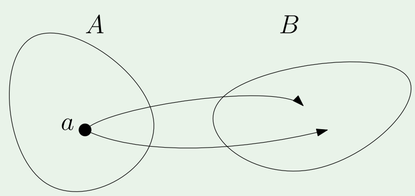
$$|s_n - \tilde{s}_n| \leq \underbrace{\sum_{k=n_\epsilon+1}^{n} |a_k| + \sum_{1 \leq k \leq n} |\tilde{a}_k|}_{\text{finitely many terms and all indices are } > n_\epsilon} < \epsilon \quad (\tilde{a}_k \notin \{a_1, \dots, a_{n_\epsilon}\})$$

As $\lim_{n\to\infty} s_n = s \in \mathbb{R}$ we deduce that $\lim_{n\to\infty} \tilde{s}_n = s$. □

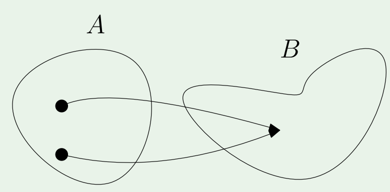
§18 Lec 18: Feb 17, 2021

§18.1 Functions

Definition 18.1 (Function) — Let A, B be two non-empty sets. A function $f : A \to B$ is a way of associating to each element $a \in A$ exactly one element in B denoted $f(a)$.



not a function



A function

A is called the domain of f.
B is called the range of f.

$f(A) = \{f(a) : a \in A\}$ is called the image of A under f. If $A' \subseteq A$ then $f(A') = \{f(a) : a \in A'\}$ is called the image of A' under f.

If $f(A) = B$ then we say that f is surjective/onto. In this case, $\forall b \in B \quad \exists a \in A$ s.t. $f(a) = b$.

We say that f is injective if it satisfies: if $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$ then $a_1 = a_2$.

We say that f is bijective if f is injective and surjective.

Remark 18.2. The injectivity and surjectivity of a function depend not only on the law f, but also on the domain and the range.

Example 18.3

$f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = 2n$ which is injective but not surjective.

$$f(n) = f(m) \implies 2n = 2m \implies n = m$$

$g : \mathbb{R} \to \mathbb{R}$, $g(x) = 2x$ bijective.

Example 18.4

$f : [0, \infty) \to [0, \infty)$, $f(x) = x^2$ bijective, $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$ not injective, not surjective.

Definition 18.5 (Composition) — Let A, B, C be non-empty sets and $f : A \to B$, $g : B \to C$ be two functions. The composition of g with f is a function $g \circ f : A \to C$, $(g \circ f)(a) = g(f(a))$.

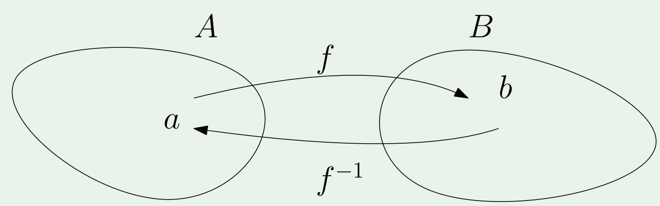
Remark 18.6. The composition of two functions need not be commutative.

$$\begin{aligned} f : \mathbb{Z} \to \mathbb{Z}, &\quad f(n) = 2n \\ g : \mathbb{Z} \to \mathbb{Z}, &\quad g(n) = n+1 \\ g \circ f : \mathbb{Z} \to \mathbb{Z}, &\quad (g \circ f)(n) = g(f(n)) = 2n+1 \\ f \circ g : \mathbb{Z} \to \mathbb{Z}, &\quad (f \circ g)(n) = f(g(n)) = 2(n+1) \end{aligned}$$

Exercise 18.1. The composition of functions is associate: if $f : A \to B$, $g : B \to C$, $h : C \to D$ are three functions, then

$$(h \circ g) \circ f = h \circ (g \circ f)$$

Definition 18.7 (Inverse Function) — Let $f : A \to B$ be a bijective function. The inverse of f is a function $f^{-1} : B \to A$ defined as follows: if $b \in B$ then $f^{-1}(b) = a$ where a is the unique element in A s.t. $f(a) = b$. The existence of a is guaranteed by surjectivity and the uniqueness by injectivity.



So

$$\begin{aligned} f \circ f^{-1} : B \to B \\ \left(f \circ f^{-1}\right)(b) = b \end{aligned}$$

and

$$\begin{aligned} f^{-1} \circ f : A \to A \\ \left(f^{-1} \circ f\right)(a) = a \end{aligned}$$

Exercise 18.2. Let $f : A \to B$ and $g : B \to C$ be two bijective functions. Then $g \circ f : A \to C$ is a bijection and

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

Definition 18.8 (Preimage) — Let $f : A \to B$ be a function. If $B' \subseteq B$ then the preimage of B' is $f^{-1}(B') = \{a \in A : f(a) \in B'\}$. The preimage of a set is well defined whether or not f is bijective. In fact, if $B' \subseteq B$ s.t. $B' \cap f(A) = \emptyset$ then $f^{-1}(B') = \emptyset$.

Exercise 18.3. Let $f : A \to B$ be a function and let $A_1, A_2 \subseteq A$ and $B_1, B_2 \subseteq B$. Then

1. $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$

2. $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$
3. $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$
4. $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$
5. The following are equivalent:
   i) f is injective.
   ii) $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ for all subsets $A_1, A_2 \subseteq A$.

§18.2 Cardinality

Definition 18.9 (Equipotent) — We say that two sets A and B have the same cardinality (or the same cardinal number) if there exists a bijection $f : A \to B$. In this case we write $A \sim B$.

Exercise 18.4. Show that $\sim$ is an equivalence relation on sets.

Definition 18.10 (Finite Set, Countable vs. Uncountable) — We say that a set A is finite if $A = \emptyset$ (in which case we say that it has cardinality 0) or $A \sim \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ (in which case we say that A has cardinality n).
We say that A is countable if $A \sim \mathbb{N}$. I this case we say that A has cardinality $\aleph_0$.
We say that A is at most countable if A is finite or countable. If A is not at most countable we say that A is uncountable.

Lemma 18.11
Let A be a finite set and let $B \subseteq A$. Then B is finite.

Proof. If $B = \emptyset$ then B is finite. Assume now that $B \neq \emptyset \implies A \neq \emptyset$. As A is finite, $\exists n \in \mathbb{N}$ and $\exists f : A \to \{1, \ldots, n\}$ bijective. Then $f|_B : B \to f(B)$ is bijective.

WE merely have to relabel the elements in $f(B)$. Let $b_1 \in B$ be such that $f(b_1) = \min f(B)$.
Define $g(b_1) = 1$. If $B \setminus \{b_1\} \neq \emptyset$, let $b_2 \in B$ be such that $f(b_2) = \min f(B \setminus \{b_1\})$. Define $g(b_2) = 2$. Keep going. The process terminates in at most n steps. □

Example 18.12
$f : \mathbb{N} \cup \{0, -1, -2, \ldots, -k\} \to \mathbb{N}$ where $k \in \mathbb{N}$

$$f(n) = n + k + 1 \text{ is bijective}$$

So the cardinality of $\mathbb{N} \cup \{0, -1, \ldots, -k\}$ is $\aleph_0$.

Example 18.13

$f : \mathbb{Z} \to \mathbb{N}$

$$f(n) = \begin{cases} 2n+2,\ n \geq 0 \\ -2n-1,\ n < 0 \end{cases} \quad \text{is bijective}$$

So the cardinality of $\mathbb{Z}$ is $\aleph_0$.

Example 18.14

$f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

$$f(n,m) = \frac{(n+m-1)(n+m-2)}{2} + n$$

is bijective. So the cardinality of $\mathbb{N} \times \mathbb{N}$ is $\aleph_0$.

n \ m	1	2	3	4
1	$(1,1)$	$(2,2)$	$(3,3)$	$(4,4)$
2	$(2,1)$	$(2,2)$	$(2,3)$	$(2,4)$
3	$(3,1)$	$(3,2)$	$(3,3)$	$(3,4)$
4	$(4,1)$	$(4,2)$	$(4,3)$	$(4,4)$

Cont'd in Lec 19.

§19 Lec 19: Feb 19, 2021

§19.1 Functions & Cardinality (Cont'd)

From the last example of Lec 18, $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $f(n,m) = \frac{(n+m-1)(n+m-2)}{2} + n$, f is bijective.
We prove that f is surjective by induction. For $k \in \mathbb{N}$ let $P(k)$ denoted that statement

$$\exists (n,m) \in \mathbb{N} \times \mathbb{N} \text{ s.t. } f(n,m) = k$$

Base step: Note that $f(1,1) = \frac{1 \cdot 0}{2} + 1 = 1$. So $P(1)$ holds.
Inductive step: Fix $k \geq 1$ and assume that $P(k)$ holds. Then $\exists (n,m) \in \mathbb{N} \times \mathbb{N}$ s.t. $f(n,m) = k$.

$$\begin{aligned}
&\implies \frac{(n+m-1)(n+m-2)}{2} + n + 1 = k + 1 \\
&\implies \frac{[(n+1)+(m-1)-1]\,[(n+1)+(m-1)-2]}{2} + n + 1 = k + 1 \\
&\implies f(n+1, m-1) = k + 1
\end{aligned}$$

This works if $(n+1, m-1) \in \mathbb{N} \times \mathbb{N} \iff m - 1 \in \mathbb{N} \iff m \geq 2$. So if $m \geq 2$ we found $(n+1, m-1) \in \mathbb{N} \times \mathbb{N}$ s.t. $f(n+1, m-1) = k+1$. Assume now $m = 1$. Then

$$\begin{aligned}
&\implies f(n,1) = k \iff \frac{n(n-1)}{2} + n = k \iff \frac{(n+1)n}{2} = k \\
&\implies \frac{(n+1)n}{2} + 1 = k + 1 \\
&\implies \frac{[1+(n+1)-1]\,[1+(n+1)-2]}{2} + 1 = k + 1 \\
&\implies f(1, n+1) = k + 1
\end{aligned}$$

So if $m = 1$ we found $(1, n+1) \in \mathbb{N} \times \mathbb{N}$ s.t. $f(1, n+1) = k+1$. This proves $P(k+1)$ holds.
By induction, $\forall k \in \mathbb{N}\, \exists (n,m) \in \mathbb{N} \times \mathbb{N}$ s.t. $f(n,m) = k$, i.e. f is surjective.

Let $(n,m), (a,b) \in \mathbb{N} \times \mathbb{N}$ s.t. $f(n,m) = f(a,b)$. We want to show that $(n,m) = (a,b)$, thus proving that f is injective.

Case 1:

$$\left.\begin{aligned}
&\tfrac{(n+m-1)(n+m-2)}{2} = \tfrac{(a+b-1)(a+b-2)}{2} \\
&f(n,m) = f(a,b)
\end{aligned}\right\} \implies n = a$$

Then $(n+m-1)(n+m-2) = (n+b-1)(n+b-2)$

$$\begin{aligned}
&\implies n^2 + n(2m-3) + m^2 - 3m + 2 = n^2 + n(2b-3) + b^2 - 3b + 2 \\
&\implies 2n(m-b) + (m-b)(m+b) - 3(m-b) = 0 \\
&\implies \left.\begin{aligned} &(m-b)(2n+m+b-3) = 0 \\ &2n+m+b-3 \geq 2+1+1-3 \geq 1 \end{aligned}\right\} \implies m = b
\end{aligned}$$

Case 2: $\frac{(n+m-1)(n+m-2)}{2} = \frac{(a+b-1)(a+b-2)}{2} + r$ for some $r \in \mathbb{N}$.

Exercise 19.1. Show that this cannot occur.

Lemma 19.1

Let A be a countable set. Let B be an infinite subset of A.Then B is countable.

Proof. A is countable $\implies \exists f : \mathbb{N} \to A$ bijection. This means we can enumerate the elements of A :

$$A = \{a_1(= f(1)), a_2(= f(2)), a_3(= f(3)), \ldots\}$$

Let $k_1 = \min\{n : a_n \in B\}$. Define $g(1) = a_{k_1}$. Then $B \setminus \{a_{k_1}\} \neq \emptyset$. Let $k_2 = \min\{n : a_n \in B \setminus \{a_{k_1}\}\}$. Define $g(2) = a_{k_2}$.

We proceed inductively. Assume we found $k_1 < \ldots < k_j$ such that $a_{k_1}, \ldots, a_{k_j} \in B$ and $g(1) = a_{k_1}, \ldots, g(j) = a_{k_j}$. Then $B \setminus \{a_{k_1}, \ldots, a_{k_j}\} \neq \emptyset$. Let $k_{j+1} = \min\{n : a_n \in B \setminus \{a_{k_1}, \ldots, a_{k_j}\}\}$. Define $g(j+1) = a_{k_{j+1}}$.

By construction, $g : \mathbb{N} \to B$ is bijective. □

Lemma 19.2

Let A be a finite set and let B be a proper subset of A. Then A and B are not equipotent, that is, there is no bijective function $f : A \to B$.

Proof. If $B = \emptyset \implies A \neq \emptyset$. There is no function $f : A \to B$. Assume $B \neq \emptyset$. Assume towards a contradiction that there exists a bijection $f : A \to B$.

As $B \subsetneq A$, $\exists a_0 \in A \setminus B$.

For $n \geq 1$ let $a_n = \underbrace{(f \circ f \circ \ldots \circ f)}_{n \text{ times}}(a_0)$. Note $a_{n+1} = f(a_n)\, \forall n \geq 0$. Note $a_n \in B\, \forall n \geq 1$.

We will show

Claim 19.1. $a_n \neq a_m$ for $n \neq m$.

If the claim holds then B (and so A) would contain countably many elements. Contradiction, since A is finite!

To prove the claim we argue by contradiction. Assume that there exists $n, k \in \mathbb{N}$ s.t. $a_{n+k} = a_n$.

Write

$$\left.\begin{aligned} a_{n+k} &= \underbrace{(f \circ f \circ \ldots \circ f)}_{n \text{ times}}(a_k) \\ a_n &= \underbrace{(f \circ f \circ \ldots \circ f)}_{n \text{ times}}(a_0) \\ f \text{ injective} &\implies \underbrace{f \circ f \circ \ldots \circ f}_{n \text{ times}} \text{ injective} \end{aligned}\right\} \implies B \ni a_k = a_0 \in A \setminus B$$

which is a contradiction! This proves the claim and completes the proof of the lemma. □

Lemma 19.3

Every infinite set has a countable subset.

Proof. Let A be an infinite set $\implies A \neq \emptyset \implies \exists a_1 \in A$. Then $A \setminus \{a_1\} \neq \emptyset \implies \exists a_2 \in A \setminus \{a_1\}$.

We proceed inductively. Having found $a_1, \ldots, a_n \in A$ distinct, $A \setminus \{a_1, \ldots, a_n\} \neq \emptyset \implies \exists a_{n+1} \in A \setminus \{a_1, \ldots, a_n\}$. This gives a sequence $\{a_n\}_{n\geq 1}$ of distinct elements in A. □

Theorem 19.4

A set A is infinite if and only if there is a bijection between A and a proper subset of A.

Proof. "$\impliedby$" Assume that there is a bijection $f : A \to B$ where $B \subsetneq A$. By Lemma 19.2, A must be infinite.
"$\implies$" Assume that A is infinite. By Lemma 19.3, there exists a countable subset B of A. Write $B = \{a_1, a_2, a_3, \ldots\}$ with $a_n \neq a_m$ if $n \neq m$. Then $A \setminus \{a_1\}$ is a proper subset of A. Define $f : A \to A \setminus \{a_1\}$ via

$$f(a) = \begin{cases} a, \text{ if } a \in A \setminus B \\ a_{j+1}, \text{ if } a = a_j \text{ for some } j \geq 1 \end{cases}$$

This is a bijective function.
Assume $f(a) = f(b)$.
Case 1: $a, b \in A \setminus B$. Then $f(a) = a$, $f(b) = b$ and so $f(a) = f(b) \implies a = b$.
Case 2: $a, b \in B \implies \exists i, j \in \mathbb{N}$ s.t. $a = a_i$, $b = a_j$

$$f(a) = f(b) \implies a_{i+1} = a_{j+1} \implies i + 1 = j + 1 \implies i = j \implies a = b$$

Case 3: $a \in A \setminus B$, $b \in B$. Then $f(a) \in A \setminus B$ and $f(b) \in B$, which cannot occur.
Case 4: $a \in B$ and $b \in A \setminus B$. Argue as for Case 3.

Exercise 19.2. f is surjective. □

Theorem 19.5 (Schröder – Bernstein)

Assume that A and B are two sets such that there exists two injective functions $f : A \to B$ and $g : B \to A$. Then A and B are equipotent.

Example 19.6

$$\begin{aligned} f &: \mathbb{N} \to \mathbb{N} \times \mathbb{N}, \quad f(n) = (1, n) \text{ injective} \\ g &: \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad g(n, m) = 2^n \cdot 3^m \text{ injective} \end{aligned}$$

By Schröder – Bernstein, $\mathbb{N} \sim \mathbb{N} \times \mathbb{N}$.

§20 Lec 20: Feb 22, 2021

§20.1 Countable vs. Uncountable Sets

Proof. (Schröder – Bernstein) We will decompose each of the sets A and B into disjoint subsets:

$$A = A_1 \cup A_2 \cup A_3 \text{ with } A_i \cap A_j = \emptyset \text{ if } i \neq j$$
$$B = B_1 \cup B_2 \cup B_3 \text{ with } B_i \cap B_j = \emptyset \text{ if } i \neq j$$

and we will show that $f : A_1 \to B_1, f : A_2 \to B_2, g : B_3 \to A_3$ are bijections.

Then $h : A \to B$ given by

$$h(a) = \begin{cases} f(a), & \text{if } a \in A_1 \cup A_2 \\ \left(g|_{B_3}\right)^{-1}(a), & \text{if } a \in A_3 \end{cases}$$

is a bijection. Exc!

For $a \in A$ consider the set

$$S_a = \left\{ \underbrace{a}_{\in A}, \underbrace{g^{-1}(a)}_{\in B}, \underbrace{f^{-1} \circ g^{-1}(a)}_{\in A}, \underbrace{g^{-1} \circ f^{-1} \circ g^{-1}(a)}_{\in B}, \ldots \right\}$$

Note that the preimage under f or g is either $\emptyset$ or it contains exactly one point (because f and g are injective).

There are three possibilities:

1. The process defining S_a does not terminate. We can always find a preimage.
2. The process defining S_a terminates in A, that is, the last element $x \in S_a$ is $x = a$ or $x = f^{-1} \circ g^{-1} \circ \ldots \circ g^{-1}(a)$ and $g^{-1}(x) = \emptyset$.
3. The process defining S_a terminates in B, that is, the last element $x \in S_a$ is $x = g^{-1}(a)$ or $x = g^{-1} \circ f^{-1} \circ \ldots \circ g^{-1}(a)$ and $f^{-1}(x) = \emptyset$.

We define

$$A_1 = \{a \in A : \text{ the process defining } S_a \text{ does not terminate}\}$$
$$A_2 = \{a \in A : \text{ the process defining } S_a \text{ terminates in } A\}$$
$$A_3 = \{a \in A : \text{ the process defining } S_a \text{ terminates in } B\}$$

Similarly, for $b \in B$ we define the set

$$T_b = \left\{ \underbrace{b}_{\in B}, \underbrace{f^{-1}(b)}_{\in A}, \underbrace{g^{-1} \circ f^{-1}(b)}_{\in B}, \underbrace{f^{-1} \circ g^{-1} \circ f^{-1}(b)}_{\in A}, \ldots \right\}$$

As before we define

$$B_1 = \{b \in B : \text{ the process defining } T_b \text{ does not terminate}\}$$
$$B_2 = \{b \in B : \text{ the process defining } T_b \text{ ends in } A\}$$
$$B_3 = \{b \in B : \text{ the process defining } T_b \text{ ends in } B\}$$

Let's show $f : A_1 \to B_1$ is a bijection. Injectivity is inherited from $f : A \to B$ is injective. Let $b \in B_1$. Then the process defining

$$T_b = \left\{b, f^{-1}(b), g^{-1} \circ f^{-1}(b), \ldots\right\} \text{ does not terminate}$$

In particular, $\exists a \in A$ s.t. $f^{-1}(b) = a$. Note that

$$S_a = \left\{a, g^{-1}(a), f^{-1} \circ g^{-1}(a), \ldots\right\} \quad = \left\{f^{-1}(b), g^{-1} \circ f^{-1}(b), f^{-1} \circ g^{-1} \circ f^{-1}(b), \ldots\right\}$$

does not terminate. So $a \in A_1$.

This proves $f : A_1 \to B_1$ is surjective.

Let's show $f : A_2 \to B_2$ is a bijection. Again, injectivity is inherited from $f : A \to B$ is injective.

Let $b \in B_2$. Then the process defining

$$T_b = \left\{b, f^{-1}(b), g^{-1} \circ f^{-1}(b), \ldots\right\} \text{ terminates in } A$$

In particular, $\exists a \in A$ s.t. $f^{-1}(b) = a$. Note that

$$S_a = \left\{a, g^{-1}(a), \ldots\right\} = \left\{f^{-1}(b), g^{-1} \circ f^{-1}(b), \ldots\right\}$$

terminates in $A \implies a \in A_2$. So $f : A_2 \to B_2$ is surjective.

Exercise 20.1. $g : B_3 \to A_3$ is bijective.

□

Theorem 20.1 (Union of Countable Sets)

Let $\{A_n\}_{n\geq 1}$ be a sequence of countable sets. Then

$$\bigcup_{n\geq 1} A_n = \{a : a \in A_n \text{ for some } n \geq 1\}$$

is countable.

Proof. We define

$$\begin{aligned} B_1 &= A_1 \\ B_{n+1} &= A_{n+1} \setminus \bigcup_{k=1}^{n} A_k \quad \forall n \geq 1 \end{aligned}$$

By construction,

$$\begin{cases} B_n \cap B_m = \emptyset, \, \forall n \neq m \\ \bigcup_{n\geq 1} B_n = \bigcup_{n\geq 1} A_n \end{cases}$$

Note that each B_n is at most countable.

Let $I = \{n \in \mathbb{N} : B_n \neq \emptyset\}$. Then $\bigcup_{n\geq 1} B_n = \bigcup_{n\in I} B_n$. For $n \in I$, let $f_n : B_n \to I_n$ bijection where I_n is an at most countable subset of $\mathbb{N}$.

In particular, $f_1 : B_1 \to \mathbb{N}$ bijective $\implies f_1^{-1} : \mathbb{N} \to B_1$ bijective. To show $\bigcup_{n\in I} B_n$ is countable, we will use the Schröder – Bernstein theorem.

Let $g : \mathbb{N} \to \bigcup_{n \in I} B_n$, $g(n) = f_1^{-1}(n) \in B_1 \subseteq \bigcup_{n\in I} B_n$ is injective.

Let $h : \bigcup_{n\in I} B_n \to \mathbb{N} \times \mathbb{N}$ defined as follows: if $b \in \bigcup_{n \in I} B_n \implies \exists n \in I$ s.t. $b \in B_n$.

Define $h(b) = (n, f_n(b))$. Note that h is injective. Indeed, if $h(b_1) = h(b_2)$ then $(n_1, f_{n_1}(b_1)) = (n_2, f_{n_2}(b_2))$

$$\implies \left\{ \begin{cases} n_1 = n_2 \\ f_{n_1}(b_1) = f_{n_2}(b_2) \end{cases}, f_{n_1} \text{ is injective} \right\} \implies b_1 = b_2$$

Recall there exists a bijection $\phi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. So $\phi \circ h : \bigcup_{n\in I} B_n \to \mathbb{N}$ is injective. By Schröder – Bernstein, $\bigcup_{n \in I} B_n = \bigcup_{n \geq 1} A_n \sim \mathbb{N}$. □

Proposition 20.2

Let $\{A_n\}_{n\geq 1}$ be a sequence of sets such that for each $n \geq 1$, A_n has at least two elements. Then $\prod_{n\geq 1} A_n = \left\{ \{a_n\}_{n \geq 1} : a_n \in A_n \, \forall n \geq 1 \right\}$ is uncountable.

Proof. We argue by contradiction. Assume that $\prod_{n \geq 1} A_n$ is countable. Thus we may enumerate the elements of $\prod_{n\geq 1} A_n$:

$$\begin{aligned} a_1 &= (a_{11}, a_{12}, a_{13}, \dots) \\ a_2 &= (a_{21}, a_{22}, a_{23}, \dots) \\ &\dots \\ a_n &= (a_{n1}, a_{n2}, a_{n3}, \dots) \\ &\dots \end{aligned}$$

Let $x = \{x_n\}_{n \geq 1} \in \prod_{n\geq 1} A_n$ such that $x_n \in A_n \setminus \{a_{nn}\}$. Then $x \neq a_n \, \forall n \geq 1$ since $x_n \neq a_{nn}$. This gives a contradiction. □

Remark 20.3. The same argument using binary expansion shows that the set $(0, 1)$ is uncountable.

§21 Lec 21: Feb 24, 2021

§21.1 Countable vs. Uncountable Sets (Cont'd)

Proposition 21.1

Let $\{A_n\}_{n\geq 1}$ be a sequence of sets s.t. $\forall n \geq 1$, the set A_n has at least two elements. Then $\prod_{n\geq 1} A_n$ is uncountable.

Remark 21.2.

1. The Cantor diagonal argument can be used to show that the set $(0, 1)$ is uncountable (using binary expansion).
2. We can identify

$$\begin{aligned}\left\{\{a_n\}_{n\geq 1} : a_n \in \{0,1\}\ \forall n \geq 1\right\} &= \{f : \mathbb{N} \to \{0,1\} : f \text{ function}\} \\ &= \{0,1\} \times \{0,1\} \times \dots \\ &= \{0,1\}^{\mathbb{N}}\end{aligned}$$

By the proposition, this set is uncountable. We say it has cardinality $2^{\aleph_0}$.

Theorem 21.3

Let A be any set. Then there exists no bijection between A and the power set of A, $\mathcal{P}(A) = \{B : B \subseteq A\}$.

Proof. If $A = \emptyset$ then $\mathcal{P}(A) = \{\emptyset\}$. So the cardinality of A is 0, but the cardinality of $\mathcal{P}(A)$ is 1. Thus A is not equipotent with $\mathcal{P}(A)$.

Assume $A \neq \emptyset$. We argue by contradiction. Assume that there exists $f : A \to \mathcal{P}(A)$ a bijection.

Let $B = \{a \in A : a \notin f(a)\} \subseteq A$. f is surjective $\implies \exists b \in A$ s.t. $f(b) = B$

We distinguish two cases:

Case 1: $b \in B = f(b) \implies b \notin B$ – Contradiction.

Case 2: $b \notin B = f(b) \implies b \in B$ – Contradiction.

So A is not equipotent to $\mathcal{P}(A)$ □

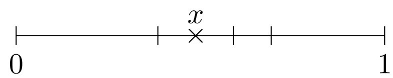
Theorem 21.4

The set $[0, 1)$ has cardinality $2^{\aleph_0}$.

Proof. We write $x \in [0, 1)$ using the binary expansion.

$$\begin{aligned}x &= 0.x_1x_2x_3\dots \quad \text{with } x_n \in \{0,1\}\ \forall n \geq 1 \\ &= \frac{x_1}{2} + \frac{x_2}{2^2} + \frac{x_3}{2^3} + \dots = \sum_{n\geq 1} \frac{x_n}{2^n}\end{aligned}$$

with the convention that no expansion ends in all ones.



$x = 0.010\dots$

E.g.

$$\begin{aligned}
x &= 0.x_1x_2x_3\dots x_n0111\dots \\
&= \frac{x_1}{2} + \dots + \frac{x_n}{2^n} + \underbrace{\frac{1}{2^{n+2}} + \frac{1}{2^{n+3}} + \dots}_{=\frac{1}{2^{n+1}}} \\
&= \frac{x_1}{2} + \dots + \frac{x_n}{2^n} + \frac{1}{2^{n+1}} = 0.x_1x_2\dots x_n1000\dots
\end{aligned}$$

Note that we can identify $[0,1)$ with

$$\begin{aligned}
\mathcal{F} &= \{f : \mathbb{N} \to \{0,1\} : \forall n \in \mathbb{N}\, \exists m > n \text{ s.t. } f(m) = 0\} \\
&\subseteq \{f : \mathbb{N} \to \{0,1\} : f \text{ function}\}
\end{aligned}$$

In particular, we have an injection $\phi : [0,1) \to \{f : \mathbb{N} \to \{0,1\}\}$. To prove the theorem, by Schröder – Bernstein, it suffices to construct an injective function $\psi : \{f : \mathbb{N} \to \{0,1\}\} \to [0,1)$. For $f : \mathbb{N} \to \{0,1\}$ we define

$$\begin{aligned}
\psi(f) &= 0.0f(1)0f(2)0f(3)\dots \\
&= \frac{f(1)}{2^2} + \frac{f(2)}{2^4} + \frac{f(3)}{2^6} + \dots \\
&= \sum_{n\geq 1} \frac{f(n)}{2^{2n}}
\end{aligned}$$

Let's show ψ is an injective. Let $f_1, f_2 : \mathbb{N} \to \{0,1\}$ s.t. $f_1 \neq f_2$. Let $n_0 = \min\{n : f_1(n) \neq f_2(n)\}$. Say, $f_1(n_0) = 1$ and $f_2(n_0) = 0$.

$$\begin{aligned}
\psi(f_1) - \psi(f_2) = \sum_{n\geq 1} \frac{f_1(n)}{2^{2n}} - \sum_{n\geq 1} \frac{f_2(n)}{2^{2n}} &= \frac{f_1(n_0) - f_2(n_0)}{2^{2n_0}} + \sum_{n\geq n_0+1} \frac{f_1(n) - f_2(n)}{2^{2n}} \\
&\geq \frac{1}{2^{2n_0}} - \sum_{n\geq n_0+1} \frac{1}{2^{2n}} \\
&= \frac{1}{2^{2n_0}} - \frac{1}{2^{2(n_0+1)}} \cdot \frac{1}{1-\frac{1}{2}} \\
&= \frac{1}{2^{2n_0+1}} > 0
\end{aligned}$$

$\implies \psi(f_1) > \psi(f_2)$
So ψ is injective.
By Schröder – Bernstein, $[0,1) \sim \{f : \mathbb{N} \to \{0,1\}\}$ and so it has cardinality $2^{\aleph_0}$. $\square$

§21.2 Metric Spaces

Definition 21.5 (Metric Space) — Let X be a non-empty set. A metric on X is a map $d : X \times X \to \mathbb{R}$ such that

1. $d(x, y) \geq 0 \, \forall x, y \in X$
2. $d(x, y) = 0 \iff x = y$
3. $d(x, y) = d(y, x) \, \forall x, y \in X$
4. $d(x, y) \leq d(x, z) + d(z, y) \, \forall x, y, z \in X$

Then we say (X, d) is a metric space.

Example 21.6

1. $X = \mathbb{R}$, $d(x, y) = |x - y|$ is a metric.
2. $X = \mathbb{R}^n$, $d_2(x, y) = \sqrt{\sum_{k=1}^n |x_k - y_k|^2}$ is a metric.
3. X is any non-empty set. The discrete metric
$$d(x, y) = \begin{cases} 1, \, x \neq y \\ 0, \, x = y \end{cases}$$
4. Let (X, d) be a metric space. Then $\tilde{d} : X \times X \to \mathbb{R}$, $\tilde{d}(x, y) = \frac{d(x,y)}{1+d(x,y)}$ is a metric.
Let's see it satisfies (4). Fix $x, y, z \in X$. As d is a metric,
$$d(x, y) \leq d(x, z) + d(z, y)$$
Note $a \mapsto \frac{a}{1+a} = 1 - \frac{1}{1+a}$ is increasing on $[0, \infty)$. Then,
$$\begin{aligned} \tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)} \leq \frac{d(x, z) + d(z, y)}{1 + d(x, z) + d(z, y)} &\leq \frac{d(x, z)}{1 + d(x, z)} + \frac{d(z, y)}{1 + d(z, y)} \\ &= \tilde{d}(x, z) + \tilde{d}(z, y) \end{aligned}$$

Definition 21.7 ((Un)Bounded Metric Space) — We say that a metric space (X, d) is bounded if $\exists M > 0$ s.t. $d(x, y) \leq M \, \forall x, y \in X$. If (X, d) is not bounded, we say that it is bounded.

Remark 21.8. If (X, d) is an unbounded metric space then $(X, \tilde{d})$ is a bounded metric space where $\tilde{d}(x, y) = \frac{d(x,y)}{1+d(x,y)}$.

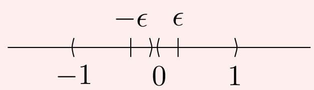
Definition 21.9 (Distance Between Sets) — Let (X, d) be a metric space and let $A, B \subseteq X$. The distance between A and B is

$$d(A, B) = \inf \{d(x, y) : x \in A, y \in B\}$$

Caution: This does not define a metric on subset of X.
In fact, $d(A, B) = 0$ does not even imply $A \cap B \neq \emptyset$.

Example 21.10

$(X, d) = (\mathbb{R}, |\cdot|)$, $A = (0, 1), B = (-1, 0)$, $d(A, b) = 0$ but $A \cap B = \emptyset$



Definition 21.11 (Distance Between Point and Set) — Let (X, d) be a metric space, $A \subseteq X, x \in X$. The distance from x to A is

$$d(x, A) = \inf \{d(x, a) : a \in A\}$$

Again, $d(x, A) = 0 \;\not\Longrightarrow\; x \in A$

§22 Lec 22: Feb 26, 2021

§22.1 Hölder & Minkowski Inequalities

Proposition 22.1 (Hölder's Inequality)

Fix $1 \le p \le \infty$ and let q denote the dual of p, that is, $\frac{1}{p} + \frac{1}{q} = 1$. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and let $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Then

$$\sum_{k=1}^{n} |x_k y_k| \le \left(\sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |y_k|^q \right)^{\frac{1}{q}}$$

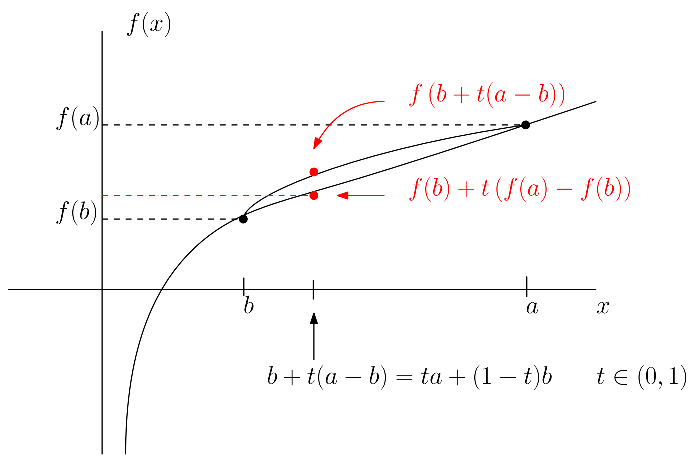
with the convention that if $p = \infty$, then $(\sum_{k=1}^{n} |x_k|^p)^{\frac{1}{p}} = sup_{1 \le k \le n} |x_k|$

Remark 22.2. If $p = 2 (\implies q = 2)$ we call this the Cauchy – Schwarz inequality.

Proof. If $p = 1$, then $q = \infty$.

$$\sum_{k=1}^{n} |x_k y_k| \le \sum_{k=1}^{n} |x_k| \cdot \sup_{1 \le l \le n} |y_l| \le \left(\sum_{k=1}^{n} |x_k| \right) \cdot \sup_{1 \le l \le n} |y_l|$$

If $p = \infty \implies (q = 1)$ a similar argument yields the claim.
Assume $1 < p < \infty$. We will use the fact that $f : (0, \infty) \to \mathbb{R}$, $f(x) = \log(x)$ is a concave function.



$$\begin{aligned} tf(a) + (1 - t)f(b) &\le f(ta + (1 - t)b) \quad \forall (a, b) \in (0, \infty)\, \forall t \in (0, 1) \\ t \log(a) + (1 - t) \log(b) &\le \log(ta + (1 - t)b) \\ \log(a^t) + \log(b^{1-t}) &\le \log(ta + (1 - t)b) \\ \log\left(a^t b^{1-t}\right) &\le \log(ta + (1 - t)b) \\ a^t b^{1-t} &\le ta + (1 - t)b \end{aligned}$$

We will apply this inequality with $a = \frac{|x_k|^p}{\sum_{l=1}^n |x_l|^p}$, $b = \frac{|y_k|^q}{\sum_{l=1}^n |y_l|^q}$.

$$t = \frac{1}{p} \implies 1 - t = 1 - \frac{1}{p} = \frac{1}{q}$$

We get

$$\frac{|x_k|}{\left(\sum_{l=1}^n |x_l|^p\right)^{\frac{1}{p}}} \cdot \frac{|y_k|}{\left(\sum_{l=1}^n |y_l|^q\right)^{\frac{1}{q}}} \le \frac{1}{p} \frac{|x_k|^p}{\sum_{l=1}^n |x_l|^p} + \frac{1}{q} \frac{|y_k|^q}{\sum_{l=1}^n |y_l|^q}$$

Sum over $1 \le k \le n$

$$\sum_{k=1}^n \frac{|x_k| \cdot |y_k|}{\left(\sum_{l=1}^n |x_l|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{l=1}^n |y_l|^q\right)^{\frac{1}{q}}} \le \frac{1}{p} \sum_{k=1}^n \frac{|x_k|^p}{\sum_{l=1}^n |x_l|^p} + \frac{1}{q} \sum_{k=1}^n \frac{|y_k|^q}{\sum_{l=1}^n |y_l|^q} \quad = \frac{1}{p} + \frac{1}{q} = 1$$

$\implies \sum_{k=1}^n |x_k y_k| \le \left(\sum_{l=1}^n |x_l|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{l=1}^n |y_l|^q\right)^{\frac{1}{q}}$. □

Corollary 22.3 (Minkowski's Inequality)
Fix $1 \le p \le \infty$ and let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Then

$$\left(\sum_{k=1}^n |x_k + y_k|^p\right)^{\frac{1}{p}} \le \left(\sum_{k=1}^n |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^n |y_k|^p\right)^{\frac{1}{p}}$$

Proof. If $p = 1$, this follows from the triangle inequality:

$$\text{LHS} = \sum_{k=1}^n |x_k + y_k| \le \sum_{k=1}^n |x_k| + |y_k| = \text{RHS}$$

If $p = \infty$,

$$\text{LHS} = sup_{1 \le k \le n} |x_k + y_k| \le \sup_{1 \le k \le n} |x_k| + \sup_{1 \le k \le n} |y_k| = \text{RHS}$$

Assume $1 < p < \infty$.

$$\begin{aligned}
\sum_{k=1}^n |x_k + y_k|^p &= \sum_{k=1}^n |x_k + y_k| \, |x_k + y_k|^{p-1} \\
&\le \sum_{k=1}^n (|x_k| + |y_k|) \, |x_k + y_k|^{p-1} \\
&= \sum_{k=1}^n |x_k| \cdot |x_k + y_k|^{p-1} + \sum_{k=1}^n |y_k| \, |x_k + y_k|^{p-1} \\
\text{(Hölder)} &\le \left(\sum_{k=1}^n |x_k|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{k=1}^n |x_k + y_k|^{(p-1)\cdot q}\right)^{\frac{1}{q}} \\
&\quad + \left(\sum_{k=1}^n |y_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^n |x_k + y_k|^{(p-1)q}\right)^{\frac{1}{q}}
\end{aligned}$$

$\frac{1}{p} + \frac{1}{q} = 1 \implies \frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p} \implies q = \frac{p}{p-1}$
Get

$$\sum_{k=1}^{n} |x_k + y_k|^p \leq \left[\left(\sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |y_k|^p \right)^{\frac{1}{p}} \right] \cdot \left(\sum_{k=1}^{n} |x_k + y_k|^p \right)^{1-\frac{1}{p}}$$

$$\implies \left(\sum_{k=1}^{n} |x_k + y_k|^p \right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |y_k|^p \right)^{\frac{1}{p}}$$

□

Corollary 22.4
For $1 \leq p < \infty$ let $d_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,

$$d_p(x, y) = \left(\sum_{k=1}^{n} |x_k - y_k|^p \right)^{\frac{1}{p}}$$

For $p = \infty$ let $d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,

$$d_\infty(x, y) = \sup_{1 \leq k \leq n} |x_k - y_k|$$

The d_p is a metric on $\mathbb{R}^n$ $\forall 1 \leq p \leq \infty$.

Proof. The triangle inequality follows from Minkowski's inequality. □

Remark 22.5. The Hölder and Minkowski inequalities generalize to sequences. For example, say $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ are sequences of real numbers such that $\left(\sum_{n\geq 1} |x_n|^p\right)^{\frac{1}{p}} < \infty$ and $\left(\sum_{n\geq 1} |y_n|^q\right)^{\frac{1}{q}} < \infty$. Then for each fixed $N \geq 1$,

$$\underbrace{\sum_{n=1}^{N} |x_k y_k|}_{\text{increasing seq indexed by N}} \leq \left(\sum_{n=1}^{N} |x_n|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{n=1}^{N} |y_n|^q \right)^{\frac{1}{q}} \leq \left(\sum_{n\geq 1} |x_n|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{n\geq 1} |y_n|^q \right)^{\frac{1}{q}} < \infty$$

So

$$\sum_{n\geq 1} |x_k y_k| \leq \left(\sum_{n\geq 1} |x_n|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{n\geq 1} |y_n|^q \right)^{\frac{1}{q}}$$

A similar argument gives Minkowski for sequences.

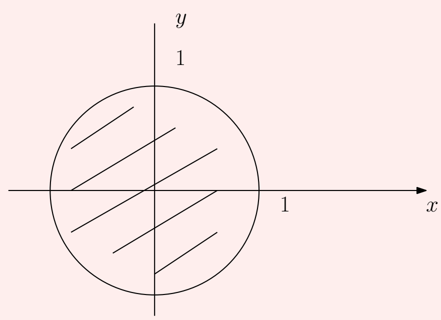
§22.2 Open Sets

Definition 22.6 (Ball/Neighborhood of a Point) — Let (X, d) be a metric space. A neighborhood of a point $a \in X$ is

$$B_r(a) = \{x \in X : d(a, x) < r\} \text{ for some } r > 0$$

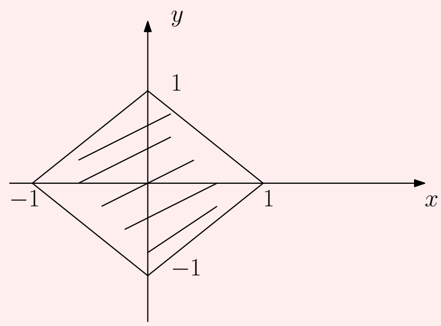
Example 22.7 1. $(\mathbb{R}^2, d_2)$

$$\begin{aligned} B_1(0) &= \left\{(x,y) \in \mathbb{R}^2 : d_2\left((x,y),(0,0)\right) < 1\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\right\} \end{aligned}$$



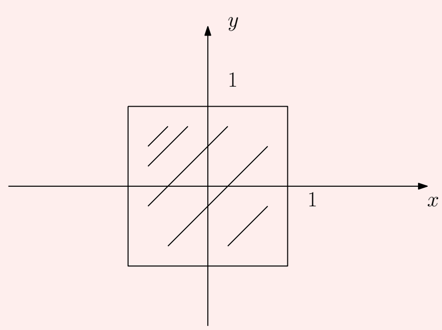
2. $(\mathbb{R}^2, d_1)$

$$B_1(0) = \left\{(x,y) \in \mathbb{R}^2 : |x| + |y| < 1\right\}$$



3. $(\mathbb{R}^2, d_\infty)$

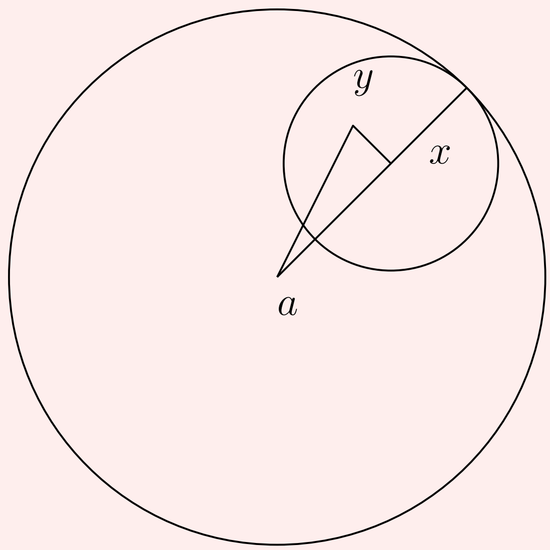
$$B_1(0) = \left\{(x,y) \in \mathbb{R}^2 : \max\left\{|x|, |y|\right\} < 1\right\}$$



Definition 22.8 (Interior Point) — Let (X, d) be a metric space and let $\emptyset \neq A \subseteq X$.
We say that a point $a \in X$ is an interior point of A if $\exists r > 0$ s.t. $B_r(a) \subseteq A$.
The set of all interior points of A is denoted $\mathring{A}$ and is called the interior of A.
We say that A is open if $A = \mathring{A}$.

Example 22.9

1. $\emptyset, X$ are open sets.
2. $B_r(a)$ is an open set $\forall a \in X$, $\forall r > 0$.
   Indeed, let $x \in B_r(a) \implies d(x, a) < r \implies \rho = r - d(x, a) > 0$



Claim 22.1. $B_\rho(x) \subseteq B_r(a)$ and so $x \in \widehat{\mathring{B_r(a)}}$

Proof. Let $y \in B_\rho(x) \implies d(x, y) < \rho$

$$d(y, a) \leq d(y, x) + d(x, a) < \rho + d(x, a) = r \implies y \in B_r(a)$$

□

Remark 22.10. $\mathring{A} \subseteq A$. To prove A is open, it suffices to show $A \subseteq \mathring{A}$.

§23 Lec 23: Mar 1, 2021

§23.1 Open Sets (Cont'd)

Proposition 23.1

Let (X, d) be a metric space and let $A, B \subseteq X$. Then

1. If $A \subseteq B$ then $\mathring{A} \subseteq \mathring{B}$
2. $\mathring{A} \cup \mathring{B} \subseteq \overset{\circ}{\widehat{A \cup B}}$
3. $\mathring{A} \cap \mathring{B} = \overset{\circ}{\widehat{A \cap B}}$
4. $\mathring{\mathring{A}} = \mathring{A}$. In particular, $\mathring{A}$ is an open set.
5. $\mathring{A}$ is the largest open set contained in A.
6. A finite intersection of open sets is an open set.
7. An arbitrary union of open sets is an open set.

Remark 23.2. An arbitrary intersection of open sets need not be open. E.g.

$$\bigcap_{n \geq 1} \underbrace{\left(-\frac{1}{n}, \frac{1}{n}\right)}_{B_{\frac{1}{n}}(0) \in (\mathbb{R}, |\cdot|)} = \{0\}$$

Note that $\{0\}$ is not an open set because it does not contain any neighborhood of 0.

Proof. (Of the proposition):

1. If $\mathring{A} = \emptyset$ this is clear. Assume $\mathring{A} \neq \emptyset$. Let $a \in \mathring{A} \implies \exists r > 0$ s.t.
$$\left.\begin{matrix} B_r(a) \subseteq A \\ A \subseteq B \end{matrix}\right\} \implies B_r(a) \subseteq B$$
So $a \in \mathring{B}$.
2. Consider:
$$\left.\begin{matrix} A \subseteq A \cup B \overset{(1)}{\implies} \mathring{A} \subseteq \overset{\circ}{\widehat{A \cup B}} \\ B \subseteq A \cup B \overset{(1)}{\implies} \mathring{B} \subseteq \overset{\circ}{\widehat{A \cup B}} \end{matrix}\right\} \implies \mathring{A} \cup \mathring{B} \subseteq \overset{\circ}{\widehat{A \cup B}}$$
3. Consider:
$$\left.\begin{matrix} A \cap B \subseteq A \overset{(1)}{\implies} \overset{\circ}{\widehat{A \cap B}} \subseteq \mathring{A} \\ A \cap B \subseteq B \overset{(2)}{\implies} \overset{\circ}{\widehat{A \cap B}} \subseteq \mathring{B} \end{matrix}\right\} \implies \overset{\circ}{\widehat{A \cap B}} \subseteq \mathring{A} \cap \mathring{B}$$
Now let $x \in \mathring{A} \cap \mathring{B}$
$$\implies \begin{cases} \exists r_1 > 0 \text{ s.t. } B_{r_1}(x) \subseteq A \\ \exists r_2 > 0 \text{ s.t. } B_{r_2}(x) \subseteq B \end{cases}$$

Let $r = \min\{r_1, r_2\} > 0$. Then $B_r(x) \subseteq B_{r_1}(x) \cap B_{r_2}(x) \subseteq A \cap B \implies x \in \overset{\circ}{\widehat{A \cap B}}$. So $\mathring{A} \cap \mathring{B} \subseteq \overset{\circ}{\widehat{A \cap B}}$

4. $\mathring{A} \subseteq A \overset{(1)}{\implies} \mathring{\mathring{A}} \subseteq \mathring{A}$. Let $x \in \mathring{A} \implies \exists r > 0$ s.t. $B_r(x) \subseteq A \overset{(1)}{\implies} B_r(x) = \overset{\circ}{\widehat{B_r(x)}} \subseteq \mathring{A} \implies x \in \mathring{\mathring{A}}$. So $\mathring{A} \subseteq \mathring{\mathring{A}}$.

5. By (4), $\mathring{A}$ is an open set contained in A. Let $B \subseteq A$ be an open set. Then by (1), $B = \mathring{B} \subseteq \mathring{A}$.

6. Using (3) and (4) we see that if $A = \mathring{A}$ and $B = \mathring{B}$ then $A \cap B = \overset{\circ}{\widehat{A \cap B}}$ is an open set.

   A simple inductive argument yields the claim.

7. Let $\{A_i\}_{i \in I}$ be a family of open sets. Let's show

$$\overset{\circ}{\widehat{\bigcup_{i \in I} A_i}} = \bigcup_{i \in I} A_i$$

   Let $x \in \bigcup_{i \in I} A_i \implies \exists i_0 \in I$ s.t.

$$\left.\begin{aligned} x &\in A_{i_0} \\ A_{i_0} &= \mathring{A}_{i_0} \end{aligned}\right\} \implies \exists r > 0 \text{ s.t. } B_r(x) \subseteq A_{i_0}$$

   So $B_r(x) \subseteq \bigcup_{i \in I} A_i \implies x \in \overset{\circ}{\widehat{\bigcup_{i \in I} A_i}}$. Thus, $\bigcup_{i \in I} A_i \subseteq \overset{\circ}{\widehat{\bigcup_{i \in I} A_i}}$. □

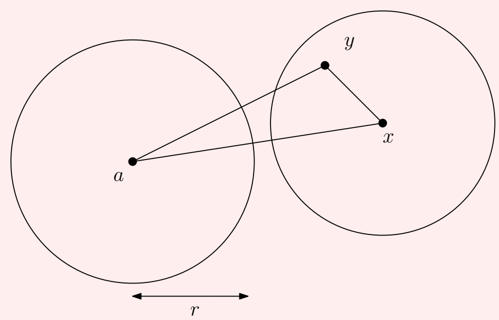
§23.2 Closed Sets

Definition 23.3 (Closed Set) — Let (X, d) be a metric space. A set $A \subseteq X$ is closed if ${}^cA$ is open.

Example 23.4

1. ϕ, X are closed.
2. If $a \in X$, $r > 0$, then ${}^cB_r(a) = \{x \in X : d(a,x) \geq r\}$ is a closed set.
3. If $a \in X$, $r > 0$, then $K_r(a) = \{x \in X : d(a,x) \leq r\}$ is a closed set.

   Let's show ${}^cK_r(a) = \{x \in X : d(a,x) > r\}$ is open. Let $x \in {}^cK_r(a) \implies d(a,x) > r$ and let $\rho = d(a,x) - r > 0$

   Claim 23.1. $B_\rho(x) \subseteq {}^cK_r(a)$

   Let $y \in B_\rho(x) \implies d(x,y) < \rho$. By the triangle inequality,

   $$d(a,y) \geq d(a,x) - d(x,y) > d(a,x) - \rho = r \implies y \in {}^cK_r(a)$$

   So $B_\rho(x) \subseteq K_r(a) \implies x \in \overset{\circ}{\widehat{{}^cK_r(a)}}$. Thus, ${}^cK_r(a)$ is an open set.
4. There are sets that are neither open nor closed. E.g. $(0,1]$ is not open and is not closed.

Definition 23.5 (Adherent Point) — Let (X,d) be a metric space and let $A \subseteq X$. A point $a \in X$ is an <u>adherent point for A</u> if

$$\forall r > 0 \text{ we have } B_r(a) \cap A \neq \emptyset$$

The set of all adherent points of A is denoted $\overline{A}$ and is called the <u>closure of A</u>.

Definition 23.6 (Isolated Point) — An adherent point a of A is called <u>isolated</u> if

$$\exists r > 0 \text{ s.t. } B_r(a) \cap A = \{a\} \quad (a \in A)$$

If every point in A is an isolated point of A then A is called an isolated set.

Definition 23.7 (Accumulation Point) — An adherent point a of A that is not isolated is called an accumulation point for A. The set of accumulation points of A is denoted A'. Note that

$$a \in A' \iff \forall r > 0 \quad B_r(a) \cap A \setminus \{a\} \neq \emptyset$$

Example 23.8

$(\mathbb{R}, |\cdot|)$, $\quad A = \left\{\frac{1}{n} : n \geq 1\right\}$. A is isolated. Indeed $B_{\frac{1}{n(n+1)}}\left(\frac{1}{n}\right) \cap A = \left\{\frac{1}{n}\right\}$.
$A' = \{0\}$ since $\forall r > 0\, B_r(0) = (-r, r)$ intersects $A \setminus \{0\} = A$.

Remark 23.9.

1. $A \subseteq \overline{A}$
2. $\overline{A} = A' \cup A$

Proposition 23.10

Let (X, d) be a metric space and let $A, B \subseteq X$. Then

1. ${}^c(\overline{A}) = \widehat{{}^cA}^{\circ}$
2. ${}^c(\mathring{A}) = \overline{{}^cA}$
3. A is closed set $\iff$ $A = \overline{A}$
4. If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$
5. $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$
6. $\overline{A} \cup \overline{B} = \overline{A \cup B}$
7. $\overline{\overline{A}} = \overline{A}$. In particular, $\overline{A}$ is a closed set.
8. $\overline{A}$ is the smallest closed set containing A.
9. A finite union of closed sets is a closed set.
10. An arbitrary intersection of closed sets is a closed set.

Remark 23.11. An arbitrary union of closed sets need not be a closed set. E.g.

$$\bigcup_{n \geq 1} \underbrace{\left[\frac{1}{n}, 1\right]}_{\text{closed}} = \underbrace{(0, 1]}_{\text{not closed}}$$

Proof. (of the proposition)

1. Consider

$$\begin{aligned} x \in {}^c(\overline{A}) \iff x \notin \overline{A} &\iff \exists r > 0 \text{ s.t. } B_r(x) \cap A = \emptyset \\ &\iff \exists r > 0 \text{ s.t. } B_r(x) \subseteq {}^cA \\ &\iff x \in \overset{\circ}{\widehat{{}^cA}} \end{aligned}$$

2. Apply (1) to ${}^cA$.

3. A is closed $\iff$ ${}^cA$ is open

$$\begin{aligned} &\iff {}^cA = \overset{\circ}{\widehat{{}^cA}} \\ &\overset{(1)}{\iff} {}^cA = {}^c(\overline{A}) \\ &\iff A = \overline{A} \end{aligned}$$

We continue in the next lecture. $\square$

§24 Lec 24: Mar 3, 2021

§24.1 Closed Sets (Cont'd)

Proposition 24.1

Let (X, d) be a metric space and let $A, B \subseteq X$. Then

1. ${}^c(\overline{A}) = \overset{\circ}{\widehat{{}^cA}}$
2. ${}^c(\mathring{A}) = \overline{{}^cA}$
3. A is closed set $\iff A = \overline{A}$
4. If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$
5. $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$
6. $\overline{A} \cup \overline{B} = \overline{A \cup B}$
7. $\overline{\overline{A}} = \overline{A}$. In particular, $\overline{A}$ is a closed set.
8. $\overline{A}$ is the smallest closed set containing A.
9. A finite union of closed sets is a closed set.
10. An arbitrary intersection of closed sets is a closed set.

Proof. (Cont'd from last lecture)

4. If $\overline{A} = \emptyset$ then clearly $\overline{A} \subseteq \overline{B}$. Assume $\overline{A} \neq \emptyset$. Let $a \in \overline{A} \implies \forall r > 0$,

$$\left.\begin{aligned} &B_r(a) \cap A \neq \emptyset \\ &A \subseteq B \end{aligned}\right\} \implies B_r(a) \cap B \neq \emptyset \, \forall r > 0$$
$$\implies a \in \overline{B}$$

So $\overline{A} \subseteq \overline{B}$

5. Have:

$$\left.\begin{aligned} A \cap B \subseteq A &\overset{(4)}{\implies} \overline{A \cap B} \subseteq \overline{A} \\ A \cap B \subseteq B &\overset{(4)}{\implies} \overline{A \cap B} \subseteq \overline{B} \end{aligned}\right\} \implies \overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$$

6. Have

$$\begin{aligned} {}^c(\overline{A \cup B}) \overset{(1)}{=} \overset{\circ}{\widehat{{}^c(A \cup B)}} = \overset{\circ}{\widehat{{}^cA \cap {}^cB}} = \overset{\circ}{\widehat{{}^cA}} \cap \overset{\circ}{\widehat{{}^cB}} &\overset{(1)}{=} {}^c(\overline{A}) \cap {}^c(\overline{B}) \\ &= {}^c(\overline{A} \cup \overline{B}) \end{aligned}$$

$$\implies \overline{A \cup B} = \overline{A} \cup \overline{B}$$

7. Clearly, $A \subseteq \overline{A} \overset{(4)}{\implies} \overline{A} \subseteq \overline{\overline{A}}$. Want to show $\overline{\overline{A}} \subseteq \overline{A}$. Let $a \in \overline{\overline{A}}$. Want to prove that $\forall r > 0\, B_r(a) \cap A \neq \emptyset$.

   Fix $r > 0$. As $a \in \overline{\overline{A}} \implies B_r(a) \cap \overline{A} \neq \emptyset$. Let $x \in B_r(a) \cap \overline{A}$
   $$x \in \overline{A} \implies \forall \rho > 0,\, B_\rho(x) \cap A \neq \emptyset$$
   Choose $\rho = r - d(a, x) > 0$. Then
   $$\left.\begin{matrix} B_\rho(x) \subseteq B_r(a) \\ B_\rho(x) \cap A \neq \emptyset \end{matrix}\right\} \implies B_r(a) \cap A \neq \emptyset$$
   So $a \in \overline{A}$.

8. Note $\overline{A}$ is a closed subset containing A. Let B be a closed set containing A.
   $$A \subseteq B \overset{(4)}{\implies} \overline{A} \subseteq \overline{B} \overset{(3)}{=} B$$

9. Let $\{A_n\}_{n=1}^N$ be a closed sets. Then ${}^cA_n$ is an open set $\forall 1 \leq n \leq N$. Then $\bigcap_{n=1}^N {}^cA_n$ is an open set. Now $\bigcap_{n=1}^N {}^cA_n = {}^c\left(\bigcup_{n=1}^N A_n\right)$ open $\implies \bigcup_{n=1}^N A_n$ closed.

10. Let $\{A_i\}_{i \in I}$ be a family of closed sets. Then ${}^cA_i$ is open $\forall i \in I$
   $$\implies \bigcup_{i \in I} {}^cA_i = {}^c\left(\bigcap_{i \in I} A_i\right) \text{ is open}$$
   $$\implies \bigcap_{i \in I} A_i \text{ is closed}$$
   ☐

§24.2 Subspaces of Metric Spaces

Definition 24.2 (Subspace of Metric Space) — Let (X, d) be a metric space and let $\emptyset \neq Y \subseteq X$. Then $d_1 : Y \times Y \to \mathbb{R}$, $d_1(x, y) = d(x, y)\, \forall x, y \in Y$ is a metric on Y and is called the induced metric on Y. (Y, d_1) is called a subspace of (X, d).

Proposition 24.3

Let (X, d) be a metric space and let $\emptyset \neq Y \subseteq X$ equipped with the induced metric d_1.

1. A set $D \subseteq Y$ is open in (Y, d_1) if and only if there exists $O \subseteq X$ open in (X, d) s.t. $D = O \cap Y$.
2. A set $F \subseteq Y$ is closed in (Y, d_1) if and only if there exists $C \subseteq X$ closed in (X, d) s.t. $F = C \cap Y$.

Proof. 1. " $\implies$ " Let $D \subseteq Y$ be open in (Y, d_1). Then $\forall a \in D\, \exists r_a > 0$ s.t. $B^y_{r_a}(a) = \{y \in Y : d(a, y) < r_a\} \subseteq D$. Note $B^y_{r_a}(a) = B^x_{r_a}(a) \cap Y$. So
$$D = \bigcup_{a \in D} B^y_{r_a}(a) = \bigcup_{a \in D} \left[B^x_{r_a}(a) \cap Y\right] = \underbrace{\left(\bigcup_{a \in D} B^x_{r_a}(a)\right)}_{\text{open in } (X,d)} \cap Y$$

" $\Longleftarrow$ " Assume that $D = O \cap Y$ for O open in (X, d). Let $a \in D \subseteq O \implies \exists r > 0$ s.t. $B_r^x(a) \subseteq O$

$$\implies B_r^y(a) = B_r^x(a) \cap Y \subseteq O \cap Y = D \implies a \text{ is an interior point of } D \text{ in the } (Y, d_1)$$

So D is open in (Y, d_1).

2. $F \subseteq Y$ is closed in $(Y, d_1) \iff Y \setminus F$ is open in $(Y, d_1) \overset{(1)}{\iff} \exists O$ open set in (X, d) s.t. $Y \setminus F = O \cap Y$. But

$$\begin{aligned} F &= Y \setminus (Y \setminus F) = Y \setminus (O \cap Y) = Y \cap {}^c(O \cap Y) = Y \cap ({}^cO \cup {}^cY) \\ &= (Y \cap {}^cO) \cup \underbrace{(Y \cap {}^cY)}_{=\emptyset} = Y \cap \underbrace{{}^cO}_{\text{closed in } (X,d)} \end{aligned}$$

□

Example 24.4 1. $[0, 1)$ is not an open set in $(\mathbb{R}, |\cdot|)$, but it is open in $([0, 2), |\cdot|)$. Say $[0, 1) = (-1, 1) \cap [0, 2)$.

2. $(0, 1]$ is not a closed set in $(\mathbb{R}, |\cdot|)$, but it is closed in $((0, 2), |\cdot|)$. Say $(0, 1] = [-1, 1] \cap (0, 2)$.

Proposition 24.5

Let (X, d) be a metric space and let $\emptyset \neq Y \subseteq X$ equipped with the induced metric. The followings are equivalent:

1. Any $A \subseteq Y$ that is open (closed) in Y is also open(closed) in X.
2. Y is open(closed) in X.

Proof. 1) $\implies$ 2) Take $A = Y$.
2) $\implies$ 1) Assume Y is open in X. Let $A \subseteq Y$ be open in $Y \implies \exists O$ open in X s.t. $A = \underbrace{O}_{\text{open in X}} \cap \underbrace{Y}_{\text{open in X}}$ open in X. □

Proposition 24.6

Let (X, d) be a metric space and let $\emptyset \neq Y \subseteq X$ equipped with the induced metric. For a set $A \subseteq Y$,

$$\overline{A}^Y = \overline{A}^X \cap Y$$

Proof. Have:

$$\begin{aligned} a \in \overline{A}^Y &\iff \forall r > 0 \quad B_r^y(a) \cap A \neq \emptyset \\ &\iff \forall r > 0 \quad B_r^x(a) \cap \underbrace{Y \cap A}_{=A} \neq \emptyset \\ &\iff a \in \overline{A}^X \cap Y \end{aligned}$$

□

§24.3 Complete Metric Spaces

Definition 24.7 (Sequential Limit) — Let (X, d) be a metric space and let $\{x_n\}_{n\geq 1} \subseteq X$. We say $\{x_n\}_{n\geq 1}$ converges to a point $x \in X$ if

$$\forall \epsilon > 0 \quad \exists n_\epsilon \in \mathbb{N} \text{ s.t. } d(x_n, x) < \epsilon \quad \forall n \geq n_\epsilon$$

Then x is called the limit of $\{x_n\}_{n\geq 1}$ and we write $x = \lim_{n\to\infty} x_n$ or $x_n \xrightarrow[n\to\infty]{d} x$.

Exercise 24.1. The limit of a convergent sequence is unique.

Exercise 24.2. A sequence of $\{x_n\}_{n\geq 1}$ converges to $x \in X$ if and only if every subsequences of $\{x_n\}_{n\geq 1}$ converges to x.

> **Remark 24.8.** If $x_n \xrightarrow[n\to\infty]{d} x$ and $y_n \xrightarrow[n\to\infty]{d} y$, then $d(x_n, y_n) \xrightarrow[n\to\infty]{} d(x, y)$.

Indeed,

$$\begin{aligned} |d(x_n, y_n) - d(x, y)| &\leq |d(x_n, y_n) - d(x_n, y)| + |d(x_n, y) - d(x, y)| \\ &\leq d(y_n, y) + d(x_n, x) \xrightarrow[n\to\infty]{} 0 \end{aligned}$$

Definition 24.9 (Cauchy Sequence (MS)) — Let (X, d) be a metric space. We say that $\{x_n\}_{n\geq 1} \subseteq X$ is Cauchy if

$$\forall \epsilon > 0 \quad \exists n_\epsilon \in \mathbb{N} \text{ s.t. } d(x_n, x_m) < \epsilon \quad \forall n, m \geq n_\epsilon$$

Exercise 24.3. Every convergent sequence is Cauchy.

Caution: Not every Cauchy sequence is convergent in an arbitrary metric space.

Example 24.10

1. $(X, d) = ((0, 1), |\cdot|)$, $x_n = \frac{1}{n}\ \forall n \geq 2$ is Cauchy but does not converge in X.
2. $(X, d) = (\mathbb{Q}, |\cdot|)$, $x_1 = 3$, $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}\ \forall n \geq 1$. Then $\{x_n\}_{n\geq 1}$ is Cauchy but does not converge in X.

Definition 24.11 (Complete Metric Space) — A metric space (X, d) is complete if every Cauchy sequence in X converges in X.

Example 24.12

$(\mathbb{R}, |\cdot|)$ is a complete metric space.

Exercise 24.4. Show that a Cauchy sequence with a convergent subsequence converges.

§25 Lec 25: Mar 5, 2021

§25.1 Complete Metric Spaces (Cont'd)

Lemma 25.1

Let (X, d) be a metric space and let $\emptyset \neq F \subseteq X$. The following equivalent:

1. $a \in \overline{F}$
2. There exists $\{a_n\}_{n\geq 1} \subseteq F$ s.t. $a_n \xrightarrow[n\to\infty]{d} a$

Proof. 1) $\implies$ 2) Assume $a \in \overline{F}$. Then

$$\forall r > 0, \quad B_r(a) \cap F \neq \emptyset$$

For $n \geq 1$, take $r = \frac{1}{n}$. Then $B_{\frac{1}{n}}(a) \cap F \neq \emptyset$. Let $a_n \in B_{\frac{1}{n}}(a) \cap F$. Consider $\{a_n\}_{n\geq 1} \subseteq F$. We have $\forall n \geq 1$,

$$d(a_n, a) < \frac{1}{n} \xrightarrow[n\to\infty]{} 0 \implies a_n \xrightarrow[n\to\infty]{d} a$$

2) $\implies$ 1) Assume $\exists \{a_n\}_{n\geq 1} \subseteq F$ s.t. $a_n \xrightarrow[n\to\infty]{d} a$. Fix $r > 0$. Then $\exists n_r \in \mathbb{N}$ s.t. $d(a_n, a) < r \, \forall n \geq n_r$. In particular, $\forall n \geq n_r$, $a_n \in B_r(a) \cap F \implies B_r(a) \cap F \neq \emptyset$. As r was arbitrary, we get $a \in \overline{F}$. □

Theorem 25.2

Let (X, d) be a metric space. The following are equivalent:

1. (X, d) is a complete metric space.
2. For every sequence $\{F_n\}_{n\geq 1}$ of non-empty closed subset of X, that is nested (that is, $F_{n+1} \subseteq F_n \, \forall n \geq 1$), and satisfies $\delta(F_n) \xrightarrow[n\to\infty]{} 0$, we have $\bigcap_{n\geq 1} F_n = \{a\}$ for some $a \in X$.

Proof. 1) $\implies$ 2) Assume (X, d) is complete. As $F_n \neq \emptyset \, \forall n \geq 1$, $\exists a_n \in F_n$.

Claim 25.1. $\{a_n\}_{n\geq 1}$ is Cauchy.

Let $\epsilon > 0$. As $\delta(F_n) \xrightarrow[n\to\infty]{} 0$, $\exists n_\epsilon \in \mathbb{N}$ s.t. $\delta(F_n) < \epsilon \, \forall n \geq n_\epsilon$. Let $m, n \geq n_\epsilon$. Since $\{F_n\}_{n\geq 1}$ is nested, $F_n \subseteq F_{n_\epsilon}$, $F_m \subseteq F_{n_\epsilon}$. So

$$d(a_n, a_m) \leq \delta(F_{n_\epsilon}) < \epsilon$$

So this proves the claim.

As (X, d) is complete, $\exists a \in X$ s.t. $a_n \xrightarrow[n\to\infty]{d} a$. For $\forall n \geq 1$, $\{a_m\}_{m\geq n} \subseteq F_n \implies a \in \overline{F_n} = F_n$. So $a \in \bigcap_{n\geq 1} F_n$.

It remains to show a is the only point in $\bigcap_{n\geq 1} F_n$. Assume, toward a contradiction, that $\exists y \neq a$ s.t. $y \in \bigcap_{n\geq 1} F_n$. Then $y \in F_n \, \forall n \geq 1 \implies d(y,a) \leq \delta(F_n) \underset{n\to\infty}{\longrightarrow} 0 \implies y = a$ – Contradiction!
2) $\implies$ 1) Want to show (X,d) is complete. Let $\{x_n\}_{n\geq 1} \subseteq X$ be a Cauchy sequence. To prove that $\{x_n\}_{n\geq 1}$ converges in X, it suffices to show that $\{x_n\}_{n\geq 1}$ admits a subsequence that converges in X.

$\{x_n\}_{n\geq 1}$ is Cauchy $\implies \exists n_1 \in \mathbb{N}$ s.t. $d(x_n, x_m) < \frac{1}{2^2} \, \forall n,m \geq n_1$. Let $k_1 = n_1$ and select x_{k_1}.

$\{x_n\}_{n\geq 1}$ is Cauchy $\implies \exists n_2 \in \mathbb{N}$ s.t. $d(x_n, x_m) < \frac{1}{2^3}, \forall n,m \geq n_2$. Let $k_2 = \max\{n_2, k_1+1\}$ and select x_{k_2}.

Proceeding inductively, we find a strictly increasing sequence $\{k_n\}_{n\geq 1} \subseteq \mathbb{N}$ s.t.

$$d(x_l, x_m) < \frac{1}{2^{n+1}} \quad \forall l,m \geq k_n$$

For $n \geq 1$, let $F_n = K_{\frac{1}{2^n}}(X_{k_n}) = \left\{x \in X : d(x, x_{k_n}) < \frac{1}{2^n}\right\}$. Note $\emptyset \neq F_n = \overline{F_n}$ and $\delta(F_n) \leq 2 \cdot \frac{1}{2^n} \underset{n\to\infty}{\longrightarrow} 0$.

Claim 25.2. $F_{n+1} \subseteq F_n \quad \forall n \geq 1$.

Let $y \in F_{n+1} \implies d(y, x_{k_{n+1}} \leq \frac{1}{2^{n+1}}$. By the triangle inequality,

$$d(y, x_{k_n}) \leq d(y, x_{k_{n+1}}) + d\left(x_{k_{n+1}}, x_{k_n}\right) \leq \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} = \frac{1}{2^n}$$

So $y \in F_n$. As $y \in F_{n+1}$ was arbitrary, we get $F_{n+1} \subseteq F_n$.
By hypothesis, $\bigcap_{n\geq 1} F_n = \{a\}$ for some $a \in X$. As $\forall n \geq 1$, $a \in F_n$ we have $d(a, x_{k_n}) \leq \frac{1}{2^n} \underset{n\to\infty}{\longrightarrow} 0$

$$\left.\begin{aligned} & x_{k_n} \xrightarrow[n\to\infty]{d} a \\ & \{x_n\}_{n\geq 1} \text{ is Cauchy} \end{aligned}\right\} \implies x_n \xrightarrow[n\to\infty]{d} a$$

□

§25.2 Examples of Complete Metric Spaces

Recall $(\mathbb{R}, |\cdot|)$ is a complete metric space.

Lemma 25.3

Assume (A, d_1) and (B, d_2) are complete metric spaces. We define $d : (A \times B) \times (A \times B) \to \mathbb{R}$ via

$$d\left((a_1, b_1), (a_2, b_2)\right) = \sqrt{d_1^2(a_1, a_2) + d_2^2(b_1, b_2)}$$

Then $(A \times B, d)$ is a complete metric space.

Exercise 25.1. Show that d is a metric on $A \times B$.

Proof. Let's show $A \times B$ is complete. Let $\{(a_n, b_n)\}_{n\geq 1} \subseteq A \times B$ be a Cauchy sequence.

Fix $\epsilon > 0$, $\exists n_\epsilon \in \mathbb{N}$ s.t. $d\left((a_n, b_n), (a_m, b_m)\right) < \epsilon \, \forall n, m \geq n_\epsilon$.

$$\implies \sqrt{d_1^2(a_n, a_m) + d_2^2(b_n, b_m)} < \epsilon \quad \forall n, m \geq n_\epsilon$$
$$\implies \begin{cases} d_1(a_n, a_m) < \epsilon & \forall n, m \geq n_\epsilon \\ d_2(b_n, b_m) < \epsilon & \forall n, m \geq n_\epsilon \end{cases}$$

So

$$\begin{cases} \{a_n\}_{n\geq 1} \text{ is Cauchy sequence in } A \\ \{b_n\}_{n\geq 1} \text{ is Cauchy sequence in } B \end{cases}$$

As A and B are complete metric spaces, $\exists a \in A$, $\exists b \in B$ s.t. $a_n \xrightarrow[n\to\infty]{d_1} a$ and $b_n \xrightarrow[n\to\infty]{d_2} b$.

Claim 25.3. $(a_n, b_n) \xrightarrow[n\to\infty]{d} (a, b)$.

Indeed,

$$\begin{aligned} d\left((a_n, b_n), (a, b)\right) &= \sqrt{d_1^2(a_n, a) + d_2^2(b_n, b)} \\ &\leq d_1(a_n, a) + d_2(b_n, b) \xrightarrow[n\to\infty]{} 0 \end{aligned}$$

$\implies (a_n, b_n) \xrightarrow[n\to\infty]{d} (a, b)$. □

Corollary 25.4

For $n \geq 2$, $(\mathbb{R}^n, d_2)$ is a complete metric space.

Proof. Use induction. □ Exc!

Exercise 25.2. Show that for all $n \geq 2$, $(\mathbb{R}^n, d_p)$ is a complete metric space $\forall 1 \leq p \leq \infty$.

We define

$$l^2 = \left\{ \{x_n\}_{n\geq 1} \subseteq \mathbb{R} : \sum_{n\geq 1} |x_n|^2 < \infty \right\}$$

We define a metric on l^2 as follows: for $x = \{x_n\}_{n\geq 1}$ and $y = \{y_n\}_{n\geq 1} \in l^2$,

$$d_2(x, y) = \sqrt{\sum_{n\geq 1} |x_n - y_n|^2}$$

The fact this is a metric follows from Minkowski's inequality.

Claim 25.4. $\left(l^2, d_2\right)$ is a complete metric space.

Proof. Let $\left\{x^{(d)}\right\}_{k\geq 1}$ be a Cauchy sequence in l^2.

$$\begin{aligned} x^{(1)} &= \left\{x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \dots\right\} \\ x^{(2)} &= \left\{x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \dots\right\} \\ &\dots \\ x^{(n)} &= \left\{x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \dots\right\} \end{aligned}$$

We continue in the next lecture. □

§26 Lec 26: Mar 8, 2021

§26.1 Examples of Complete Metric Spaces (Cont'd)

Recall

$$l^2 = \left\{ \{x_n\}_{n\geq 1} \subseteq \mathbb{R} : \sum_{n\geq 1} |x_n|^2 < \infty \right\}$$

We define a metric $d_2 : l^2 \times l^2 \to \mathbb{R}$ via

$$d_2 \left(\{x_n\}_{n\geq 1}, \{y_n\}_{n\geq 1} \right) = \sqrt{\sum_{n\geq 1} |x_n - y_n|^2}$$

Then (l^2, d_2) is a complete metric space. To see this, let $\{x^{(k)}\}_{k\geq 1}$ be a Cauchy sequence in l^2. Then $\forall \epsilon > 0 \, \exists k_\epsilon \in \mathbb{N}$ s.t. $d_2 \left(x^{(k)}, x^{(l)} \right) < \epsilon \, \forall k, l \geq k_\epsilon$. So

$$d_2 \left(x^{(k)}, x^{(l)} \right) = \sqrt{\sum_{n\geq 1} \left| x_n^{(k)} - x_n^{(l)} \right|^2} < \epsilon \quad \forall k, l \geq k_\epsilon$$

$$\implies \sum_{n\geq 1} \left| x_n^{(k)} - x_n^{(l)} \right|^2 < \epsilon^2 \quad k, l \geq k_\epsilon$$

$$\implies \forall n \geq 1 \text{ we have } \left| x_n^{(k)} - x_n^{(l)} \right| < \epsilon \quad \forall k, l \geq k_\epsilon$$

So $\forall n \geq 1$, the sequence $\left\{ x_n^{(k)} \right\}_{k\geq 1}$ is Cauchy in $(\mathbb{R}, |\cdot|)$. As $(\mathbb{R}, |\cdot|)$ is complete, $\exists x_n \in \mathbb{R}$ s.t. $x_n^{(k)} \xrightarrow[k\to\infty]{\mathbb{R}} x_n$.

Let $x = \{x_n\}_{n\geq 1}$

Claim 26.1. $x \in l^2$ and $x^{(k)} \xrightarrow[k\to\infty]{l^2} x$.

Note $d_2 \left(x^{(k)}, x \right) = \sqrt{\sum_{n\geq 1} \left| x_n^{(k)} - x_n \right|^2}$. While $\left| x_n^{(k)} - x_n \right| \xrightarrow[k\to\infty]{} 0 \, \forall n \geq 1$, the limit theorems do not apply to yield

$$\sum_{n\geq 1} \left| x_n^{(k)} - x_n \right|^2 \xrightarrow[k\to\infty]{} 0$$

Instead, we argue as follows:

Fix $\epsilon > 0$. As $\{x^{(k)}\}_{k\geq 1}$ is Cauchy in l^2, $\exists k_\epsilon \in \mathbb{N}$ s.t. $d_2 \left(x^{(k)}, x^{(l)} \right) < \epsilon \, \forall k, l \geq k_\epsilon$. In particular, $\sum_{n\geq 1} \left| x_n^{(k)} - x_n^{(l)} \right|^2 < \epsilon^2 \, \forall k, l \geq k_\epsilon$. So for each fixed $N \in \mathbb{N}$ we have

$$\sum_{n=1}^{N} \left| x_n^{(k)} - x_n^{(l)} \right|^2 < \epsilon^2 \quad \forall k, l \geq k_\epsilon$$

Note $\lim_{l\to\infty}\left|x_n^{(k)}-x_n^{(l)}\right| = \left|x_n^{(k)}-x_n\right|$ $\forall n \ge 1$, $\forall k \ge k_\epsilon$. By the limit theorems,

$$\lim_{l\to\infty}\sum_{n=1}^{N}\left|x_n^{(k)}-x_n^{(l)}\right|^2 \le \epsilon^2 \quad \forall k \ge k_\epsilon$$
$$\implies \sum_{n=1}^{N}\left|x_n^{(k)}-x_n\right|^2 \le \epsilon^2 \quad \forall k \ge k_\epsilon$$

Note $\left\{\sum_{n=1}^{N}\left|x_n^{(k)}-x_n\right|^2\right\}_{N\ge 1}$ is an increasing sequence bounded above by ϵ^2. So

$$\sum_{n\ge 1}\left|x_n^{(k)}-x_n\right|^2 \le \epsilon^2 \quad \forall k \ge k_\epsilon$$

$\implies d_2\left(x^{(k)},x\right) \le \epsilon \quad \forall k \ge k_\epsilon$.
So $x^{(k)} \xrightarrow[k\to\infty]{l^2} x$. Finally, $x \in l^2 \iff d_2(x,0) < \infty$. But

$$d_2(x,0) \le \underbrace{d_2(x,x^{(k)})}_{\le\epsilon\,\forall k\ge k_\epsilon} + \underbrace{d_2\left(x^{(k)},0\right)}_{<\infty \text{ since } x^{(k)}\in l^2} < \infty$$

Exercise 26.1. 1. Fix $1 \le p < \infty$ and let

$$l^p = \left\{\{x_n\}_{n\ge 1} \subseteq \mathbb{R} : \sum_{n\ge 1}|x_n|^p < \infty\right\}$$

We define $d_p : l^p \times l^p \to \mathbb{R}$ via

$$d_p\left(\{x_n\}_{n\ge 1}, \{y_n\}_{n\ge 1}\right) = \left(\sum_{n\ge 1}|x_n-y_n|^p\right)^{\frac{1}{p}}$$

Then (l^p, d_p) is a complete metric space.

2. Define $l^\infty = \left\{\{x_n\}_{n\ge 1} \subseteq \mathbb{R} : \sup_{n\ge 1}|x_n| < \infty\right\}$. We define $d_\infty : l^\infty \times l^\infty \to \mathbb{R}$ via

$$d_\infty\left(\{x_n\}_{n\ge 1}, \{y_n\}_{n\ge 1}\right) = \sup_{n\ge 1}|x_n-y_n|$$

Show (l^∞, d_∞) is a complete metric space.

§26.2 Connected Sets

Definition 26.1 (Separated Set) — Let (X,d) be a metric space and let $A, B \subseteq X$. We say that A and B are separated if

$$\overline{A} \cap B = \emptyset \text{ and } A \cap \overline{B} = \emptyset$$

Remark 26.2. Separated sets are disjoint: $A \cap B \subseteq \overline{A} \cap B = \emptyset$. But disjoint sets need not be separated. For example,

$$(X, d) = (\mathbb{R}, |\cdot|), \quad A = (-1, 0), \quad B = [0, 1)$$

Then $A \cap B = \emptyset$ but $\overline{A} \cap B = \{0\} \neq \emptyset$ so A, B are not separated.

Remark 26.3. If A and B are separated and $A_1 \subseteq A$ and $B_1 \subseteq B$, then A_1 and B_1 are separated.

Lemma 26.4

Let (X, d) be a metric space and let $A, B \subseteq X$. If $d(A, B) > 0$ then A and B are separated.

Proof. Assume, towards a contradiction that A and B are not separated. Then, $\overline{A} \cap B \neq \emptyset$ or $A \cap \overline{B} \neq \emptyset$. Say $\overline{A} \cap B \neq \emptyset$. Let $a \in \overline{A} \cap B$.

$$\left.\begin{array}{l} a \in B \\ a \in \overline{A} \implies d(a, A) = 0 \end{array}\right\} \implies d(A, B) = 0 \quad \text{– Contradiction!}$$ □

Remark 26.5. Two sets A and B can be separated even if $d(A, B) = 0$.

Example 26.6

$A = (0, 1)$ and $B = (1, 2)$ separated, but $d(A, B) = 0$.

Proposition 26.7

1. Two closed sets A and B are separated $\iff A \cap B = \emptyset$.
2. Two open sets A and B are separated $\iff A \cap B = \emptyset$.

Proof. Two separated sets are disjoint. So we only have to prove " $\Longleftarrow$ " in both cases.

1. Assume $A \cap B = \emptyset$. Then A closed $\implies A = \overline{A}$ and so $\overline{A} \cap B = A \cap B = \emptyset$. Similarly, B closed $\implies \overline{B} = B$ and so $\overline{B} \cap A = B \cap A = \emptyset$. So A and B are separated.

2. Assume $A \cap B = \emptyset \implies A \subseteq {}^cB$ where ${}^cB$ is closed since B is open.

$$\implies \overline{A} \subseteq \overline{{}^cB} = {}^cB \implies \overline{A} \cap B = \emptyset$$

A similar argument shows that $\overline{B} \cap A = \emptyset$ and so A and B are separated. □

Proposition 26.8
1. If an open set D is the union of two separated sets A and B, then A and B are both open.
2. If a closed set F is the union of two separated sets A and B, then A and B are both closed.

Proof.
1. If $A = \emptyset$, then since $D = A \cup B$ we have $B = D$ and so A and B are open.

   Assume $A \neq \emptyset$. We want to show A is open $\iff A = \mathring{A}$. Let $a \in A \subseteq D$ and D open $\implies \exists r > 0$ s.t. $B_{r_1}(a) \subseteq D$. A and B are separated $\implies A \cap \overline{B} = \emptyset$. So $a \in A \subseteq {}^c\left(\overline{B}\right) = \mathring{\widehat{{}^cB}}$

   $$\implies \exists r_2 > 0 \text{ s.t. } B_{r_2}(a) \subseteq {}^cB$$

   Let $r = \min\{r_1, r_2\}$. Then

   $$B_r(a) \subseteq D \cap {}^cB = (A \cup B) \cap {}^cB = A$$

   so $a \in \mathring{A}$.

   This shows A is open. A similar argument shows B is open.

2. Let's show A is closed $\iff \overline{A} = A$.

   $$\left.\begin{array}{l} A \subseteq F \\ F \text{ closed} \iff F = \overline{F} \end{array}\right\} \implies \overline{A} \subseteq \overline{F} = F$$

   So $\overline{A} = \overline{A} \cap F = \overline{A} \cap (A \cup B) = \underbrace{(\overline{A} \cap A)}_{=A} \cup (\underbrace{\overline{A} \cap B}_{=\emptyset}) = A$.

   Similarly, one can show that $\overline{B} = B$ and so B is closed. □

§27 Lec 27: Mar 10, 2021

§27.1 Connected Sets (Cont'd)

Definition 27.1 (Connected/Disconnected Set) — Let (X, d) be a metric space and let $A \subseteq X$. We say that A is disconnected if it can be written as the union of two non-empty separated sets, that is,

$$\exists B, C \subseteq X \text{ s.t. } B \neq \emptyset,\ C \neq \emptyset,\ \overline{B} \cap C = \overline{C} \cap B = \emptyset,\ A = B \cup C$$

We say that A is connected if it's not disconnected.

Lemma 27.2

Let (X, d) be a metric space and let $Y \subseteq X$ be equipped with the induced metric d_1. Then Y is connected in (Y, d_1) if and only if Y is connected in (X, d).

Proof. " $\Longrightarrow$ " Assume that Y is connected in (Y, d_1). We argue by contradiction. Assume that Y is not connected in (X, d). Then $\exists A, B \subseteq X$, $A \neq \emptyset$, $B \neq \emptyset$, $\overline{A}^X \cap B = \overline{B}^X \cap A = \emptyset$, $Y = A \cap B$.

Claim 27.1. A, B are separated in (Y, d_1). Then $Y = A \cup B$ is disconnected in (Y, d_1). Contradiction!

Indeed,

$$\overline{A}^Y \cap B = \left(\overline{A}^X \cap Y\right) \cap B = \overline{A}^X \cap \underbrace{Y \cap B}_{=B} = \overline{A}^X \cap B = \emptyset$$

$$\overline{B}^Y \cap A = \left(\overline{B}^X \cap Y\right) \cap A = \overline{B}^X \cap \underbrace{(Y \cap A)}_{=A} = \overline{B}^X \cap A = \emptyset$$

So A and B are separated in (Y, d_1).
" $\Longleftarrow$ " Assume Y is connected in (X, d). We argue by contradiction. Assume that Y is disconnected in (Y, d_1). So $\exists A, B \subseteq Y$, $A \neq \emptyset$, $B \neq \emptyset$, $\overline{A}^Y \cap B = \overline{B}^Y \cap A = \emptyset$, $Y = A \cup B$.

Claim 27.2. A, B are separated in (X, d). Then $Y = A \cup B$ is disconnected in (X, d). Contradiction!

Indeed,

$$\overline{A}^X \cap B = \overline{A}^X \cap (Y \cap B) = \left(\overline{A}^X \cap Y\right) \cap B = \overline{A}^Y \cap B = \emptyset$$

$$\overline{B}^X \cap A = \overline{B}^X \cap (Y \cap A) = \left(\overline{B}^X \cap Y\right) \cap A = \overline{B}^Y \cap A = \emptyset$$

So A and B are separated in (X, d). $\square$

Proposition 27.3

Let (X, d) be a metric space. Then X is connected if and only if the only subsets of X that are both open and closed are $\emptyset$ and X.

Proof. " $\implies$ " Assume X is connected. We argue by contradiction. Assume $\exists \emptyset \neq A \subsetneq X$ s.t. A is both open and closed. Let

$$\begin{aligned} &B = X \setminus A \neq \emptyset \text{ (since } A \neq X) \\ &B \neq X \text{ (since } A \neq \emptyset) \\ &B \text{ is open (since A is closed)} \\ &B \text{ is closed (since A is open)} \end{aligned}$$

As A and B are closed and $A \cap B = A \cap (X \setminus A) = \emptyset$, we have that A and B are separated. So

$$\left.\begin{aligned} &X = A \cup (X \setminus A) = A \cup B \\ &A \neq \emptyset,\ B \neq \emptyset, A \text{ and } B \text{ are separated} \end{aligned}\right\} \implies X \text{ is disconnected – Contradiction!}$$

" $\impliedby$ " Assume that the only subsets of X that are both open and closed in (X, d) are $\emptyset$ and X. We argue by contradiction. Assume that X is disconnected. Then $\exists A, B \subseteq X$ s.t. $A \neq \emptyset$, $B \neq \emptyset$, $\overline{A} \cap B = \overline{B} \cap A = \emptyset$, $X = A \cup B$. As X is open (and closed) we get that A and B are both open (and closed).

$$\left.\begin{aligned} &A \text{ and } B \text{ are both open and closed} \\ &A \neq \emptyset,\ B \neq \emptyset \end{aligned}\right\} \implies A = B = X$$

But then $\overline{A} \cap B = \overline{X} \cap X = X \cap X = X \neq \emptyset$. Contradiction! □

Corollary 27.4

Let (X, d) be a metric space and let $\emptyset \neq A \subseteq X$. The following are equivalent:

1. A is disconnected.
2. $A \subseteq D_1 \cup D_2$ with D_1, D_2 open in (X, d), $A \cap D_1 \neq \emptyset$, $A \cap D_2 \neq \emptyset$, $A \cap D_1 \cap D_2 = \emptyset$.
3. $A \subseteq F_1 \cup F_2$ with F_1, F_2 closed in (X, d), $A \cap F_1 \neq \emptyset$, $A \cap F_2 \neq \emptyset$, $A \cap F_1 \cap F_2 = \emptyset$.

Proof. We'll show 1) $\implies$ 3) $\implies$ 2) $\implies$ 1).
1) $\implies$ 3) Assume A is disconnected. By the Proposition 27.3, there exists $\emptyset \neq B \subsetneq A$ s.t. B is both open and closed in A. Let $C = A \setminus B$. Then $C \neq \emptyset$, $C \neq A$, and C is both open and closed in A.

$$\begin{aligned} B \text{ closed in } A &\implies \exists F_1 \subseteq X \text{ closed in } (X, d) \text{ s.t. } B = A \cap F_1 \neq \emptyset \\ C \text{ closed in } A &\implies \exists F_2 \subseteq X \text{ closed in } (X, d) \text{ s.t. } C = A \cap F_2 \neq \emptyset \end{aligned}$$

Note that $A \cap F_1 \cap F_2 = (A \cap F_1) \cap (A \cap F_2) = B \cap C = B \cap (A \setminus B) = \emptyset$.

3) $\implies$ 2) Assume $A \subseteq F_1 \cup F_2$, F_1, F_2 closed in (X, d), $A \cap F_1 \neq \emptyset$, $A \cap F_2 \neq \emptyset$, $A \cap F_1 \cap F_2 = \emptyset$. Define $D_1 = {}^cF_1$ open in (X, d) and $D_2 = {}^cF_2$ open in (X, d).

$$A \subseteq F_1 \cup F_2 = {}^cD_1 \cup {}^cD_2 = {}^c(D_1 \cap D_2) \implies A \cap (D_1 \cap D_2) = \emptyset$$
$$\emptyset = A \cap F_1 \cap F_2 = A \cap ({}^cD_1 \cap {}^cD_2) = A \cap {}^c(D_1 \cup D_2) \implies A \subseteq D_1 \cup D_2$$

Let's show $A \cap D_1 \neq \emptyset$. We argue by contradiction. Assume $A \cap D_1 = \emptyset \implies A \subseteq {}^cD_1 = F_1$. But the $\emptyset = \underbrace{A \cap F_1}_{=A} \cap F_2 = A \cap F_2 \neq \emptyset$. Contradiction! This shows $A \cap D_1 \neq \emptyset$. A similar argument gives $A \cap D_2 \neq \emptyset$.

2) $\implies$ 1) Assume $A \subseteq D_1 \cup D_2$, D_1, D_2 open in (X, d), $A \cap D_1 \neq \emptyset$, $A \cap D_2 \neq \emptyset$, $A \cap D_1 \cap D_2 = \emptyset$. Let

$$B = A \cap D_1 \neq \emptyset \text{ open in } A \text{ (since } D_1 \text{ is open in } X)$$
$$C = A \cap D_2 \neq \emptyset \text{ open in } A \text{ (since } D_2 \text{ is open in } X)$$
$$B \cap C = (A \cap D_1) \cap (A \cap D_2) = A \cap D_1 \cap D_2 = \emptyset$$

So

$$\left.\begin{array}{l} B \text{ and } C \text{ are separated in } A \\ A \subseteq D_1 \cup D_2 \implies A = (D_1 \cup D_2) \cap A = (D_1 \cap A) \cup (D_2 \cap A) = B \cup C \\ B \neq \emptyset, \quad C \neq \emptyset \end{array}\right\} \implies$$

$\implies$ A is disconnected in $A \implies A$ is disconnected in X. □

Proposition 27.5

Let (X, d) be a metric space and let $A \subseteq X$ be disconnected. Let $F_1, F_2 \subseteq X$ be closed in (X, d) s.t. $A \subseteq F_1 \cup F_2$, $A \cap F_1 \neq \emptyset$, $A \cap F_2 \neq \emptyset$, $A \cap F_1 \cap F_2 = \emptyset$. If $B \subseteq A$ is connected then $B \subseteq F_1$ or $B \subseteq F_2$.

§28 Lec 28: Mar 12, 2021

§28.1 Connected Sets (Cont'd)

Proposition 28.1

Let (X, d) be a metric space and let $A \subseteq X$ be disconnected. Let F_1, F_2 be closed in X s.t. $A \subseteq F_1 \cup F_2$, $A \cap F_1 \neq \emptyset$, $A \cap F_2 \neq \emptyset$, $A \cap F_1 \cap F_2 = \emptyset$. Let $B \subseteq A$ be connected. Then $B \subseteq F_1$ or $B \subseteq F_2$.

Proof. We argue by contradiction. Assume $B \nsubseteq F_1$ and $B \nsubseteq F_2$.

$$\left.\begin{array}{l} \left.\begin{array}{l} B \subseteq A \subseteq F_1 \cup F_2 \\ B \nsubseteq F_1 \end{array}\right\} \implies B \cap F_2 \neq \emptyset \\ \left.\begin{array}{l} B \subseteq F_1 \cup F_2 \\ B \nsubseteq F_2 \end{array}\right\} \implies B \cap F_1 \neq \emptyset \\ B \cap F_1 \cap F_2 \subseteq A \cap F_1 \cap F_2 = \emptyset \\ B \subseteq F_1 \cup F_2 \end{array}\right\} \implies B \text{ is disconnected -- Contradiction!}$$

□

Remark 28.2. One can replace the closed sets (in X) F_1 and F_2 by open sets (in X) D_1 and D_2 and the same conclusion holds.

Proposition 28.3

Let (X, d) be a metric space and let $A \subseteq X$ be connected. Then if $A \subseteq B \subseteq A^{-X}$, then B is connected.

Proof. We argue by contradiction. Assume B is disconnected. Then $\exists F_1, F_2 \subseteq X$, closed in X, s.t.

$$\begin{cases} B \subseteq F_1 \cup F_2 \\ B \cap F_1 \neq \emptyset \\ B \cap F_2 \neq \emptyset \\ B \cap F_1 \cap F_2 = \emptyset \end{cases}$$

and

$$\left.\begin{array}{l} A \subseteq B \subseteq F_1 \cup F_2 \\ A \text{ connected} \end{array}\right\} \implies A \subseteq F_1 \text{ or } A \subseteq F_2$$

Say $A \subseteq F_1 \implies B \subseteq A^{-X} \subseteq F_1^{-X} = F_1$. Then $\emptyset = \underbrace{B \cap F_1}_{=B} \cap F_2 = B \cap F_2 \neq \emptyset$.

Contradiction! □

§28.2 Connected Subsets

Proposition 28.4

Let (X,d) be a metric space and let $\{A_i\}_{i\in I}$ be a family of connected subsets of X. Assume that each two of these sets are not separated, that is, $\forall i,j\in I$, $i\neq j$, we have $\overline{A_i}\cap A_j\neq\emptyset$ or $A_i\cap\overline{A_j}\neq\emptyset$. Then $\bigcup_{i\in I}A_i$ is connected.

Proof. We argue by contradiction. Assume $\bigcup_{i\in I}A_i$ is disconnected $\implies \exists B,C$ non-empty separated sets s.t.

$$\bigcup_{i\in I}A_i = B\cup C$$

Fix $i\in I$. Then $A_i\subseteq B\cup C$.

$$\left.\begin{aligned}&\implies A_i=(B\cup C)\cap A_i=(B\cap A_i)\cup(C\cap A_i)\\ &B,C \text{ separated} \implies B\cap A_i,\, C\cap A_i \text{ separated}\\ &A_i \text{ is connected}\end{aligned}\right\} \implies \begin{cases}B\cap A_i=\emptyset\\ \quad\text{or}\\ C\cap A_i=\emptyset\end{cases}$$

Then

$$\left.\begin{aligned}A_i&\subseteq B\cup C\\ A_i\cap B&=\emptyset\end{aligned}\right\}\implies A_i\subseteq C$$
$$\left.\begin{aligned}A_i&\subseteq B\cup C\\ A_i\cap C&=\emptyset\end{aligned}\right\}\implies A_i\subseteq B$$

So for each $i\in I$, the set A_i satisfies $A_i\subseteq B$ or $A_i\subseteq C$. As $\bigcup_{i\in I}A_i=B\cup C \implies \exists i,j\in I$ s.t. $A_i\cap B\neq\emptyset$ and $A_j\cap C\neq\emptyset$

$$\left.\begin{aligned}&\implies A_i\subseteq B \text{ and } A_j\subseteq C\\ &B \text{ and } C \text{ are separated}\end{aligned}\right\}\implies A_i,\ A_j \text{ are separated – Contradiction!}$$ □

Corollary 28.5

Let (X,d) be a metric space and let $\{A_i\}_{i\in I}$ be connected subsets of X. Assume $\forall i\neq j$ we have $A_i\cap A_j\neq\emptyset$. Then $\bigcup_{i\in I}A_i$ is connected.

Proposition 28.6

$\mathbb{R}$ is connected.

Proof. Assume, towards a contradiction, that $\mathbb{R}$ is disconnected. Then $\exists A,B$ non-empty subsets of $\mathbb{R}$, both open and closed in $\mathbb{R}$, disjoint, such that $\mathbb{R}\subseteq A\cup B$.

$$A\neq\emptyset \implies \exists a_1\in A$$
$$B\neq\emptyset \implies \exists b_1\in B$$

Let $\alpha_1 = \frac{a_1+b_1}{2} \in \mathbb{R} = A \cup B \implies \alpha_1 \in A$ or $\alpha_1 \in B$. If

$$\alpha_1 \in A \text{ let } (a_2, b_2) := (\alpha_1, b_1)$$
$$\alpha_1 \in B \text{ let } (a_2, b_2) := (a_1, \alpha_1)$$

Let $\alpha_2 = \frac{a_2+b_2}{2} \in \mathbb{R} = A \cup B \implies \alpha_2 \in A$ or $\alpha_2 \in B$. If

$$\alpha_2 \in A \text{ let } (a_3, b_3) := (\alpha_2, b_2)$$
$$\alpha_2 \in B \text{ let } (a_3, b_3) := (a_2, \alpha_2)$$

Continuing this process, we find

- an increasing sequence $\{a_n\}_{n\geq 1} \subseteq A$ bounded above by b_1.
- a decreasing sequence $\{b_n\}_{n\geq 1} \subseteq B$ bounded below by a_1.

So $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ converge in $\mathbb{R}$. Let

$$a = \lim_{n\to\infty} a_n \in \overline{A} = A$$
$$b = \lim_{n\to\infty} b_n \in \overline{B} = B$$

Note that by contradiction, $b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2}\ \forall n \geq 1$

$$\implies |b_{n+1} - a_{n+1}| = \frac{|b_n - a_n|}{2} = \ldots = \frac{|b_1 - a_1|}{2^n} \underset{n\to\infty}{\longrightarrow} 0$$
$$\implies |b - a| = 0 \implies a = b \in A \cap B = \emptyset$$

Contradiction! □

Proposition 28.7

The only non-empty connected subsets of $\mathbb{R}$ are the intervals.

Proof. The argument in the previous proof extends easily to show that intervals are connected subset of $\mathbb{R}$.

It remains to show that if $\emptyset \neq A \subseteq \mathbb{R}$ is connected, then A is an interval. Let

$$\alpha = \inf A \quad (\alpha = -\infty \text{ if } A \text{ is unbounded below})$$
$$\beta = \sup A \quad (\beta = \infty \text{ if } A \text{ is unbounded above})$$

Claim 28.1. $(\alpha, \beta) \subseteq A$. This shows A is an interval.

We argue by contradiction. Assume $\exists c \in (\alpha, \beta) \setminus A$. Let $D_1 = (-\infty, c)$ open in $\mathbb{R}$ and $D_2 = (c, \infty)$ open in $\mathbb{R}$.

$$\left.\begin{array}{l} A \subseteq \mathbb{R} \setminus \{c\} = D_1 \cup D_2 \\ A \cap D_1 \cap D_2 = \emptyset \\ A \cap D_1 \neq \emptyset \text{ (because } \inf A = \alpha < c) \\ A \cap D_2 \neq \emptyset \text{ (because } \sup A = \beta > c) \end{array}\right\} \implies A \text{ is disconnected – Contradiction!}$$

□

Proposition 28.8

Let (X, d) be a metric space. Assume that for every pair of points in X, there exists a connected subset of X that contains them. Then X is connected.

Proof. Assume, towards a contradiction, that X is disconnected. Then there exists two non-empty separated sets $A, B \subseteq X$ s.t. $X = A \cup B$.

$$\left.\begin{aligned} A \neq \emptyset \implies \exists a \in A \\ B \neq \emptyset \implies \exists b \in B \end{aligned}\right\} \implies \exists C \subseteq X \text{ connected s.t. } \{a, b\} \subseteq C$$

$$\left.\begin{aligned} &C \subseteq X = A \cup B \\ &C \text{ connected} \\ &X \text{ closed} \implies A, B \text{ closed} \end{aligned}\right\} \implies \left.\begin{aligned} &\underbrace{C \subseteq A}_{b \in A \cap B} \text{ or } \underbrace{C \subseteq B}_{a \in B \cap A} \\ &A \cap B = \emptyset \end{aligned}\right\} \implies \text{Contradiction!} \qquad \square$$

Let (X, d) be a metric space. For $a, b \in X$, we write $a \sim b$ if there exists a connected subset of X, $A_{ab} \subseteq X$ s.t. $\{a, b\} \subseteq A_{ab}$.

Exercise 28.1. $\sim$ defines an equivalence relation of X.

For $a \in X$, let C_a denote the equivalence class of a.

Exercise 28.2.

1. C_a is a connected subset of X.
2. C_a is the largest connected set containing a.
3. C_a is closed in X.
4. If $a \nsim b$ then C_a and C_b are separated.

We can decompose $X = \bigcup_{a \in X} C_a$ as a union of connected components.

We will continue the class with Professor Visan again in Spring 2021 through 131BH – Honors Real Analysis II.