Math 131ABH – Honors Real Analysis

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About the notes

This is math 131AH & 131BH – Undergraduate Honors Real Analysis sequence at UCLA. We meet weekly on MWF from 10:00am – 10:50am for lectures. There are two textbooks associated to the class, *Principles of Mathematical Analysis* by *Rudin* and *Metric Spaces* by *Copson*. Keep in mind that there are a total of 57 lectures; the first 28 are for 131AH, and the rest of them is from 131BH. Thus, the lecture number would be adjusted accordingly for each class. All the typos/errors in the notes are my responsibility, and please let me know through my email if you spot any of them. Additional details with regard to note taking in live lecture and other course notes can also be found at my blog site.

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131AH Lectures

§1 Lec 1: Jan 4, 2021

§1.1 Logical Statments & Basic Set Theory

Let A and B be two statements. We write

- A if A is true.
- not A if A is false.
- A and B if both A and B are true.
- A or B if A is true or B is true or both A and B are true (inclusive "or" – it is not either A or B).
- $\underbrace{A \implies B}$: if ($A$ and B) or (not A) – We read this "A implies B" or "If A then B". In this case, B is at least as true as A. In particular, a false statement can imply anything.

Example 1.1

Consider the following statement: If x is a natural number (i.e., $x \in \mathbb{N} = \{1, 2, 3, \ldots\}$, then $x \geq 1$. In this case, $A =$ "x is a natural number", $B =$ "$x \geq 1$". Taking $x = 3$, we get a $T \implies T$. Taking $x = \pi$ we get $F \implies T$. If $x = 0$, we get $F \implies F$.

Example 1.2

Consider the statement: $\underbrace{\text{If a number is less than 10}}_{A}$, $\underbrace{\text{then it's less than 20}}_{B}$.

Taking

$$\begin{aligned} \text{number } &= 5, & T &\implies T \\ &= 15, & F &\implies T \\ &= 25, & F &\implies F \end{aligned}$$

We write $\underbrace{A \iff B}$ if A and B are true together or false together. We read this as "A is equivalent to B" or "A if and only if B". Compare these notions to similar ones from set theory. Let X is an ambient space. Let A and B be subsets of X. Then

$$\begin{aligned} A^c &= \{x \in X; x \notin A\} \\ A \cap B &= \{x \in X; x \in A \text{ and } x \in B\} \\ A \cup B &= \{x \in X; x \in A \text{ or } x \in B \text{ or } x \in A \cap B\} \\ A \subseteq B &\text{ corresponds to } A \implies B \\ A = B & \quad\quad A \iff B \end{aligned}$$

<u>Truth table</u>:

A	B	not A	A and B	A or B	A $\implies$ B	A $\iff$ B
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Example 1.3

Using the truth table show that $A \implies B$ is logically equivalent to (not A) or B.

A	B	A $\implies$ B	not A	(not A) or B
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Homework 1.1. Using the truth table prove De Morgan's laws:

$$\text{not (A and B)} = \text{(not A) or (not B)}$$
$$\text{not (A or B)} = \text{(not A) and (not B)}$$

Compare this to

$$(A \cap B)^c = A^c \cup B^c$$
$$(A \cup B)^c = A^c \cap B^c$$

Exercise 1.1. Negate the following statement: If A then B.
Solution:

$$\begin{aligned} \text{not}(A \implies B) &= \text{not}\left(\text{(not A) or B}\right) \\ &= [\text{not(not A) and (not B)}] \\ &= A \text{ and (not B)} \end{aligned}$$

The negation is "A is true and B is false".

Example 1.4

Negate the following sentence: If I speak in front of the class, I am nervous.
I speak in front of the class and I am not nervous.

Quantifiers:

- $\forall$ reads "for all" or "for any"
- $\exists$ reads "there is" or "there exists"

The negation of $\forall A, B$ is true is $\exists A$ s.t. B is false.
The negation of $\exists A, B$ is true is $\forall A, B$ is false.

Example 1.5

Negate the following: Every student had coffee or is late for class.
$\forall$ student (had coffee) or (is late for class)
$\exists$ student s.t. not[(had coffee) or (is late for class)]
$\exists$ student s.t. not (had coffee) and not (is late for class)
Ans: There is a student that did not have coffee and is not late for class.

§2 Lec 2: Jan 6, 2021

§2.1 Mathematical Induction

The natural numbers – $\mathbb{N} = \{1, 2, 3, \ldots\}$; they satisfy the Peano axioms:

N1) $1 \in \mathbb{N}$

N2) If $n \in \mathbb{N}$ then $n + 1 \in \mathbb{N}$

N3) 1 is not the successor of any natural number.

N4) If $n, m \in \mathbb{N}$ such that $n + 1 = m + 1$ then $n = m$

N5) Let $S \subseteq \mathbb{N}$. Assume that S satisfies the following two conditions:

(i) $1 \in S$

(ii) If $n \in S$ then $n + 1 \in S$

Then $S = \mathbb{N}$.

Axiom N5) forms the basis for mathematical induction. Assume we want to prove that a property $P(n)$ holds for all $n \in \mathbb{N}$. Then it suffices to verify two steps:
Step 1 (base step): $P(1)$ holds.
Step 2 (inductive step): If $P(n)$ is true for some $n \geq 1$, then $P(n + 1)$ is also true, i.e., $P(n) \implies P(n + 1) \forall n \geq 1$.
Indeed, if we let

$$S = \{n \in \mathbb{N} : P(n) \text{ holds}\}$$

then Step 1 implies $1 \in S$ and Step 2 implies if $n \in S$ then $n + 1 \in S$. By Axiom N5 we deduce $S = \mathbb{N}$.

Example 2.1

Prove that
$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \qquad \forall n \in \mathbb{N}$$

Solution: We argue by mathematical induction. For $n \in \mathbb{N}$ let $P(n)$ denote the statement
$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Step 1 (Base step): $P(1)$ is the statement
$$1^2 = \frac{1 \cdot 2 \cdot 3}{6}$$

which is true, so $P(1)$ holds.
Step 2 (Inductive step): Assume that $P(n)$ holds for some $n \in \mathbb{N}$. We want to know $P(n+1)$ holds. We know
$$1^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Let's add $(n+1)^2$ to both sides of $P(n)$
$$\begin{aligned} 1^2 + \ldots + n^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= (n+1)\left[\frac{n(2n+1)}{6} + n + 1\right] \\ &= \frac{(n+1)(n+2)(2n+3)}{6} \end{aligned}$$

So $P(n+1)$ holds.
Collecting the two steps, we conclude $P(n)$ holds $\forall n \in \mathbb{N}$. □

Example 2.2

Prove that $2^n > n^2$ for all $n \geq 5$.
Solution: We argue by mathematical induction. For $n \geq 5$ let $P(n)$ denote the statement $2^n > n^2$.
Step 1 (base step): $P(5)$ is the statement

$$32 = 2^5 > 5^2 = 25$$

which is true. So $P(5)$ holds.
Step 2 (Inductive step): Assume $P(n)$ is true for some $n \geq 5$ and we want to prove $P(n+1)$. We know

$$2^n > n^2$$

Let us manipulate the above inequality to get $P(n+1)$

$$\begin{gathered} 2^n > n^2 \\ 2^{n+1} > 2n^2 = (n+1)^2 + n^2 - 2n - 1 \\ 2^{n+1} > (n+1)^2 + (n-1)^2 - 2 \end{gathered}$$

As $n \geq 5$ we have $(n-1)^2 - 2 \geq 4^2 - 2 = 14 \geq 0$. So

$$2^{n+1} > (n+1)^2$$

So $P(n+1)$ holds.
Collecting the two steps, we conclude that $P(n)$ holds $\forall n \geq 5$. $\square$

Remark 2.3. Each of the two steps are essential when arguing by induction. Note that $P(1)$ is true. However, our proof of the second step fails if $n = 1 : (1-1)^2 - 2 = -2 < 0$. Note that our proof of the second step is valid as soon as

$$(n-1)^2 - 2 \geq 0 \iff (n-1)^2 \geq 2 \iff n - 1 \geq 2 \iff n \geq 3$$

However, $P(3)$ fails.

Example 2.4

Prove by mathematical induction that the number $4^n + 15n - 1$ is divisible by 9 for all $n \geq 1$.

Solution: We'll argue by induction. For $n \geq 1$, let $P(n)$ denote the statement that "$4^n + 15n - 1$ is divisible by 9". We write this $9/(4^n + 15n - 1)$.

Step 1: $4^1 + 15 \cdot 1 - 1 = 18 = 9 \cdot 2$. This is divisible by 9, so $P(1)$ holds.

Step 2: Assume $P(n)$ is true for some $n \geq 1$. We want to show $P(n+1)$ holds.

$$\begin{aligned} 4^{n+1} + 15(n+1) - 1 &= 4(4^n + 15n - 1) - 60n + 4 + 15n + 14 \\ &= 4(4^n + 15n - 1) - 45n + 18 \\ &= 4(4^n + 15n - 1) - 9(5n - 2) \end{aligned}$$

By the inductive hypothesis, $9/(4^n + 15n - 1) \implies 9/4(4^n + 15n - 1)$. Also $9/9\underbrace{(5n-2)}_{\in \mathbb{N}}$. So

$$9/\left[4(4^n + 15n - 1) - 9(5n - 2)\right]$$

So $P(n+1)$ holds. Collecting the two steps, we conclude $P(n)$ holds $\forall n \in \mathbb{N}$. $\square$

Example 2.5

Compute the following sum and then use mathematical induction to prove your answer: for $n \geq 1$

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2n-1)(2n+1)}$$

Solution: Note that $\frac{1}{(2n-1)(2n+1)}=\frac{1}{2}\left[\frac{1}{2n-1}-\frac{1}{2n+1}\right]\ \forall n\geq 1$. So

$$\begin{aligned}\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\ldots+\frac{1}{(2n-1)(2n+1)}&=\frac{1}{2}\left\{\frac{1}{1}-\frac{1}{3}+\frac{1}{3}\ldots+\frac{1}{2n-1}-\frac{1}{2n+1}\right\}\\&=\frac{1}{2}\frac{2n}{2n+1}=\frac{n}{2n+1}\end{aligned}$$

For $n\geq 1$, let $P(n)$ denote the statement

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$

Step 1: $P(1)$ becomes $\frac{1}{1\cdot 3}=\frac{1}{3}$, which is true. So $P(1)$ holds.
Step 2: Assume $P(n)$ holds for some $n\geq 1$. We want to show $P(n+1)$. We know

$$\frac{1}{1\cdot 3}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$

Let's add $\frac{1}{(2n+1)(2n+3)}$ to both sides

$$\begin{aligned}\frac{1}{1\cdot 3}+\ldots+\frac{1}{(2n+1)(2n+3)}&=\frac{n}{2n+1}+\frac{1}{(2n+1)(2n+3)}\\&=\frac{2n^2+3n+1}{(2n+1)(2n+3)}\\&=\frac{(n+1)(2n+1)}{(2n+1)(2n+3)}\\&=\frac{n+1}{2n+3}\end{aligned}$$

So $P(n+1)$ holds.
Collecting the two steps, we conclude $P(n)$ holds for $\forall n\geq 1$. □

§3 Lec 3: Jan 8, 2021

§3.1 Equivalence Relation

The set of integers is $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n : n \in \mathbb{N}\}$.

Definition 3.1 (Equivalence Relation) — An equivalence relation $\sim$ on a non-empty set A satisfies the following three properties:

- Reflexivity: $a \sim a, \forall a \in A$
- Symmetry: If $a, b \in A$ are such that $a \sim b$, then $b \sim a$
- Transitivity: If $a, b, c \in A$ are such that $a \sim b$ and $b \sim c$, then $a \sim c$.

Example 3.2

$=$ is an equivalence relation on $\mathbb{Z}$.

Example 3.3

Let $q \in \mathbb{N}, q > 1$. For $a, b \in \mathbb{Z}$ we write $a \sim b$ if $q/(a-b)$. This is an equivalence relation on $\mathbb{Z}$. Indeed, it suffices to check 3 properties:

- Reflexivity: If $a \in \mathbb{Z}$ then $a - a = 0$, which is divisible by q. So $q/(a-a) \iff a \sim a$.
- Symmetry: Let $a, b \in \mathbb{Z}$ such that $a \sim b \iff q/(a-b)$ which means there exists $k \in \mathbb{Z}$ s.t. $a - b = kq \implies b - a = \underbrace{-k}_{\in \mathbb{Z}} \cdot q$. So $q/(b-a) \iff b \sim a$.
- Transitivity: Let $a, b, c \in \mathbb{Z}$ such that $a \sim b$ and $b \sim c$, $a \sim b \iff q/(a-b) \implies \exists n \in \mathbb{Z}$ s.t. $a - b = q \cdot n$. And $b \sim c \iff q/(b-c) \implies \exists m \in \mathbb{Z}$ s.t. $b - c = q \cdot m$. So, we must have $a - c = q\underbrace{(n+m)}_{\in \mathbb{Z}}$. So $q/(a-c) \iff a \sim c$.

§3.2 Equivalence Class

Definition 3.4 (Equivalence Class) — Let $\sim$ denote an equivalence relation on a non-empty set A. The equivalence class of an element $a \in A$ is given by

$$C(a) = \{b \in A : a \sim b\}$$

Proposition 3.5 (Properties of Equivalence Classes)

Let $\sim$ denote an equivalence relation on a non-empty set A. Then

1. $a \in C(a) \quad \forall a \in A$.
2. If $a, b \in A$ are such that $a \sim b$, then $C(a) = C(b)$.
3. If $a, b \in A$ are such that $a \not\sim b$, then $C(a) \cap C(b) = \emptyset$.
4. $A = \bigcup_{a \in A} C(a)$

Proof. 1. By reflexivity, $a \sim a \quad \forall a \in A \implies a \in C(a) \quad \forall a \in A$.

2. Assume $a, b \in A$ with $a \sim b$. Let's show $C(a) \subseteq C(b)$. Let $c \in C(a)$ be arbitrary. Then $a \sim c$ (by definition). As $a \sim b$ (by hypothesis), which implies $b \sim a$ (by symmetry). By transitivity, we obtain $b \sim c \implies c \in C(b)$. This proves that $C(a) \subseteq C(b)$.

   A similar argument shows that $C(b) \subseteq C(a)$. Putting the two together, we obtain $C(a) = C(b)$.

3. We argue by contradiction. Assume that $a, b \in A$ are such that $a \not\sim b$, but $C(a) \cap C(b) \neq \emptyset$. Let $c \in C(a) \cap C(b)$.

$$c \in C(a) \implies a \sim c$$
$$c \in C(b) \implies b \sim c \implies c \sim b \quad \text{(by symmetry)}$$

   By transitivity, $a \sim b$. This contradicts the hypothesis $a \not\sim b$. This proves that if $a \not\sim$ then $C(a) \cap C(b) = \emptyset$.

4. Clearly, $C(a) \subseteq A \quad \forall a \in A$, we get

$$\bigcup_{a \in A} C(a) \subseteq A$$

   Conversely, $A = \bigcup_{a \in A} \{a\} \subseteq \bigcup_{a \in A} C(a)$. Putting everything together, we obtain $A = \bigcup_{a \in A} C(a)$. □

Example 3.6

Take $q = 2$ in our previous example: for $a, b \in \mathbb{Z}$ we write $a \sim b$ if $2/(a - b)$. The equivalence classes are

$$C(0) = \{a \in \mathbb{Z} : 2/(a - 0)\} = \{2n : n \in \mathbb{Z}\}$$
$$C(1) = \{a \in \mathbb{Z} : 2/(a - 1)\} = \{2n + 1 : n \in \mathbb{Z}\}$$
$$\mathbb{Z} = C(0) \cup C(1)$$

Let $F = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : b \neq 0\}$. If $(a, b), (c, d) \in F$ we write $(a, b) \sim (c, d)$ if $ad = bc$.

Example 3.7

$(1, 2) \sim (2, 4) \sim (3, 6) \sim (-4, -8)$.

Lemma 3.8

$\sim$ is an equivalence relation on F.

Proof. We have to check 3 properties:

- Reflexivity: Fix $(a,b) \in F$. As $ab = ba$ we have $(a,b) \sim (a,b)$
- Symmetry: Let $(a,b),(c,d) \in F$ such that

$$(a,b) \sim (c,d) \iff ad = bc \iff cb = da \iff (c,d) \sim (a,b)$$

- Transitivity: Let $(a,b),(c,d),(e,f) \in F$ such that $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$.

$$\begin{aligned}(a,b) \sim (c,d) &\iff ad = bc \implies adf = bcf \\ (c,d) \sim (e,f) &\iff cf = de \implies cfb = deb\end{aligned}$$

$$\implies adf = deb \implies \underbrace{d}_{\neq 0}(af - be) = 0, \text{ so } af = be \iff (a,b) \sim (e,f).$$

□

For $(a,b) \in F$, we denote its equivalence class by $\frac{a}{b}$. We define addition and multiplication of equivalence classes as follows:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}; \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

We have to check that these operations are well-defined. Specifically, if $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$ then

$$(ad+bc, bd) \sim (a'd'+b'c', b'd') \quad (1)$$

$$(ac, bd) \sim (a'c', b'd') \quad (2)$$

Let's check (1). We want to show

$$(ad+bc)b'd' = bd(a'd'+b'c')$$

We know

$$\begin{aligned}(a,b) \sim (a',b') &\iff ab' = ba' \quad | \cdot dd' \\ (c,d) \sim (c',d') &\iff cd' = dc' \quad | \cdot bb'\end{aligned}$$

Adding the two (after multiplying the two terms) together, we have

$$\begin{aligned}ab'dd' + cd'bb' &= ba'dd' + dc'bb' \\ (ad+bc)b'd' &= bd(a'd'+b'c')\end{aligned}$$

This proves addition is well defined.

Hw: Check (2)

The set of <u>rational numbers</u> is

$$\mathbb{Q} = \left\{ \frac{a}{b} : (a,b) \in F \right\}$$

§4 Lec 4: Jan 11, 2021

§4.1 Field & Ordered Field

Definition 4.1 (Field) — A field is a set F with at least two elements with two operators: addition (denoted $+$) and multiplication (denoted $\cdot$) that satisfy the following

- A1) Closure: if $a, b \in F$ then $a + b \in F$
- A2) Commutativity: if $a, b \in F$ then $a + b = b + a$
- A3) Associativity: if $a, b, c \in F$ then $(a + b) + c = a + (b + c)$
- A4) Identity: $\exists 0 \in F$ s.t. $a + 0 = 0 + a = a \; \forall a \in F$
- A5) Inverse: $\forall a \in F \exists (-a) \in F$ s.t. $a + (-a) = -a + a = 0$
- M1) Closure: if $a, b \in F$ then $a \cdot b \in F$
- M2) Commutativity: if $a, b \in F$ then $a \cdot b = b \cdot a$
- M3) Associativity: if $a, b, c \in F$ then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- M4) Identity: $\exists 1 \in F$ s.t. $a \cdot 1 = 1 \cdot a = a \; \forall a \in F$
- M5) Inverse: $\forall a \in F \setminus \{0\} \, \exists a^{-1} \in F$ s.t. $a \cdot a^{-1} = a^{-1} \cdot a = 1$
- D) Distributivity: if $a, b, c \in F$ then $(a + b) \cdot c = a \cdot c + b \cdot c$

Example 4.2
$(\mathbb{N}, +, \cdot)$ is not a field. A4 fails.

Example 4.3
$(\mathbb{Z}, +, \cdot)$ is not a field. M5 fails.

Example 4.4
$(\mathbb{Q}, +, \cdot)$ is a field.

Hw

Recall:

$$\mathbb{Q} = \left\{ \frac{a}{b} : (a, b) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \right\}$$

where $\frac{a}{b}$ denotes the equivalence class of $(a, b) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ with respect to the equivalence relation

$$(a, b) \sim (c, d) \iff a \cdot d = b \cdot c$$

Note $\frac{1}{2} = \frac{2}{4}$ because $(1, 2) \sim (2, 4)$. We defined

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Additive identity $\frac{0}{1}$ equivalence class $(0,1)$.
Multiplicative identity $\frac{1}{1}$ equivalence class of $(1,1)$.
Additive inverse: $\frac{a}{b} \in \mathbb{Q}$ has inverse $-\frac{a}{b}$
Multiplicative inverse: $\frac{a}{b} \in \mathbb{Q} \setminus \left\{\frac{0}{1}\right\}$ has inverse $\frac{b}{a}$.

Proposition 4.5

Let $(F, +, \cdot)$ be a field. Then

1. The additive and multiplicative identities are unique.
2. The additive and multiplicative inverses are unique.
3. If $a, b, c \in F$ s.t. $a + b = a + c$ then $b = c$. In particular, if $a + b = a$ then $b = 0$.

3'. If $a, b, c \in F$ s.t. $a \neq 0$ and $a \cdot b = a \cdot c$ then $b = c$. In particular, $a \neq 0$ and $a \cdot b = a$ then $b = 1$.

4. $a \cdot 0 = 0 \cdot a = 0 \ \forall a \in F$.
5. If $a, b \in F$ then $(-a) \cdot b = a \cdot (-b) = -(a \cdot b)$
6. If $a, b \in F$ then $(-a) \cdot (-b) = a \cdot b$
7. If $a \cdot b = 0$ then $a = 0$ or $b = 0$.

Proof. 1. We'll show the additive identity is unique. Assume

$$\exists 0, 0' \in F \text{ s.t. } \forall a \in F, \begin{cases} a + 0 = 0 + a = a & (i) \\ a + 0' = 0' + a = a & (ii) \end{cases}$$

Take $a = 0'$ in (i) and $a = 0$ in (ii) to get

$$\left.\begin{matrix} 0' + 0 = 0' \\ 0' + 0 = 0 \end{matrix}\right\} \implies 0 = 0'$$

2. We'll show that the additive inverse is unique. Let $a \in F$. Assume $\exists (-a), a' \in F$ s.t.

$$\begin{cases} -a + a = a + (-a) = 0 \\ a' + a = a + a' = 0 \end{cases}$$

We have

$$a' + a = 0 \qquad | + (-a)$$

$$(a' + a) + (-a) = 0 + (-a) \overset{A3,A4}{\implies} a' + (a + (-a)) = -a$$
$$\overset{A5}{\implies} a' + 0 = -a \overset{A4}{\implies} a' = -a$$

3. Assume $a + b = a + c \quad | + (-a)$ to the left

$$-a + (a + b) = -a + (a + c)$$
$$\overset{A3}{\implies} (-a + a) + b = (-a + a) + c$$
$$\overset{A5}{\implies} 0 + b = 0 + c \overset{A4}{\implies} b = c$$

So if $a + b = a = a + 0$, then $b = 0$.

4.

$$a \cdot 0 \overset{A4}{=} a \cdot (0+0) \overset{D}{=} a \cdot 0 + a \cdot 0 \overset{(3)}{\Longrightarrow} a \cdot 0 = 0$$
$$0 \cdot a \overset{A4}{=} (0+0) \cdot a = 0 \cdot a + 0 \cdot a \overset{(3)}{\Longrightarrow} 0 \cdot a = 0$$

5. $(-a) \cdot b + a \cdot b \overset{D}{=} (-a+a) \cdot \overset{A5}{=} 0 \cdot b \overset{(4)}{=} 0 \implies (-a) \cdot b = -(a \cdot b)$. Similarly, $a \cdot (-b) = -(a \cdot b)$.

6. $(-a) \cdot (-b) + [-(a \cdot b)] \overset{(5)}{=} (-a) \cdot (-b) + (-a) \cdot b \overset{D}{=} (-a)(-b+b) \overset{A5}{=} (-a) \cdot 0 \overset{(4)}{=} 0$. So $(-a) \cdot (-b) = a \cdot b$.

7. Assume $a \cdot b = 0$. Assume $a \neq 0$. Want to show $b = 0$. As $a \neq 0$ then $\exists a^{-1} \in F$ s.t. $a \cdot a^{-1} = a^{-1} \cdot a = 1$.

$$a \cdot b = 0 \quad | \cdot a^{-1} \text{ to the left}$$
$$a^{-1} \cdot (a \cdot b) = a^{-1} \cdot 0 \overset{M3,(4)}{\Longrightarrow} (a^{-1} \cdot a) \cdot b = 0 \overset{M5}{\Longrightarrow} 1 \cdot b = 0 \overset{M4}{\Longrightarrow} b = 0 \qquad \square$$

Definition 4.6 (Order Relation) — An <u>order relation</u> $<$ on a non-empty set A satisfies the following properties:

- <u>Trichotomy</u>: if $a, b \in A$ then one and only one of the following statement holds: $a < b$ or $a = b$ or $b < a$.
- <u>Transitivity</u>: if $a, b, c \in A$ such that $a < b$ and $b < c$, then $a < c$.

Example 4.7

For $a, b \in \mathbb{Z}$ we write $a < b$ if $b - a \in \mathbb{N}$. This is an order relation.

<u>Notation</u>: We write

$$a > b \text{ if } b < a$$
$$a \leq b \text{ if } [a < b \text{ or } a = b]$$
$$a \geq b \text{ if } b \leq a$$

Definition 4.8 (Ordered Field) — Let $(F, +, \cdot)$ be a field. We say $(F, +, \cdot)$ is an <u>ordered field</u> if it is equipped with an order relation $<$ that satisfies the following

01) if $a, b, c \in F$ such that $a < b$ then $a + c < b + c$.

02) if $a, b, c \in F$ such that $a < b$ and $0 < c$ then $a \cdot c < b \cdot c$.

<u>Note</u>:

To check something is an ordered field, we have to check that it satisfies the properties of order relation and ordered field.

§5 Lec 5: Jan 13, 2021

§5.1 Ordered Field (Cont'd)

Proposition 5.1

Let $(F, +, \cdot, <)$ be an ordered field. Then,

1. $a > 0 \iff -a < 0$.
2. If $a, b, c \in F$ are such that $a < b$ and $c < 0$, then $ac > bc$.
3. If $a \in F \setminus \{0\}$ then $a^2 = a \cdot a > 0$. In particular, $1 > 0$.
4. If $a, b \in F$ are such that $0 < a < b$ then $0 < b^{-1} < a^{-1}$.

Proof. 1. Let's prove " $\implies$ ". Assume $a > 0$.

$$\overset{01}{\implies} a + (-a) > 0 + (-a) \overset{A5,A4}{\implies} 0 > -a$$

Let's prove " $\impliedby$ ". Assume $-a < 0$

$$\overset{01}{\implies} -a + a < 0 + a \overset{A5,A4}{\implies} 0 < a$$

2. Assume $a < b$ and $c < 0$

$$\left.\begin{array}{l} a < b \\ c < 0 \overset{01}{\implies} -c > 0 \end{array}\right\} \overset{02}{\implies} a \cdot (-c) < b \cdot (-c)$$

$$\begin{aligned} &\overset{01}{\implies} -ac + (ac + bc) < -bc + (ac + bc) \\ &\overset{A3,A2}{\implies} (-ac + ac) + bc < -bc + (bc + ac) \\ &\overset{A5,A3}{\implies} 0 + bc < (-bc + bc) + ac \\ &\overset{A4,A5}{\implies} bc < 0 + ac \\ &\overset{A4}{\implies} bc < ac \end{aligned}$$

3. By trichotomy, exactly one of the following hold:

$$a > 0 \overset{02}{\implies} a \cdot a > 0 \cdot a \implies a^2 > 0$$

or

$$a < 0 \overset{2)}{\implies} a \cdot a > 0 \cdot a \implies a^2 > 0$$

4. First we show that if $a > 0$ then $a^{-1} > 0$. Let's argue by contradiction. Assume $\exists a \in F$ s.t. $a > 0$ but $a^{-1} < 0$. Then

$$\left.\begin{array}{l} a > 0 \\ a^{-1} < 0 \end{array}\right\} \overset{(2)}{\implies} a \cdot a^{-1} < 0 \overset{M5}{\implies} 1 < 0$$

This contradicts (3). So if $a > 0$ then $a^{-1} > 0$.

Say

$$0 < a < b \quad | \cdot a^{-1} \cdot b^{-1}$$

$$\begin{aligned} &\overset{02}{\Longrightarrow} 0 \cdot (a^{-1} \cdot b^{-1}) < a \cdot (a^{-1} \cdot b^{-1}) < b \cdot (a^{-1} \cdot b^{-1}) \\ &\overset{M3,M2}{\Longrightarrow} 0 < (a \cdot a^{-1}) \cdot b^{-1} < b \cdot (b^{-1} \cdot a^{-1}) \\ &\overset{M5,M3}{\Longrightarrow} 0 < 1 \cdot b^{-1} < (b \cdot b^{-1}) \cdot a^{-1} \\ &\overset{M4,M5}{\Longrightarrow} 0 < b^{-1} < 1 \cdot a^{-1} \\ &\overset{M4}{\Longrightarrow} 0 < b^{-1} < a^{-1} \end{aligned}$$

□

Theorem 5.2 (Ordered Field)
Let $(F, +, \cdot)$ be a field. The following are equivalent

1) F is an ordered field.
2) There exists $P \subseteq F$ that satisfies the following properties
   01') For every $a \in F$ one and only one of the following statements holds: $a \in P$ or $a = 0$ or $-a \in P$.
   02') If $a, b \in P$ then $a + b \in P$ and $a \cdot b \in P$.

Proof. Let's show 1) $\implies$ 2). Define $P = \{a \in F : a > 0\}$. Let's check (01'). Fix $a \in F$. By trichotomy for the order relation on F we get that exactly one of the following statements is true:

- $a > 0 \implies a \in P$.
- $a = 0$.
- $a < 0 \implies -a > 0 \implies -a \in P$.

Let's check (02'). Fix $a, b \in P$.

$$\left.\begin{aligned} a \in P &\implies a > 0 \\ b \in P &\implies b > 0 \end{aligned}\right\} \overset{01}{\Longrightarrow} a + b > 0 + b \overset{A4}{=} b > 0 \implies a + b \in P$$

And

$$\left.\begin{aligned} a \in P &\implies a > 0 \quad | \cdot b \\ b \in P &\implies b > 0 \end{aligned}\right\} \overset{02}{\Longrightarrow} a \cdot b > 0 \cdot b = 0 \implies a \cdot b \in P$$

Let's check that 2) $\implies$ 1).
For $a, b \in F$ we write $a < b$ if $b - a \in P$. Let's check this is an order relation.

- Trichotomy: Fix $a, b \in F$. By 01') exactly one of the following hold:

$$\begin{aligned} b - a \in P &\implies a < b \\ b - a = 0 &\implies a = b \\ -(b - a) \in P &\implies a - b \in P \implies b < a \end{aligned}$$

- Transitivity Assume $a, b, c \in F$ s.t. $a < b$ and $b < c$

$$\left.\begin{aligned} a < b &\implies b - a \in P \\ b < c &\implies c - b \in P \end{aligned}\right\} \stackrel{02'}{\implies} (b-a) + (c-b) \in P \implies c - a \in P \implies a < c$$

Now let's check that with this order relation, F is an ordered field. We have to check 01 and 02.

01) Fix $a, b, c \in F$ s.t. $a < b \implies b - a \in P \implies b - a \in P \implies (b + c) - (a + c) \in P \implies a + c < b + c$.

02) Fix $a, b, c \in F$ s.t. $a < b$ and $0 < c$

$$\left.\begin{aligned} a < b &\implies b - a \in P \\ 0 < c &\implies c - 0 = c \in P \end{aligned}\right\} \stackrel{02'}{\implies} (b-a) \cdot c \in P \stackrel{D}{\implies} b \cdot c - a \cdot c \in P \implies a \cdot c < b \cdot c$$

□

We extend the order relation $<$ from $\mathbb{Z}$ to the field $(\mathbb{Q}, +, \cdot)$ by writing $\frac{a}{b} > 0$ if $a \cdot b > 0$. Let's see this is well defined. Specifically, we need to show that if $\frac{a}{b} = \frac{c}{d}$, i.e., $(a, b) \sim (c, d)$ and $a \cdot b > 0$ then $c \cdot d > 0$.

$$\begin{aligned} (a, b) \sim (c, d) &\implies a \cdot d = b \cdot c \quad | \cdot (ad) \\ &\implies 0 < (ad)^2 = (ab) \cdot (cd) \text{ where } a \cdot d \neq 0 \end{aligned}$$

So

$$\left.\begin{aligned} &0 < (ab) \cdot (cd) \\ &0 < ab \end{aligned}\right\} \implies cd > 0 \implies \frac{c}{d} > 0$$

Let $P = \left\{ \frac{a}{b} \in \mathbb{Q} : \frac{a}{b} > 0 \right\}$. By the theorem, to prove that $\mathbb{Q}$ is an ordered field, it suffices to show that P satisfies (01') and (02').

Hw: check (01') and (02')

§6 Lec 6: Jan 15, 2021

§6.1 Least Upper Bound & Greatest Lower Bound

Definition 6.1 (Boundedness – Maximum and Minimum) — Let $(F, +, \cdot, <)$ be an ordered field. Let $\emptyset \neq A \subseteq F$. We say that A is bounded above if $\exists M \in F$ s.t. $a \leq M \forall a \in A$. Then M is called an upper bound for A. If moreover, $M \in A$ then we say that M is the maximum of A.
We say that A is bounded below if $\exists m \in F$ s.t. $m \leq a \forall a \in A$. Then m is called a lower bound for A. If moreover, $m \in A$ then we say that m is the minimum of A.
We say that A is bounded if A is bounded both above and below.

Example 6.2
$A = \left\{1 + \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$ bounded.

- 3 is an upper bound for A.
- $\frac{3}{2}$ is the maximum of A.
- 0 is a lower bound for A ; 0 is the minimum of A.

Example 6.3
$A = \left\{x \in \mathbb{Q} : 0 < x^4 \leq 16\right\}$ bounded.

- 2 is the maximum of A.
- -2 is the minimum of A.

Example 6.4

$A = \{x \in \mathbb{Q} : x^2 < 2\}$ bounded.

- 2 is an upper bound for A.
- -2 is lower bound for A.
- A does not have a maximum. Indeed, let $x \in A$. We'll construct $y \in A$ s.t. $y > x$. Define $y = x + \frac{2-x^2}{2+x}$.

$$\left.\begin{array}{l} x \in A \implies x \in \mathbb{Q} \implies 2 - x^2, 2 + x \in \mathbb{Q} \\ x \in A \implies 2 + x > 0 \implies \frac{1}{2+x} \in \mathbb{Q} \end{array}\right\} \implies \frac{2-x^2}{2+x} \in \mathbb{Q} \implies y \in \mathbb{Q}(i)$$

Also note

$$\left.\begin{array}{l} 2 - x^2 > 0 (\text{as } x \in A) \\ 2 + x > 0 \implies \frac{1}{2+x} > 0 \end{array}\right\} \implies \frac{2-x^2}{2+x} > 0$$

So $y = x + \frac{2-x^2}{2+x} > x$ (ii). Let's compute $y^2 = \left(\frac{2x+x^2+2-x^2}{2+x}\right)^2 = \frac{2(x^2+4x+4)+2x^2-4}{x^2+4x+4} =$ $2 + \underbrace{\frac{2(x^2-2)}{(x+2)^2}}_{<0}$. So $y^2 < 2$. (iii)

So collecting (i) – (iii) we get $y \in A$ and $y > x$.

Homework 6.1. Show that the maximum and minimum of a set are unique, if they exist.

Definition 6.5 (Least Upper Bound) — Let $(F, +, \cdot, <)$ be an ordered field. Let $\emptyset \neq A \subseteq F$ and assume A is bounded above. We say that L is the least upper bound of A if it satisfies:

1. L is an upper bound of A.
2. If M is an upper bound of A then $L \leq M$.

We write $L = \sup A$ and we say L is the supremum of A.

Lemma 6.6

The least upper bound of a set is unique, if it exists.

Proof. Say that a set $\emptyset \neq A \subseteq F$, A bounded above, admits two least upper bounds L, M.

L is a least upper bound $\overset{(1)}{\implies}$ L is an upper bound for A.

M is a least upper bound $\overset{(2)}{\implies}$ $M \leq L$.

M is a least upper bound for A $\overset{(1)}{\implies}$ M is an upper bound for A $\implies$ L is a least upper bound for A $\overset{(2)}{\implies}$ $L \leq m$. So $L = M$. □

Definition 6.7 (Greatest Lower Bound) — Let $(F, +, \cdot, <)$ be an ordered field. Let $\emptyset \neq A \subseteq F$ and assume A is bounded below. We say that l is the greatest lower bound of A if it satisfies

1. l is a lower bound of A.
2. If m is a lower bound of A then $m \leq l$.

We write $l = \inf A$ and we say l is the infimum of A.

Homework 6.2. Show that the greatest lower bound of a set is unique if it exists.

Definition 6.8 (Bound Property) — Let $(F, +, \cdot, <)$ be an ordered field. Let $\emptyset \neq S \subseteq F$. We say that S has the the least upper bound property if it satisfies the following: For any non-empty subset A of S is bounded above, there exists a least upper bound of A and $\sup A \in S$.
We say that S has the greatest lower bound property if it satisfies the following: $\forall \emptyset \neq A \subseteq S$ with A bounded below, $\exists \inf A \in S$.

Example 6.9
$(\mathbb{Q}, +, \cdot, <)$ is an ordered field.
$\emptyset \neq \mathbb{N} \subseteq \mathbb{Q}$, $\mathbb{N}$ has the least upper bound property. Indeed if $\emptyset \neq A \subseteq \mathbb{N}$, A bounded above, then the largest elements in A is the least upper bound of A and $\sup A \in \mathbb{N}$.
$\mathbb{N}$ also has the greatest lower bound property.

Example 6.10
$(\mathbb{Q}, +, \cdot, <)$ is an ordered field.
$\emptyset \neq \mathbb{Q} \subseteq \mathbb{Q}$, $\mathbb{Q}$ does not have the least upper bound property.
Indeed, $\emptyset \neq A = \left\{x \in \mathbb{Q} : x \geq 0 \text{ and } x^2 < 2\right\} \subseteq \mathbb{Q}$. A is bounded above by 2. However, $\sup A = \sqrt{2} \notin \mathbb{Q}$.

Proposition 6.11
Let $(F, +, \cdot, <)$ be an ordered field. Then F has the least upper bound property if and only if it has the greatest lower bound property.

Proof. ($\implies$) Assume F has the least upper bound property. Let $\emptyset \neq A \subseteq F$ bounded below. WTS $\exists \inf A \in F$. A is bounded below $\implies$ $\exists m \in F$ s.t. $m \leq a \forall a \in A$. Let $B = \{b \in F : b \text{ is a lower bound for } A\}$. Note $B \neq \emptyset$ (as $m \in B$), $B \subseteq F$, B is bounded above (every element in A is an upper bound for B) and F has the least upper bound property $\implies$ $\sup B \in F$.

Claim 6.1. $\sup B = \inf A$ (to be proven in Lec 7). □

§7 Lec 7: Jan 20, 2021

§7.1 Least Upper & Greatest Lower Bound (Cont'd)

Proof. (Cont'd of proposition 6.11)

Claim 7.1. $\sup B = \inf A$.

Method 1:

- $\sup B$ is a lower bound for A. Indeed, let $a \in A$. We know that $a \geq b \quad \forall b \in B$. $\sup B$ is the least upper bound for $B \implies a \geq \sup B$. As $a \in A$ was arbitrary, we conclude that $\sup B \leq a \quad \forall a \in A$ and so $\sup B$ is a lower bound for A.
- If l is a lower bound for A then $l \leq \sup B$. Well, l is a lower bound for $A \implies l \in B$ and $\sup B$ is an upper bound for B. So $l \leq \sup B$.

Collecting the two bullet points above, we find that $\inf A = \sup B$.
Method 2: Let $\emptyset \neq A \subseteq F$ s.t. A is bounded below. Let $B = \{-a : a \in A\}$. Note $B \subseteq F$ by A5. $B \neq \emptyset$ because $A \neq \emptyset$. B is bounded above: indeed if m is a lower bound for A then $-m$ is an upper bound for B.

$$m \leq a \quad \forall a \in A \implies -m \geq -a \quad \forall a \in A$$

F has the least upper bound property. Altogether, it implies that $\sup B \in F$. In Hw3, you show $-\sup B = \inf A \in F$ (by A5). □

Homework 7.1. Prove the " $\Longleftarrow$ " direction.

Theorem 7.1 (Existence of $\mathbb{R}$)
There exists an ordered field with the least upper bound property. We denote it $\mathbb{R}$ and we call it the set of real numbers. $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield. Moreover, we have the following uniqueness property: If $(F, +, \cdot, <)$ is an ordered field with the least upper bound property, then F is order isomorphic with $\mathbb{R}$, that is, there exists a bijection $\phi : \mathbb{R} \to F$ such that

i) $\phi(x \underbrace{+}_{\mathbb{R}} y) = \phi(x) \underbrace{+}_{F} \phi(y)$

ii) $\phi(x \underbrace{\cdot}_{\mathbb{R}} y) = \phi(x) \underbrace{\cdot}_{F} \phi(y)$

iii) If $x \underbrace{<}_{\mathbb{R}} y$ then $\phi(x) \underbrace{<}_{F} \phi(y)$

Theorem 7.2 (Archimedean Property)
$\mathbb{R}$ has the Archimedean property, that is, $\forall x \in \mathbb{R} \quad \exists n \in \mathbb{N}$ s.t. $x < n$.

Proof. We argue by contradiction. Assume

$$\exists x_0 \in \mathbb{R} \text{ s.t. } x_0 \geq n \quad \forall n \in \mathbb{N}$$

Then $\emptyset \neq \mathbb{N} \subseteq \mathbb{R}$. $\mathbb{N}$ is bounded above by x_0. $\mathbb{R}$ has the least upper bound property $\implies \exists L = \sup \mathbb{N} \in \mathbb{R}$.

$$\left.\begin{aligned} L &= \sup \mathbb{N} \\ L - 1 &< L \end{aligned}\right\} \implies L-1 \text{ is not an upper bound for } \mathbb{N}$$

$\implies \exists n_0 \in \mathbb{N}$ s.t. $n_0 > L - 1$. So $\sup \mathbb{N} = L < n_0 + 1 \in \mathbb{N}$, which is a contradiction. ☐

Remark 7.3. $\mathbb{Q}$ has the Archimedean property.

If $r \in \mathbb{Q}$ is s.t. then choose $n = 1$. For $r \in \mathbb{Q}$ is s.t. $r > 0$, then write $r = \frac{p}{q}$ with $p, q \in \mathbb{N}$. Choose $n = p + 1$ since $\frac{p}{q} < p + 1$.

Corollary 7.4

If $a, b \in \mathbb{R}$ such that $a > 0, b > 0$ then there exists $n \in \mathbb{N}$ s.t. $n \cdot a > b$.

Proof. Apply the Archimedean Property to $x = \frac{b}{a}$. ☐

Corollary 7.5

If $\epsilon > 0$ there exists $n \in \mathbb{N}$ s.t. $\frac{1}{n} < \epsilon$.

Proof. Apply the Archimedean property to $x = \frac{1}{\epsilon}$. ☐

Lemma 7.6

For any $a \in \mathbb{R}$ there exists $N \in \mathbb{Z}$ s.t. $N \leq a \leq N + 1$.

Proof. Case 1: $a = 0$. Take $N = 0$.
Case 2: $a > 0$. Consider $A = \{n \in \mathbb{Z} : n \leq a\} \subseteq \mathbb{R}$, $A \neq \emptyset (0 \in A)$. A is bounded above by a. $\mathbb{R}$ has the least upper bound property. So $\exists L = \sup A \in \mathbb{R}$.

$$L - 1 < L = \sup A \implies L - 1 \text{ is not an upper bound for } A$$

$\implies \exists N \in A$ s.t. $L - 1 < N \implies L < N + 1$ but $L = \sup A$, so $N + 1 \notin A$. So

$$\left.\begin{aligned} N \in A &\implies N \leq a \\ N + 1 \notin A &\implies N + 1 > a \end{aligned}\right\} \implies N \leq a < N + 1$$

Case 3: $a < 0 \implies -a > 0$. By case 2, $\exists n \in \mathbb{Z}$ s.t. $n \leq -a < n+1$. So $-n-1 < a \leq -n$. If $a = -n$, let $N = -n$ and so $N \leq a < N + 1$. If $a < -n$ let $N = -n - 1$ and so $N \leq a < N + 1$. ☐

Definition 7.7 (Dense Set) — We say that a subset A of $\mathbb{R}$ is dense in $\mathbb{R}$ if for every $x, y \in \mathbb{R}$ such that $x < y$ there exists $a \in A$ such that $x < a < y$.

Lemma 7.8

$\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof. Let $x, y \in \mathbb{R}$ such that $x < y$. Since $y - x > 0$ by corollary 7.5, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < y - x \implies \frac{1}{n} + x < y$.
Consider $nx \in \mathbb{R}$. By the lemma 7.6, $\exists m \in \mathbb{Z}$ s.t.

$$m \leq nx < m + 1 \implies \frac{m}{n} \leq x < \frac{m+1}{n}$$

Then

$$x < \frac{m+1}{n} = \frac{m}{n} + \frac{1}{n} \leq x + \frac{1}{n} < y$$

w where $\frac{m+1}{n} \in \mathbb{Q}$. □

Lemma 7.9

$\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

§8 Lec 8: Jan 22, 2021

§8.1 Construction of the Reals

Recall that we say a set $A \subseteq \mathbb{R}$ is dense if for every $x, y \in \mathbb{R}$ s.t. $x < y$, there exists $a \in A$ s.t. $x < a < y$. Last time we proved

Lemma 8.1

$\mathbb{Q}$ is dense in $\mathbb{R}$.

Remark 8.2. For any two rational numbers $r_1, r_2 \in \mathbb{Q}$ s.t. $r_1 < r_2$, there exists $s \in \mathbb{Q}$ s.t. $r_1 < s < r_2$.

Indeed if $r_1 < 0 < r_2$ then we may take $s = 0$.
Assume $0 < r_1 < r_2$. Write $r_1 = \frac{a}{b}, a_2 = \frac{c}{d}$ with $a, b, c, d \in \mathbb{N}$. Take $s = \frac{ad+bc}{2bd} \in \mathbb{Q}$. Note $r_1 < s < r_2$.

$$r_1 < s \iff \frac{a}{b} < \frac{ad+bc}{2bd} \iff 2ad < ad+bc \iff ad < bc \iff \frac{a}{b} < \frac{c}{d} \iff r_1 < r_2$$

Homework 8.1. Construct s in the remaining cases.

Lemma 8.3

$\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

Proof. Let $x, y \in \mathbb{R}$ s.t. $x < y \implies x + \sqrt{2} < y + \sqrt{2}$. $\mathbb{Q}$ is dense in $\mathbb{R}$. So $\exists q \in \mathbb{Q}$ s.t. (since $\mathbb{Q}$ is dense in $\mathbb{R}$)

$$x + \sqrt{2} < q < y + \sqrt{2} \implies x < q - \sqrt{2} < y$$

Claim 8.1. $q - \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$.

Otherwise, $\exists r \in \mathbb{Q}$ s.t. $q - \sqrt{2} = r \implies \sqrt{2} = q - r \in \mathbb{Q}$, contradiction. □

Theorem 8.4 (Construction of $\mathbb{R}$(Existence))

There exists an ordered field with the least upper bound property. We denote it $\mathbb{R}$ and call it the set of real numbers. $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.

Proof. We will construct an ordered field with the least upper bound property using Dedekind cuts. The elements of the field are certain subsets of $\mathbb{Q}$ called cuts.

Definition 8.5 ((Dedekind) Cuts) — A cut is a set $\alpha \subseteq \mathbb{Q}$ that satisfies:

a) $\emptyset \neq \alpha \neq \mathbb{Q}$

b) If $q \in \alpha$ and $p \in \mathbb{Q}$ s.t. $p < q$ then $p \in \alpha$.

c) For every $q \in \alpha$ there exists $r \in \alpha$ s.t. $r > q$ (α has no maximum)

Intuitively, we think of a cut as $\mathbb{Q} \cap (\infty, a)$. Of course, at this point we haven't yet constructed $\mathbb{R}$...
Note that if $\mathbb{Q} \ni q \notin \alpha$ then $q > p \forall p \in \alpha$. Indeed, otherwise, if $\exists p_0 \in \alpha$ s.t. $q \leq p_0$ then by ii) we would have $q \in \alpha$. Contradiction.
We define

$$F = \{\alpha : \alpha \text{ is a cut}\}$$

We will show F is an ordered field with the least upper bound property.
Order: For $\alpha, \beta \in F$ we write $\alpha < \beta$ if α is a proper subset of β, that is, $\alpha \subsetneq \beta$

- Transitivity: If $\alpha, \beta, \gamma \in F$ s.t. $\alpha < \beta$ and $\beta < \gamma$ then $\alpha \subsetneq \beta \subsetneq \gamma \implies \alpha \subsetneq \gamma \implies \alpha < \gamma$.

- Trichotomy: First note that at most one of the following hold

$$\alpha < \beta, \quad \alpha = \beta, \quad \beta < \alpha$$

  To prove trichotomy, it thus suffices to show that at least one of the following holds: $\alpha < \beta, \alpha = \beta, \beta < \alpha$. We show this by contradiction: Assume $\alpha < \beta, \alpha = \beta, \beta < \alpha$ all fail. Then we have

$$\left.\begin{matrix} \alpha \nsubseteq \beta \\ \alpha \neq \beta \\ \beta \nsubseteq \alpha \end{matrix}\right\} \implies \begin{cases} \exists p \in \alpha \setminus \beta \\ \exists q \in \beta \setminus \alpha \end{cases}$$

  Now

$$p \notin \beta \implies p > r \quad \forall r \in \beta \tag{1}$$
$$q \notin \alpha \implies q > s \quad \forall s \in \alpha \tag{2}$$

  Take $r = q$ in (1) and $s = p$ in (2) to get $p > q > p$. Contradiction!

So $<$ defines an order relation on F.
Let's show that F has the least upper bound property. Let $\emptyset \neq A \subseteq F$ bounded above by $\beta \in F$. Define

$$\gamma = \bigcup_{\alpha \in A} \alpha$$

Claim 8.2. $\gamma \in F$.

- $\gamma \neq \emptyset$ because $A \neq \emptyset$ and $\emptyset \neq \alpha \in A$.

- $\gamma \neq \mathbb{Q}$ because β being an upper bound for A

$$\implies \beta \geq \alpha \forall \alpha \in A \implies \beta \supseteq \alpha \forall \alpha \in A \implies \beta \supseteq \bigcup_{\alpha \in A} \alpha = \gamma$$

  As $\beta \neq \mathbb{Q} \implies \gamma \neq \mathbb{Q}$.

- Let $q \in \gamma$ and let $p \in \mathbb{Q}$ s.t. $p < q$. As $q \in \gamma \implies \exists \alpha \in A$ s.t. $q \in \alpha$ and $\mathbb{Q} \ni p < q$. So $p \in \alpha \implies p \in \gamma$.
- Let $q \in \gamma \implies \exists \alpha \in A$ s.t. $q \in \alpha \implies \exists r \in \alpha$ s.t. $q < r$. Then $r \in \gamma$ and $q < r$.

Collecting all these properties, we deduce $\gamma \in F$.

Claim 8.3. $\gamma = \sup A$.

- Note $\alpha \subseteq \gamma \forall \alpha \in A \implies \alpha \leq \gamma \forall \alpha \in A$. So γ is an upper bound for A.
- Let δ be an upper bound for $A \implies \delta \geq \alpha \forall \alpha \in A \implies \delta \supseteq \alpha \forall \alpha \in A$. So $\delta \supseteq \bigcup_{\alpha \in A} \alpha = \gamma \implies \delta \geq \gamma$.

<u>Addition</u>: If $\alpha, \beta \in F$ we define

$$\alpha + \beta = \{p + q : p \in \alpha \text{ and } q \in \beta\}$$

Let's check A1, namely, $\alpha + \beta \in F$.

- Note $\alpha + \beta = \emptyset$ because $\alpha \neq \emptyset \implies \exists p \in \alpha$ and $\beta \neq \emptyset \implies \exists q \in \beta$ which implies $p + q \in \alpha + \beta$.
- Note $\alpha + \beta \neq \mathbb{Q}$. Indeed $\alpha\mathbb{Q} \implies \exists r \in \mathbb{Q} \setminus \alpha \implies r > p \forall p \in \alpha$ and $\beta \neq \mathbb{Q} \implies \exists s \in \mathbb{Q} \setminus \beta \implies s > q \forall q \in \beta$ which implies $r + s > p + q \forall p \in \alpha$ and $\forall q \in \beta \implies r + s \notin \alpha + \beta$
- Let $r \in \alpha + \beta$ and $s \in \mathbb{Q}$ s.t. $s < r$

$$\begin{gathered} r \in \alpha + \beta \implies r = p + q \text{ for some } p \in \alpha \text{ and some } q \in \beta \\ s < r \implies s < p + q \implies \underbrace{s - p}_{\in \mathbb{Q}} < \underbrace{q}_{\in \beta} \implies s - p \in \beta \end{gathered}$$

  So $s = p + (s - p) \in \alpha + \beta$.
- Let $r \in \alpha + \beta \implies r = p + q$ for some $p \in \alpha$ and some $q \in \beta$

$$\left. \begin{aligned} \alpha \in F &\implies \exists p' \in \alpha \ni p' > p \\ \beta \in F &\implies \exists q' \in \beta \ni q' > q \end{aligned} \right\} \implies \alpha \ni p' + q' \in \beta > p + q = r$$

  So $p' + q' \in \alpha + \beta$ s.t. $p' + q' > r$.

So collecting all these properties, we see that $\alpha + \beta \in F$. □

§9 Lec 9: Jan 25, 2021

§9.1 Construction of the Reals (Cont'd)

Recall: A cut is set $\alpha \subseteq \mathbb{Q}$ such that

i) $\emptyset \neq \alpha \neq \mathbb{Q}$

ii) If $q \in \alpha$ and $p \in \mathbb{Q}$ with $p < q$ then $p \in \alpha$

iii) $\forall q \in \alpha \quad \exists r \in \alpha$ s.t. $r > q$.

We defined

$$F = \{\alpha : \alpha \text{ is a cut}\}$$

We defined an order relation on F : for $\alpha, \beta \in F$ we write $\alpha < \beta \iff \alpha \subsetneq \beta$. We showed that F has the least upper bound property with respect to this order relation. We defined an addition operation on F : for $\alpha, \beta \in F$

$$\alpha + \beta = \{p + q : p \in \alpha \text{ and } q \in \beta\}$$

We checked A1. Let's check A2: for $\alpha, \beta \in F$

$$\begin{aligned}
\alpha + \beta &= \{p + q : p \in \alpha, q \in \beta\} \\
&= \{q + p : q \in \beta, p \in \alpha\} \text{ (since addition in } \mathbb{Q} \text{ satisfies A2)} \\
&= \beta + \alpha
\end{aligned}$$

Let's check A3: for $\alpha, \beta, \gamma \in F$

$$\begin{aligned}
(\alpha + \beta) + \gamma &= \{s + r : s \in \alpha + \beta, r \in \gamma\} \\
&= \{(p + q) + r : p \in \alpha, q \in \beta, r \in \gamma\} \\
&= \{p + (q + r) : p \in \alpha, q \in \beta, r \in \gamma\} \text{ (since addition in } \mathbb{Q} \text{ satisfies A3} \\
&= \{p + t : p \in \alpha, t \in \beta + \gamma\} \\
&= \alpha + (\beta + \gamma)
\end{aligned}$$

Let's check A4: Let $0^* = \{q \in \mathbb{Q} : q < 0\}$.

Claim 9.1. $0^* \in F$

- Note $0^* \neq \emptyset$ since $-1 \in 0^*$
- Note $0^* = \mathbb{Q}$ since $2 \notin 0^*$
- Let $q \in 0^*$ and let $p \in \mathbb{Q}$ and $p < q$

  $$\left.\begin{aligned} q \in 0^* \implies q < 0 \\ p < q \end{aligned}\right\} \implies p < 0$$

  So $p \in 0^*$.
- Let $q \in 0^* \implies q < 0 \implies \exists r \in \mathbb{Q}$ s.t. $q < r < 0$. So $r \in 0^*$ and $r > q$.

Collecting all these properties we got $0^* \in F$.

Claim 9.2. $\alpha + 0^* = \alpha \qquad \forall \alpha \in F$.

- Let's check $\alpha + 0^* \subseteq \alpha$.

  Let $r \in \alpha + 0^* \implies r = p + q$ for some $p \in \alpha$ and some $q \in 0^*$. $q \in 0^* \implies q < 0$. So

  $$\left.\begin{array}{l} \mathbb{Q} \ni r = p + q < p \\ p \in \alpha \in F \end{array}\right\} \implies r \in \alpha$$

  As r was arbitrary in $\alpha + 0^*$ we find $\alpha + 0^* \subseteq \alpha$.

- Let's check $\alpha \subseteq \alpha + 0^*$. Let $p \in \alpha \implies \exists r \in \alpha$ s.t. $r > p$. We write

  $$p = \underbrace{r}_{\in \alpha} + \underbrace{(p - r)}_{\in 0^*} \in \alpha + 0^*$$

  As $p \in \alpha$ was arbitrary, this shows $\alpha \subseteq \alpha + 0^*$

Collecting everything, we get $\alpha + 0^* = \alpha$.
Let's check A5: Fix $\alpha \in F$. Define

$$\beta = \{q \in \mathbb{Q} : \exists r \in \mathbb{Q} \text{ with } r > 0 \ni -q - r \notin \alpha\}$$

Claim 9.3. $\beta \in F$.

- Note that $\beta \neq \emptyset$.

  As $\alpha \neq \mathbb{Q} \implies \exists p \in \mathbb{Q} \setminus \alpha$. Then $-(p+1) \in \beta$ because $-[-(p+1)] - 1 = (p+1) - 1 = p \notin \alpha$.

- Note that $\beta \neq \mathbb{Q}$.

  As $\alpha \neq \emptyset \implies \exists p \in \alpha$. Then $-p \notin \beta$ because $\forall r \in \mathbb{Q}, r > 0$ we have

  $$\left.\begin{array}{l} -(-p) - r = p - r < p \\ p \in \alpha \in F \end{array}\right\} \implies p - r \in \alpha$$

  So $-p \notin \beta$.

- Let $q \in \beta$ and let $p \in \mathbb{Q}$ s.t. $p < q$

  $$q \in \beta \implies \exists r \in \mathbb{Q}, r > 0 \ni -q - r \notin \alpha \implies -q - r > s \forall s \in \alpha$$

  So $-p - r > -q - r > s \forall s \in \alpha \implies -p - r \notin \alpha \implies p \in \beta$.

- Let $q \in \beta$. Want to find $s \in \beta$ s.t. $s > q$.

  $$\begin{aligned} q \in \beta &\implies \exists r \in \mathbb{Q} \ni r > 0 \text{ and } -q - r \notin \alpha \\ &\implies -\left(2 + \frac{r}{2}\right) - \frac{r}{2} = -q - r \notin \alpha \\ &\implies q + \frac{r}{2} \in \beta \end{aligned}$$

  Let $s = q + \frac{r}{2}$.

Collecting all the properties, we get $\beta \in F$.

Claim 9.4. $\alpha + \beta = 0^*$.

- Let's check that $\alpha + \beta \subseteq 0^*$.

  Let $s \in \alpha + \beta \implies s = p + q$ with $p \in \alpha$ and $q \in \beta$. Since $q \in \beta \implies \exists r \in \mathbb{Q}, r > 0 \ni -q - r \notin \alpha \implies -q - r > p$. So $\underbrace{p + q}_{\in \mathbb{Q}} < -r < 0$. So $s = p + q \in 0^*$. Thus $\alpha + \beta \subseteq 0^*$.

- Let's check $0^* \subseteq \alpha + \beta$. Let $r \in 0^* \implies r \in \mathbb{Q}, r < 0$.

  Claim 9.5. $\exists N \in \mathbb{N}$ s.t. $N \cdot \left(-\frac{r}{2}\right) \in \alpha$ but $(N+1)\left(-\frac{r}{2}\right) \notin \alpha$.

  Let's prove this by contradiction. Assume

  $$\left\{n\left(-\frac{r}{2}\right) : n \in \mathbb{N}\right\} \subseteq \alpha$$

  We will show that in this case $\mathbb{Q} \subseteq \alpha$ thus reaching a contradiction.

  Fix $q \in \mathbb{Q}$. By the Archimedean property for $\mathbb{Q}$, $\exists n \in \mathbb{N}$ s.t. $n > \underbrace{q \cdot \left(-\frac{2}{r}\right)}_{\in \mathbb{Q}}$. So

  $$\left.\begin{array}{l} n \cdot \left(-\frac{r}{2}\right) > q \\ n \cdot \left(-\frac{r}{2}\right) \in \alpha \in F \end{array}\right\} \implies q \in \alpha$$

  As $q \in \mathbb{Q}$ was arbitrary, this shows $\mathbb{Q} \subseteq \alpha$. Contradiction!

  Write $r = \underbrace{N\left(-\frac{r}{2}\right)}_{\in \alpha} + (N+2) \cdot \frac{r}{2}$ and note that $(N+2)\frac{r}{2} \in \beta$ since

  $$-(N+2) \cdot \frac{r}{2} - \frac{r}{2} = (N+1) \cdot \left(-\frac{r}{2}\right) \notin \alpha$$

  As $r \in 0^*$ was arbitrary, this shows $0^* \subseteq \alpha + \beta$. Thus, $\alpha + \beta = 0^*$.

Let's check 01: if $\alpha, \beta, \gamma \in F$ s.t. $\alpha < \beta \implies \alpha \subsetneq \beta$ then $\alpha + \gamma \subsetneq \beta + \gamma \implies \alpha + \gamma < \beta + \gamma$. WE define multiplication on F as follows: for $\alpha < \beta \in F$ with $\alpha > 0$, $\beta > 0$ we define

$$\alpha \cdot \beta = \{q \in \mathbb{Q} : q < r \cdot s \text{ for some } 0 < r \in \alpha \text{ and some } 0 < s \in \beta\}$$

For $\alpha \in F$ we define $\alpha \cdot 0^* = 0^*$. We define

$$\alpha \cdot \beta = \begin{cases} (-\alpha) \cdot (-\beta), & \text{if } \alpha < 0, \beta < 0 \\ -\left[(-\alpha) \cdot \beta\right], & \text{if } \alpha < 0, \beta > 0 \\ -\left[\alpha \cdot (-\beta)\right], & \text{if } \alpha > 0, \beta < 0 \end{cases}$$

You checked M1 through M5 for positive cuts. This extends readily to all cuts.

Homework 9.1. Check (D) and (02).

We identify a rational number $r \in \mathbb{Q}$ with the cut

$$r^* = \{q \in \mathbb{Q} : q < r\}$$

One can check that

$$\begin{aligned} r^* + s^* &= (r+s)^* \\ r^* \cdot s^* &= (r \cdot s)^* \\ r < s &\iff r^* < s^* \end{aligned}$$

§10 Lec 10: Jan 27, 2021

§10.1 Sequences

Definition 10.1 (Sequence) — A sequence of real number is a function $f : \{n \in \mathbb{Z} : n \geq m\} \to \mathbb{R}$ where m is a fixed integer (m is usually 0 or 1). We write the sequence as $f(m), f(m+1), f(m+2), \ldots$ or as $\{f(n)\}_{n\geq m}$ or as $\{f_n\}_{n\geq m}$.

Example 10.2

1. $\{a_n\}_{n\geq 1}$ with $a_n = 3 - \frac{1}{n}$ bounded, strictly increasing.
2. $\{a_n\}_{n\geq 1}$ with $a_n = (-1)^n$ bounded, not monotone.
3. $\{a_n\}_{n\geq 0}$ with $a_n = n^2$ bounded below, strictly increasing.
4. $\{a_n\}_{n\geq 0}$ with $a_n = \cos\left(\frac{n\pi}{3}\right)$ bounded, not monotone.

Definition 10.3 (Boundedness of Sequence) — We say that a sequence $\{a_n\}_{n\geq 1}$ of real numbers is bounded below/bounded above/bounded if the set $\{a_n : n \geq 1\}$ is bounded below/bounded above/bounded.

We say that the sequence $\{a_n\}_{n\geq 1}$ is

- increasing if $a_n \leq a_{n+1} \quad \forall n \geq 1$
- strictly increasing if $a_n < a_{n+1} \quad \forall n \geq 1$
- decreasing if $a_n \geq a_{n+1} \quad \forall n \geq 1$
- strictly decreasing if $a_n > a_{n+1} \quad \forall n \geq 1$.
- monotone if it's either increasing or decreasing

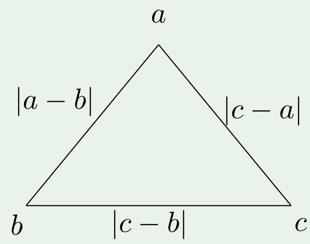
To define the notion of convergence of a sequence, we need a notion of distance between two real numbers.

Definition 10.4 (Absolute Value) — For $x \in \mathbb{R}$, the absolute value of x is

$$|x| = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$$

This function satisfies the following:

1. $|x| \geq 0 \quad \forall x \in \mathbb{R}$
2. $|x| = 0 \iff x = 0$
3. $|x + y| < |x| + |y| \quad \forall x, y \in \mathbb{R}$ (the triangle inequality)



$$\underbrace{|c - b|}_{x+y} \leq \underbrace{|c - a|}_{x} + \underbrace{|a - b|}_{y}$$

4. $|x \cdot y| = |x| \cdot |y| \quad \forall x, y \in \mathbb{R}$

Homework 10.1. $||x| - |y|| \leq |x - y| \quad \forall x, y \in \mathbb{R}$.

We think of $|x - y|$ as the distance between $x, y \in \mathbb{R}$.

Definition 10.5 (Convergent Sequence) — We say that a sequence $\{a_n\}_{n \geq 1}$ of real numbers converges if

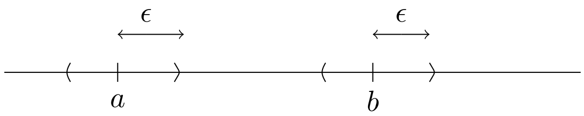
$$\exists a \in \mathbb{R} \ni \forall \epsilon > 0 \exists n_\epsilon \in \mathbb{N} \ni |a_n - a| < \epsilon \quad \forall n \geq n_\epsilon$$

We say that a is the limit of $\{a_n\}_{n \geq 1}$ and we write $a = \lim_{n \to \infty} a_n$ or $a_n \overset{n \to \infty}{\longrightarrow} a$

Lemma 10.6

The limit of a convergent sequence is unique.

Proof. We argue by contradiction. Assume that $\{a_n\}_{n \geq 1}$ is a convergent sequence and assume that there exist $a, b \in \mathbb{R}$ $a \neq b$ and $a = \lim_{n \to \infty} a_n$ and $b = \lim_{n \to \infty} a_n$.



Let $0 < \epsilon < \frac{|b-a|}{2}$ (we can choose such an ϵ because $\mathbb{Q}$ is dense in $\mathbb{R}$)

$$a = \lim_{n \to \infty} a_n \implies \exists n_1(\epsilon) \in \mathbb{N} \ni |a_n - a| < \epsilon \forall n \geq n_1(\epsilon)$$
$$b = \lim_{n \to \infty} a_n \implies \exists n_2(\epsilon) \in \mathbb{N} \ni |a_n - b| < \epsilon \forall n \geq n_2(\epsilon)$$

Set $n_\epsilon = \max\{n_1(\epsilon), n_2(\epsilon)\}$. Then for $n \geq n_\epsilon$ we have

$$|b-a| = |b-a_n+a_n-a| \leq \underbrace{|b-a_n|}_{<\epsilon} + \underbrace{|a_n-a|}_{<\epsilon} < 2\epsilon < |b-a|$$

Contradiction! □

Exercise 10.1. Show that the sequence given by $a_n = \frac{1}{n} \forall n \geq 1$ converges to 0.

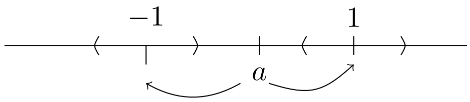
Proof. Let $\epsilon > 0$. By the Archemedean Property, $\exists n_\epsilon \in \mathbb{N} \ni n_\epsilon > \frac{1}{\epsilon}$. Then for $n \geq n_\epsilon$ we have

$$\left|0 - \frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{n_\epsilon} < \epsilon$$

By definition, $\lim_{n\to\infty} \frac{1}{n} = 0$. □

Exercise 10.2. Show that the sequence given by $a_n = (-1)^n \forall n \geq 1$ does not converge.

Proof. We argue by contradiction.



Assume $\exists a \in \mathbb{R}$ s.t. $a = \lim_{n\to\infty}(-1)^n$.
Let $0 < \epsilon < 1$. Then $\exists n_\epsilon \in \mathbb{N}$ s.t.

$$|a - (-1)^n| < \epsilon \quad \forall n \geq n_\epsilon$$

Taking $n = 2n_\epsilon$ we get $|a-1| < \epsilon$ and $n = 2n_\epsilon + 1$ we get $|a+1| < \epsilon$. By the triangle inequality,

$$2 = |1+1| = |1-a+a+1| \leq |1-a| + |a+1| < 2\epsilon < 2$$

Contradiction! □

Lemma 10.7

A convergent sequence is bounded.

Proof. Let $\{a_n\}_{n\geq 1}$ be a convergent sequence and let $a = \lim_{n\to\infty} a_n$.

$$\exists n_1 \in \mathbb{N} \ni |a - a_n| < 1 \quad \forall n \geq n_1$$

So $|a_n| \leq |a_n - a| + |a| < 1 + |a| \quad \forall n \geq n_1$. Let

$$M = \max\{1+|a|, |a_1|, |a_2|, \ldots, |a_{n_1} - 1|\}$$

Clearly, $|a_n| \leq M \quad \forall n \geq 1$ so $\{a_n\}_{n\geq 1}$ is bounded. □

Theorem 10.8

Let $\{a_n\}_{n\geq 1}$ be a convergent sequence and let $a = \lim_{n\to\infty} a_n$. Then for any $k \in \mathbb{R}$, the sequence $\{ka_n\}_{n\geq 1}$ converges and $\lim_{n\to\infty} ka_n = ka$.

Proof. If $k = 0$ then $ka_n = 0 \quad \forall n \geq 1$. So $\lim_{n\to\infty} ka_n = 0 = k \cdot a$
Assume $k \neq 0$. Let $\epsilon > 0$.
<u>Aside</u>: want to find $n_\epsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\epsilon$

$$|ka_n - ka| < \epsilon \iff |a_n - a| < \frac{\epsilon}{|k|}$$

As $a = \lim_{n\to\infty} a_n, \exists n_{\epsilon,k} \in \mathbb{N}$ s.t.

$$|a_n - a| < \frac{\epsilon}{|k|} \quad \forall n \geq n_{\epsilon,k}$$

So $|ka_n - ka| = |k| \cdot |a_n - a| < |k| \cdot \frac{\epsilon}{|k|} = \epsilon$. □

§11 Lec 11: Jan 29, 2021

§11.1 Convergent and Divergent Sequences

Theorem 11.1 (Properties of Convergent Sequences)
Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two convergent sequences of real numbers and let $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$. Then

1. the sequence $\{a_n + b_n\}_{n\geq 1}$ converges and $\lim_{n\to\infty}(a_n + b_n) = a + b$,
2. the sequence $\{a_n \cdot b_n\}$ converges and $\lim_{n\to\infty}(a_n b_n) = a \cdot b$,
3. if $a \neq 0$ and $a_n \neq 0 \forall n \geq 1$ then $\left\{\frac{1}{a_n}\right\}_{n\geq 1}$ converges and $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$,
4. if $a \neq 0$ and $a_n \neq 0 \forall n \geq 1$, then $\left\{\frac{b_n}{a_n}\right\}_{n\geq 1}$ converges and $\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{b}{a}$.
5. for any $k \in \mathbb{R}$, $\{ka_n\}_{n\geq 1}$ converges and $\lim_{n\to\infty} ka_n = ka$ (from theorem 10.8)

Proof. 1. Let $\epsilon > 0$.

Aside(Goal): Want to find $n_\epsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\epsilon$

$$|(a+b) - (a_n + b_n)| < \epsilon$$
$$|(a+b) - (a_n + b_n)| \leq \underbrace{|a - a_n|}_{<\frac{\epsilon}{2}} + \underbrace{|b - b_n|}_{<\frac{\epsilon}{2}} < \epsilon$$

Now back to the main proof, as $\lim_{n\to\infty} a_n = a, \exists n_1(\epsilon) \in \mathbb{N}$ s.t.

$$|a - a_n| < \frac{\epsilon}{2} \qquad \forall n \geq n_1(\epsilon)$$

As $\lim_{n\to\infty} b_n = b, \exists n_2(\epsilon) \in \mathbb{N}$ s.t.

$$|b - b_n| < \frac{\epsilon}{2} \qquad \forall n \geq n_2(\epsilon)$$

Let $n_\epsilon = \max\{n_1(\epsilon), n_2(\epsilon)\}$. Then for $n \geq n_\epsilon$ we have $|(a+b) - (a_b + b_n)| \leq |a - a_n| + |b - b_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. By definition, $\lim_{n\to\infty}(a_b + b_n) = a + b$.

2. Let $\epsilon > 0$.

Aside(Goal): Want to find $n_\epsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\epsilon$

$$|ab - a_n b_n| < \epsilon$$
$$|ab - a_n b_n| = |(a - a_n)b + a_n(b - b_n)|$$
$$\leq \underbrace{|a - a_n| \cdot |b|}_{<\frac{\epsilon}{2}} + \underbrace{|a_n|\,|b - b_n|}_{<\frac{\epsilon}{2}} < \epsilon$$

Take $|a - a_n| < \frac{\epsilon}{2(|b|+1)}$. Take $M > 0$ s.t. $|a_n| \leq M \forall n \geq 1$

$$|b - b_n| < \frac{\epsilon}{2M}$$

Now, back to the main proof, as $\{a_n\}_{n\geq 1}$ converges, it is bounded. Let $M > 0$ such that $|a_n| \leq M \ \forall n \geq 1$. As $\lim_{n\to\infty} a_n = a, \exists n_1(\epsilon) \in \mathbb{N}$ s.t.

$$|a - a_n| < \frac{\epsilon}{2(|b|+1)} \qquad \forall n \geq n_1(\epsilon)$$

As $\lim_{n\to\infty} b_n = b, \exists n_2(\epsilon) \in \mathbb{N}$ s.t.

$$|b - b_n| < \frac{\epsilon}{2M} \qquad \forall n \geq n_2(\epsilon)$$

Set $n_\epsilon = \max\{n_1(\epsilon), n_2(\epsilon)\}$. For $n \geq n_\epsilon$ we have

$$\begin{aligned} |ab - a_n b_n| &= |(a - a_n)b + a_n(b - b_n)| \\ &\leq |a - a_n|\,|b| + |a_n|\,|b - b_n| \\ &< \frac{\epsilon}{2(|b|+1)} \cdot |b| + M \cdot \frac{\epsilon}{2M} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

By definition, $\lim_{n\to\infty}(a_n b_n) = ab$.

3. Let $\epsilon > 0$.

<u>Aside</u>(Goal): Want to find $n_\epsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\epsilon$

$$\begin{aligned} \left|\frac{1}{a} - \frac{1}{a_n}\right| &< \epsilon \\ \left|\frac{1}{a} - \frac{1}{a_n}\right| &= \frac{|a_n - a|}{|a| \cdot |a_n|} < \epsilon \\ |a_n - a| &< \epsilon |a| \cdot |a_n| \qquad (\text{!!! -- NONONO}) \end{aligned}$$

Now, back to the proof, as $a = \lim_{n\to\infty} a_n, \exists n_1(a) \in \mathbb{N}$ s.t.

$$|a - a_n| < \frac{|a|}{2} \qquad \forall n \geq n_1$$

Then, for all $n \geq n_1$ we have

$$|a_n| \geq |a| - |a - a_n| > |a| - \frac{|a|}{2} = \frac{|a|}{2}$$

As $a = \lim_{n\to\infty} a_n, \exists n_2(\epsilon, a)$ s.t.

$$|a - a_n| < \frac{\epsilon |a|^2}{2} \qquad \forall n \geq n_2(\epsilon, a)$$

Let $n_\epsilon = \max\{n_1(a), n_2(\epsilon, a)\}$. For $n \geq n_\epsilon$ we have

$$\left|\frac{1}{a} - \frac{1}{a_n}\right| = \frac{|a - a_n|}{|a| \cdot |a_n|} < \frac{\epsilon |a|^2}{2|a|} \cdot \frac{2}{|a|} = \epsilon$$

By definition, $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$.

□

Example 11.2

Find the limit of

$$\lim_{n\to\infty} \frac{n^3 + 5n + 8}{3n^3 + 2n^2 + 7}$$

which can rewritten as

$$\lim_{n\to\infty} \frac{1 + \frac{5}{n^2} + \frac{8}{n^3}}{3 + \frac{2}{n} + \frac{7}{n^3}} = \frac{1 + 5\lim \frac{1}{n^2} + 8 \lim \frac{1}{n^3}}{3 + 2\lim \frac{1}{n} + 7 \lim \frac{1}{n^3}}$$

which is equivalent to

$$= \frac{1 + 5 \cdot 0 + 8 \cdot 0}{3 + 2 \cdot 0 + 7 \cdot 0} = \frac{1}{3}$$

Theorem 11.3 (Monotone Convergence)
Every bounded monotone sequence converges.

Proof. We'll show that an increasing sequence bounded above converges. A similar argument can be used to show that a decreasing sequence bounded below converges. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers that is bounded above and $a_{n+1} \geq a_n \quad \forall n \geq 1$. As $\emptyset \neq \{a_n : n \geq 1\} \subseteq \mathbb{R}$ is bounded above and $\mathbb{R}$ has the least upper bound property, $\exists a \in \mathbb{R}$ s.t. $a = \sup\{a_n : n \geq 1\}$.

Claim 11.1. $a = \lim_{n\to\infty} a_n$.

Let $\epsilon > 0$. Then $a - \epsilon$ is not an upper bound for $\{a_n : n \geq 1\} \implies \exists n_\epsilon \in \mathbb{N}$ s.t. $a - \epsilon < a_{n_\epsilon}$. Then for $n \geq n_\epsilon$ we have

$$a - \epsilon < a_{n_\epsilon} \leq a_n \leq a < a + \epsilon \iff |a_n - a| < \epsilon$$

This proves the claim. □

Homework 11.1. Prove for the decreasing sequence.

Definition 11.4 (Divergent Sequence) — Let $\{a_n\}$ be a sequence of real numbers. We write $\lim_{n\to\infty} a_n = \infty$ and say that a_n diverges to $+\infty$ if $\forall M > 0, \quad \exists n_M \in \mathbb{N}$ s.t. $a_n > M \quad \forall n \geq n_M$.
We write $\lim_{n\to\infty} a_n = -\infty$ and say that a_n diverges to $-\infty$ if $\forall M < 0 \quad \exists n_M \in \mathbb{N}$ s.t. $a_n < M \quad \forall n \geq n_M$.

Homework 11.2.

1. Show that $\lim_{n\to\infty}(\sqrt[3]{n} + 1) = \infty$.
2. Show that the sequence given by $a_n = (-1)^n n \quad \forall n \geq 1$ does not diverge to ∞ or to $-\infty$.
3. Let $\{a_n\}_{n\geq 1}$ be a sequence of positive real numbers. Show that

$$\lim_{n\to\infty} a_n = 0 \iff \lim_{n\to\infty} \frac{1}{a_n} = \infty$$

§12 Lec 12: Feb 1, 2021

Example 12.1

Show that $\lim_{n\to\infty} \frac{n^2+1}{n+3} = \infty$.
<u>Aside</u>: Want to find $n_M \in \mathbb{N}$ s.t. $\forall n \geq n_M$ we have

$$\frac{n^2+1}{n+3} > M$$

So

$$\frac{n^2+1}{n+3} > \frac{n^2}{n+3} > \frac{n^2}{4n} = \frac{n}{4} > M$$

Now, back to the main proof, let $M > 0$. By the Archimedean property there exists $n_M \in \mathbb{N}$ s.t.

$$n_M > 4M$$

Then for $n \geq n_M$ we have

$$\frac{n^2+1}{n+3} > \frac{n^2}{n+3} > \frac{n^2}{4n} = \frac{n}{4} \geq \frac{n_M}{4} > M$$

By the definition, $\lim_{n\to\infty} \frac{n^2+1}{n+3} = \infty$.

§12.1 Cauchy Sequences

Definition 12.2 (Cauchy Sequence) — We say that a sequence of real numbers $\{a_n\}_{n\geq 1}$ is a Cauchy sequence if

$$\forall \epsilon > 0 \quad \exists n_\epsilon \in \mathbb{N} \quad \text{s.t. } |a_n - a_m| < \epsilon \quad \forall n, m \geq n_\epsilon$$

Theorem 12.3 (Cauchy Criterion - Sequence)
A sequence of real numbers is Cauchy if and only if it converges.

We will split the proof of this theorem into various lemmas and propositions.

Proposition 12.4
Any convergent sequence is a Cauchy sequence.

Proof. Let $\{a_n\}_{n\geq 1}$ be a convergent sequence and let $a = \lim_{n\to\infty} a_n$. Let $\epsilon > 0$. As $a_n \overset{n\to\infty}{\longrightarrow} a$, $\exists n_\epsilon \in \mathbb{N}$ s.t.

$$|a - a_n| < \frac{\epsilon}{2} \quad \forall n \geq n_\epsilon$$

Then for $n, m \geq n_\epsilon$, we have

$$|a_n - a_m| \leq |a_n - a| + |a - a_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

Lemma 12.5

A Cauchy sequence is bounded.

Proof. Let $\{a_n\}_{n\geq 1}$ be a Cauchy sequence. Then $\exists n_1 \in \mathbb{N}$ s.t. $|a_n - a_m| < 1 \quad \forall n, m \geq n_1$. So, taking $m = n_1$, we get

$$|a_n| \leq |a_{n_1}| + |a_n - a_{n_1}| < |a_{n_1}| + 1 \quad \forall n \geq n_1$$

Let $M = \max\{|a_1|, |a_2|, \ldots, |a_{n_1-1}|, |a_{n_1} + 1|\}$. Clearly, $|a_n| \leq M \quad \forall n \geq 1$. □

Definition 12.6 (Subsequence) — Let $\{k_n\}_{n\geq 1}$ be a sequence of natural numbers s.t. $k_1 \geq 1$ and $k_{n+1} > k_n \quad \forall n \geq 1$. Using induction, it's easy to see that $k_n \geq n \quad \forall n \geq 1$. If $\{a_n\}_{n\geq 1}$ is a sequence, we say that $\{a_{k_n}\}_{n\geq 1}$ is a subsequence of $\{a_n\}_{n\geq 1}$.

Example 12.7

The following are subsequences of $\{a_n\}_{n\geq 1}$:

$$\{a_{2n}\}_{n\geq 1}, \{a_{2n-1}\}_{n\geq 1}, \{a_{n^2}\}_{n\geq 1}, \{a_{p_n}\}_{n\geq 1}$$

where p_n denotes the n^{th} prime.

Theorem 12.8

Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. Then $\lim_{n\to\infty} a_n = a \in \mathbb{R} \cup \{\pm\infty\}$ if and only if every subsequence $\{a_{k_n}\}_{n\geq 1}$ of $\{a_n\}_{n\geq 1}$ satisfies $\lim_{n\to\infty} a_{k_n} = a$.

Proof. We will consider $a \in \mathbb{R}$. The cases $a \in \{\pm\infty\}$ can be handled by analogous arguments.

" $\Longleftarrow$ " Take $k_n = n \quad \forall n \geq 1$

" $\Longrightarrow$ " Assume $\lim_{n\to\infty} a_n = a$ and let $\{a_{k_n}\}_{n\geq 1}$ be a subsequence of $\{a_n\}_{n\geq 1}$. Let $\epsilon > 0$. As $a_n \overset{n\to\infty}{\longrightarrow} a, \quad \exists n_\epsilon \in \mathbb{N}$ s.t.

$$|a - a_n| < \epsilon \quad \forall n \geq n_\epsilon$$

Recall that $k_n \geq n \, \forall n \geq 1$. So for $n \geq n_\epsilon$ we have $k_n \geq n \geq n_\epsilon$ and so

$$|a - a_{k_n}| < \epsilon \quad \forall n \geq n_\epsilon$$

By definition,

$$\lim_{n\to\infty} a_{k_n} = a$$ □

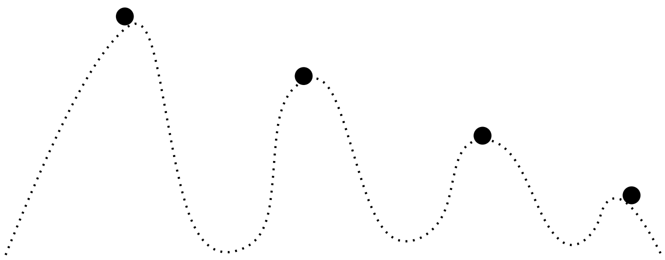
Proposition 12.9

Every sequence of real numbers has a monotone subsequence.

Proof. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. We say that the n^{th} term is <u>dominant</u> if

$$a_n > a_m \quad \forall m > n$$

We distinguished 2 cases:
<u>Case 1</u>: There are infinitely many dominant terms:



Then a subsequence formed by these dominant terms is strictly decreasing.
<u>Case 2</u>: There are none or finitely many dominant terms. Let N be larger than the largest index of the dominant terms. So $\forall n \geq N$ a_n is not dominant. Set $k_1 = N$, $a_{k_1} = a_N$. a_{k_1} is not dominant $\implies \exists k_2 > k_1$ s.t. $a_{k_2} \geq a_{k_1}$, $k_2 > k_1 = N \implies a_{k_2}$ is not dominant $\implies \exists k_3 > k_2$ s.t. $a_{k_3} \geq a_{k_2}$. Proceeding inductively we construct a subsequence $\{a_{k_n}\}_{n\geq 1}$ s.t.

$$a_{k_{n+1}} \geq a_{k_n} \quad \forall n \geq 1$$

□

Theorem 12.10 (Bolzano – Weierstrass)
Any bounded sequence has a convergent subsequence.

Proof. Let $\{a_n\}_{n\geq 1}$ be a bounded sequence. By the previous proposition, there exists $\{a_{k_n}\}_{n\geq 1}$ monotone subsequence of $\{a_n\}_{n\geq 1}$. As $\{a_n\}_{n\geq 1}$ is bounded, so is $\{a_{k_n}\}_{n\geq 1}$. As bounded monotone sequences converge, $\{a_{k_n}\}_{n\geq 1}$ converges. □

Corollary 12.11
Every Cauchy sequence has a convergent subsequence.

Lemma 12.12
A Cauchy sequence with a convergent subsequence converges.

Proof. Let $\{a_n\}_{n\geq 1}$ be a Cauchy sequence s.t. $\{a_{k_n}\}_{n\geq 1}$ is a convergent subsequence. Let $a = \lim_{n\to\infty} a_{k_n}$. Let $\epsilon > 0$. As $a_{k_n} \overset{n\to\infty}{\longrightarrow} a$, $\exists n_1(\epsilon)$ s.t. $|a - a_{k_n}| < \frac{\epsilon}{2}\, \forall n \geq n_1(\epsilon)$. As $\{a_n\}_{n\geq 1}$ is Cauchy, $\exists n_2(\epsilon)$ s.t. $|a_n - a_m| < \frac{\epsilon}{2}\, \forall n, m \geq n_2(\epsilon)$. Let $n_\epsilon = \max\{n_1(\epsilon), n_2(\epsilon)\}$. Then for $n \geq n_\epsilon$ we have

$$|a - a_n| \leq |a - a_{k_n}| + |a_{k_n} - a_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for $k_n \geq n \geq n_\epsilon$. By definition,

$$\lim_{n\to\infty} a_n = a$$

Combining the last two results, we see that a Cauchy sequence of real numbers converges.

□

§13 Lec 13: Feb 3, 2021

§13.1 Limsup and Liminf

Let $\{a_n\}_{n\geq 1}$ be a bounded sequence of real numbers (convergent or not). The asymptotic behavior of $\{a_n\}_{n\geq 1}$ depends on sets of the form $\{a_n : n \geq N\}$ for $N \in \mathbb{N}$.

As $\{a_n\}_{n\geq 1}$, the set $\{a_n : n \geq N\}$ (where $N \in \mathbb{N}$ is fixed) is a non-empty bounded subset of $\mathbb{R}$.

As $\mathbb{R}$ has the least upper bound property (and so also the greatest lower bound property), the set $\{a_n : n \geq N\}$ has an infimum and a supremum in $\mathbb{R}$.

For $N \geq 1$, let $u_N = \inf\{a_n : n \geq N\}$ and $v_N = \sup\{a_n : n \geq N\}$. Clearly, $u_N \leq v_N \quad \forall N \geq 1$. For $N \geq 1$, $\{a_n : n \geq N\} \supseteq \{a_n : n \geq N+1\}$

$$\implies \begin{cases} \inf\{a_n : n \geq N\} \leq \inf\{a_n : n \geq N+1\} \\ \sup\{a_n : n \geq N\} \geq \sup\{a_n : n \geq N+1\} \end{cases}$$

So $u_N \leq u_{N+1}$ and $v_{N+1} \leq v_N \quad \forall N \geq 1$. Thus $\{u_N\}_{N\geq 1}$ is increasing and $\{v_N\}_{N\geq 1}$ is decreasing. Moreover, $\forall N \geq 1$ we have

$$u_1 \leq u_2 \leq \ldots \leq u_N \leq v_N \leq \ldots \leq v_2 \leq v_1$$

So the sequences $\{u_N\}_{N\geq 1}$ and $\{v_N\}_{N\geq 1}$ are bounded. As monotone bounded sequence converges, $\{u_N\}_{N\geq 1}$ and $\{v_N\}_{N\geq 1}$ must converge.

Let

$$u = \lim_{N\to\infty} u_N = \sup\{u_N : N \geq 1\} := \sup_N u_N$$
$$v = \lim_{N\to\infty} v_N = \inf\{v_N : N \geq 1\} := \inf_N v_N$$

From (*), we see that

$$\begin{aligned} & u_M \leq v_N \qquad \forall M, N \geq 1 \\ \implies & \lim_{M\to\infty} u_M \leq v_N \qquad \forall N \geq 1 \\ \implies & u \leq v_N \qquad \forall N \geq 1 \\ \implies & u \leq \lim_{N\to\infty} v_N \\ \implies & u \leq v \end{aligned}$$

Moreover, if $\lim_{n\to\infty} a_n$ exists, then for all $N \geq 1$, we have

$$u_N = \inf\{a_n : n \geq N\} \leq a_n \leq \sup\{a_n : n \geq N\} = v_N \quad \forall n \geq N$$

So

$$\begin{aligned} & \implies u_N \leq \lim_{n\to\infty} a_n \leq v_N \\ & \implies u = \lim_{N\to\infty} u_N \leq \lim_{n\to\infty} a_n \leq \lim_{N\to\infty} v_N = v \end{aligned}$$

Definition 13.1 (lim sup and lim inf) — Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. We define

$$\limsup_{n\to\infty} a_n = \lim_{N\to\infty} \sup\{a_n : n \geq N\} = \lim_{N\to\infty} v_N = \inf_N v_N = \inf_N \sup_{n\geq N} a_n$$
$$\liminf_{n\to\infty} a_n = \lim_{N\to\infty} \inf\{a_n : n \geq N\} = \lim_{N\to\infty} u_N = \sup_N u_N = \sup_N \inf_{n\geq N} a_n$$

with the convention that if $\{a_n\}_{n\geq 1}$ is unbounded above then

$$\limsup_{n\to\infty} a_n = \infty$$

and if $\{a_n\}_{n\geq 1}$ is unbounded below then

$$\liminf_{n\to\infty} a_n = -\infty$$

Remark 13.2.

$$\inf\{a_n : n \geq 1\} \leq \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n \leq \sup\{a_n : n \geq 1\}$$

where $\liminf_{n\to\infty} a_n$ is the smallest value that infinitely many a_n get close to and $\limsup_{n\to\infty} a_n$ is the largest value that infinitely many a_n get close to.

Example 13.3

$a_n = 3 + \frac{(-1)^n}{n} \implies \lim_{n\to\infty} a_n = 3 \implies \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = 3$

$$\inf\{a_n : n \geq 1\} = 2 \neq 3$$
$$\sup\{a_n : n \geq 1\} = \frac{7}{2} \neq 3$$

Theorem 13.4

Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers.

1. If $\lim_{n\to\infty} a_n$ exists in $\mathbb{R} \cup \{\pm\infty\}$, then $\liminf a_n = \limsup a_n = \lim_{n\to\infty} a_n$.
2. If $\liminf a_n = \limsup a_n \in \mathbb{R} \cup \{\pm\infty\}$, then $\lim_{n\to\infty} a_n$ exists and

$$\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$$

Proof. 1. We distinguish three cases.

Case i) $\lim_{n\to\infty} a_n = -\infty$. It's enough to show $\limsup a_n = -\infty$ since $\liminf a_n \leq \limsup a_n$. Fix $M < 0$. As $\lim_{n\to\infty} a_n = -\infty$, $\exists n_M \in \mathbb{N}$ s.t. $a_n < M \quad \forall n \geq n_M$. Then for $N \geq n_M$, we have $v_N = \sup\{a_n : n \geq N\} \leq M$. Note that when taking sup(inf), $<$ can become $\leq$; e.g. $a_n = 3 - \frac{1}{n}$ where $a_n < 3 \quad \forall n \geq 1$ but $\sup_{n\geq 1} a_n = 3$.

By definition, $\limsup_{n\to\infty} a_n = \lim_{N\to\infty} v_N = -\infty$.

Case ii) $\lim_{n\to\infty} a_n = \infty$ Exercise

Case iii) $\lim_{n\to\infty} a_n = a \in \mathbb{R}$.

Fix $\epsilon > 0$. Then $\exists n_\epsilon \in \mathbb{N}$ s.t. $|a - a_n| < \epsilon \quad \forall n \geq n_\epsilon$. So

$$a - \epsilon < a_n < a + \epsilon \qquad \forall n \geq n_\epsilon$$

Thus for $N \geq n_\epsilon$ we have

$$a - \epsilon \leq \inf\{a_n : n \geq N\} \leq \sup\{a_n : n \geq N\} \leq a + \epsilon$$
$$a - \epsilon \leq u_N \leq v_N \leq a + \epsilon$$

So

$$\forall N \geq n_\epsilon \begin{cases} |u_N - a| \leq \frac{\epsilon}{2} < \epsilon \\ |v_N - a| \leq \frac{\epsilon}{2} < \epsilon \end{cases}$$

By definition,

$$\begin{cases} \liminf a_n = \lim_{N\to\infty} u_N = a \\ \limsup a_n = \lim_{N\to\infty} v_N = a \end{cases}$$

2. We distinguish three cases.

Case i) $\liminf a_n = \limsup a_n = -\infty$.

We will use $\limsup a_n = -\infty$. Fix $M < 0$. Then since $\limsup a_n = \lim_{N\to\infty} v_N = -\infty$, $\exists N_M \in \mathbb{N}$ s.t. $v_N < M \quad \forall N \geq N_M$. In particular, $v_{N_M} = \sup\{a_n : n \geq N_M\} < M$

$$\implies a_n < M \qquad \forall n \geq N_M$$

By definition, $\lim_{n\to\infty} a_n = -\infty$.

Case ii) $\liminf a_n = \limsup a_n = \infty$ — exercise

Case iii) $\liminf a_n = \limsup a_n = a \in \mathbb{R}$.

Fix $\epsilon > 0$.

$$a = \liminf a_n = \lim_{N\to\infty} u_N \implies \exists N_1(\epsilon) \in \mathbb{N} \ni |u_N - a| < \epsilon \quad \forall N \geq N_1$$

So $a - \epsilon < u_{N_1} = \inf\{a_n : n \geq N_1\} < a + \epsilon$

$$\implies a - \epsilon < a_n \qquad \forall n \geq N_1$$

And

$$a = \limsup a_n = \lim_{N\to\infty} v_N \implies \exists N_2(\epsilon) \in \mathbb{N} \ni |v_N - a| < \epsilon \quad \forall N \geq N_2$$

So $a - \epsilon < v_{N_2} = \sup\{a_n : n \geq N_2\} < a + \epsilon$.

$$\implies a_n < a + \epsilon \qquad \forall n \geq N_2$$

Thus for $n \geq \max\{N_1, N_2\}$ we have

$$a - \epsilon < a_n < a + \epsilon \iff |a_n - a| < \epsilon$$

By definition, $\lim_{n\to\infty} a_n = a$.

□

§14 Lec 14: Feb 5, 2021

§14.1 Limsup and Liminf (Cont'd)

Recall: For a sequence $\{a_n\}_{n\geq 1}$ of real numbers, we define

$$\liminf a_n = \sup_N \inf_{n\geq N} a_n = \lim_{N\to\infty} u_N \text{ where } u_N = \inf\{a_n : n \geq N\}$$
$$\limsup a_n = \inf_N \sup_{n\geq N} a_n = \lim_{N\to\infty} v_N \text{ where } v_N = \sup\{a_n : n \geq N\}$$

Last time, we proved that

$$\lim_{n\to\infty} a_n \text{ exists in } \mathbb{R}\cup\{\pm\infty\} \iff \liminf a_n = \limsup a_n$$

Theorem 14.1

Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. Then there exists a monotonic subsequence of $\{a_n\}_{n\geq 1}$ whose limit is $\limsup a_n$. Also, there exists a monotonic subsequence of $\{a_n\}_{n\geq 1}$ whose limit is $\liminf a_n$.

Proof. We will prove the statement about $\limsup a_n$. Similar arguments can be used to prove the statement about $\liminf a_n$. HW!

Note that it suffices to find a subsequence of $\{a_{k_n}\}_{n\geq 1}$ of $\{a_n\}_{n\geq 1}$ s.t.

$$\lim_{n\to\infty} a_{k_n} = \limsup a_n$$

As every sequence has a monotone subsequence, $\{a_{k_n}\}_{n\geq 1}$ has a monotone subsequence $\{a_{p_{k_n}}\}_{n\geq 1}$. Then as $\lim a_{k_n}$ exists, $\lim_{n\to\infty} a_{p_{k_n}}$ exists and

$$\lim_{n\to\infty} a_{p_{k_n}} = \lim a_{k_n} = \limsup a_n$$

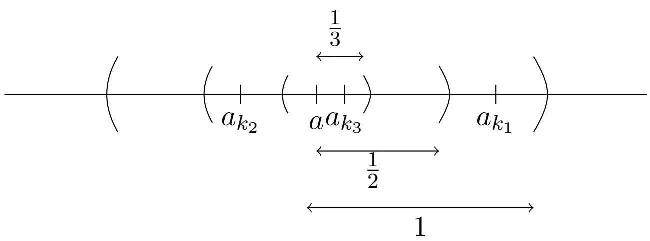
Finally, note that $\{a_{p_{k_n}}\}_{n\geq 1}$ is a subsequence of $\{a_n\}_{n\geq 1}$.

Let's find a subsequence of $\{a_n\}_{n\geq 1}$ whose limit is $\limsup a_n$.

Case 1: $\limsup a_n = -\infty$.

We showed that in this case, $\lim_{n\to\infty} a_n = -\infty$. Choose $\{a_{k_n}\}_{n\geq 1}$ to be $\{a_n\}_{n\geq 1}$.

Case 2: $\limsup a_n = a \in \mathbb{R}$.



$$a = \limsup a_n = \lim_{N\to\infty} v_N$$

Then $\exists N_1 \in \mathbb{N}$ s.t. $|a - v_N| < 1 \quad \forall N \geq N_1$. In particular,

$$\begin{aligned} & a-1 < v_{N_1} < a+1 \\ \implies & a-1 < \sup\{a_n : n \geq N_1\} \\ \implies & \exists k_1 \geq N_1 \ni \quad a-1 < a_{k_1} \\ \implies & a-1 < a_{k_1} < v_{N_1} < a+1 \end{aligned}$$

So $|a - a_{k_1}| < 1$.
As $a = \lim_{N\to\infty} v_N$, $\exists N_2 \in \mathbb{N}$ s.t. $|a - v_N| < \frac{1}{2} \quad \forall N \geq N_2$.
Let $\tilde{N}_2 = \max\{N_2, k_1 + 1\}$
In particular,

$$\left.\begin{array}{l} a - \frac{1}{2} < v_{\tilde{N}_2} < a + \frac{1}{2} \\ a - \frac{1}{2} < \sup\left\{a_n : n \geq \tilde{N}_2\right\} \\ \exists k_2 \geq \tilde{N}_2 \text{ s.t. } a - \frac{1}{2} < a_{k_2} \end{array}\right\} \implies a - \frac{1}{2} < a_{k_2} \leq v_{N_2} < a + \frac{1}{2}$$

So, $|a - a_{k_2}| < \frac{1}{2}$. To construct our subsequence we proceed inductively. Assume we have found $k_1 < k_2 < \ldots < k_n$ and $a_{k_1}, \ldots, a_{k_n}$ s.t.

$$\left|a - a_{k_j}\right| < \frac{1}{j} \quad \forall 1 \leq j \leq n$$

As $a = \lim_{N\to\infty} v_N \implies \exists N_{n+1} \in \mathbb{N}$ s.t. $|a - v_N| < \frac{1}{n+1} \quad \forall N \geq N_{n+1}$. Let $\tilde{N}_{n+1} = \max\{N_{n+1}, k_n + 1\}$. Then

$$\begin{aligned} & a - \frac{1}{n+1} < v_{\tilde{N}_{n+1}} < a + \frac{1}{n+1} \\ \implies & a - \frac{1}{n+1} < \sup\left\{a_n : n \geq \tilde{N}_{n+1}\right\} \\ \implies & \exists k_{n+1} \geq \tilde{N}_{n+1} > k_n \text{ s.t. } a - \frac{1}{n+1} < a_{k_{n+1}} \\ \implies & a - \frac{1}{n+1} < a_{k_{n+1}} \leq v_{\tilde{N}_{n+1}} < a + \frac{1}{n+1} \\ \implies & |a_{k_{n+1}} - a| < \frac{1}{n+1} \end{aligned}$$

Case 3: $\limsup a_n = \infty$. □ HW!

Definition 14.2 (Subsequential Limit) — Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. A subsequential limit of $\{a_n\}_{n\geq 1}$ is any $a \in \mathbb{R} \cup \{\pm\infty\}$ that is the limit of a subsequence of $\{a_n\}_{n\geq 1}$.

Example 14.3

1. $a_n = n\left(1 + (-1)^n\right)$

   The subsequential limits are

   $$0 = \lim_{n\to\infty} a_{2n+1}$$
   $$\infty = \lim_{n\to\infty} a_{2n}$$

2. $a_n = \cos\left(\frac{n\pi}{3}\right)$

   The subsequential limits are

   $$1 = \lim_{n\to\infty} a_{6n}$$
   $$\frac{1}{2} = \lim_{n\to\infty} a_{6n+1} = \lim_{n\to\infty} a_{6n+5}$$
   $$-\frac{1}{2} = \lim_{n\to\infty} a_{6n+2} = \lim_{n\to\infty} a_{6n+4}$$
   $$-1 = \lim_{n\to\infty} a_{6n+3}$$

Theorem 14.4

Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers and let A denote its set of subsequential limits:

$$A = \left\{a \in \mathbb{R}\cup\{\pm\infty\} : \exists \{a_{k_n}\}_{n\geq 1} \text{ subsequence of } \{a_n\}_{n\geq 1} \text{ s.t. } \lim_{n\to\infty} a_{k_n} = a\right\}$$

Then:

1. $A \neq \emptyset$.
2. $\lim_{n\to\infty} a_n$ exists (in $\mathbb{R}\cup\{\pm\infty\}$) $\iff$ A has exactly one element.
3. $\inf A = \liminf a_n$ and $\sup A = \limsup a_n$.

Proof.

1. By the previous theorem, $\liminf a_n, \limsup a_n \in A$. So $A \neq \emptyset$.
2. "$\implies$" Assume $\lim_{n\to\infty} a_n$ exists. Then if $\{a_{k_n}\}_{n\geq 1}$ is a subsequence of $\{a_n\}_{n\geq 1}$, we have

   $$\lim_{n\to\infty} a_{k_n} = \lim_{n\to\infty} a_n$$

   So $A = \{\lim_{n\to\infty} a_n\}$.

   "$\impliedby$" If A has a single element, $\liminf a_n = \limsup a_n$ and so $\lim_{n\to\infty} a_n$ exists.

3. We will prove

   Claim 14.1. $\liminf a_n \leq a \leq \limsup a_n \quad \forall a \in A$.

   Assuming the claim, let's see how to finish the proof. The claim implies

   - $\liminf a_n$ is a lower bound for $A \implies \liminf a_n \leq inf A$. On the other hand, $\liminf a_n \in A \implies \liminf a_n \geq \inf A$. Thus, $\liminf a_n = \inf A$.
   - $\limsup a_n$ is an upper bound for $A \implies \limsup a_n \geq \sup A$. But $\limsup a_n \in A \implies \limsup a_n \leq \sup A$. Thus, $\limsup a_n = \sup A$.

Let's prove the claim. Fix $a \in A \implies \exists \{a_{k_n}\}_{n\geq 1}$ subsequence of $\{a_n\}_{n\geq 1}$ s.t. $\lim_{n\to\infty} a_{k_n} = a$.

$$
\begin{aligned}
&\{a_n : n \geq N\} \supset \{a_{k_n} : n \geq N\} \\
&\implies \underbrace{\inf\{a_n : n \geq N\}}_{\text{increasing seq}} \leq \underbrace{\inf\{a_{k_n} : n \geq N\}}_{\text{increasing seq}} \leq \underbrace{\sup\{a_{k_n} : n \geq N\}}_{\text{deceasing seq}} \leq \underbrace{\sup\{a_n : n \geq N\}}_{\text{decreasing seq}} \\
&\implies \lim_{N\to\infty} \inf\{a_n : n \geq N\} \leq \lim_{N\to\infty} \inf\{a_{k_n} : n \geq N\} \leq \lim_{N\to\infty} \sup\{a_{k_n} : n \geq N\} \\
&\qquad \leq \lim_{N\to\infty} \sup\{a_n : n \geq N\} \\
&\implies \liminf a_n \leq \underbrace{\liminf a_{k_n}}_{=\lim a_{k_n}=a} \leq \underbrace{\limsup a_{k_n}}_{=\lim a_{k_n}=a} \leq \limsup a_n
\end{aligned}
$$

□

§15 Lec 15: Feb 8, 2021

§15.1 Limsup and Liminf (Cont'd)

Theorem 15.1 (Cesaro – Stolz)
Let $\{a_n\}_{n\geq 1}$ be a sequence of non-zero real numbers. Then

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \overset{1)}{\leq} \liminf |a_n|^{\frac{1}{n}} \overset{2)}{\leq} \limsup |a_n|^{\frac{1}{n}} \overset{3)}{\leq} \limsup \left|\frac{a_{n+1}}{a_n}\right|$$

In particular, if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists then $\lim_{n\to\infty} |a_n|^{\frac{1}{n}}$ exists and

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

Example 15.2
Find $\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{\frac{1}{n}}$.
If we let $a_n = n$ then $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{n} \overset{n\to\infty}{\longrightarrow} 1$. By Cesaro – Stolz, we get $\{\sqrt[n]{n}\}_{n\geq 1}$ converges and

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

Proof. We will prove inequality 3). Analogous arguments yield inequality 1). Let

$$l = \limsup |a_n|^{\frac{1}{n}} \geq 0$$
$$L = \limsup \left|\frac{a_{n+1}}{a_n}\right| \geq 0$$

We want to show $l \leq L$. If $L = \infty$, then it's clear. Henceforth we assume $L \in \mathbb{R}$. We will prove

Claim 15.1. l is a lower bound for the set

$$(L, \infty) = \{M \in \mathbb{R} : M > L\}$$

Assuming the claim for now, let's see how to finish the proof. Note (L, ∞) is a non-empty subset of $\mathbb{R}$ which is bounded below (by L). As $\mathbb{R}$ has the least upper bound property, $\inf(L, \infty)$ exists in $\mathbb{R}$. In fact,

$$\inf(L, \infty) = L$$

As l is a lower bound for (L, ∞), we must have $l \leq L$.
Let's prove the claim. Fix $M \in (L, \infty)$. We will show

$$l \leq M$$

We have $M > L = \limsup \left|\frac{a_{n+1}}{a_n}\right| = \inf_N \sup_{n\geq N} \left|\frac{a_{n+1}}{a_n}\right|$.

$$\implies \exists N_0 \in \mathbb{N} \ni \sup_{n\geq N_0} \left|\frac{a_{n+1}}{a_n}\right| < M$$
$$\implies \left|\frac{a_{n+1}}{a_n}\right| < M \quad \forall n \geq N_0$$
$$\implies |a_{n+1}| < M \cdot |a_n| \quad \forall n \geq N_0$$

A simple inductive argument yields

$$\begin{aligned}
&|a_n| < M^{n-N_0}\,|a_{N_0}| \quad \forall n > N_0 \\
&\implies |a_n|^{\frac{1}{n}} < M\left(\frac{|a_{N_0}|}{M^{N_0}}\right)^{\frac{1}{n}} \quad \forall n > N_0 \\
&\implies l = \limsup |a_n|^{\frac{1}{n}} \le \limsup M\cdot\left(\frac{|a_{N_0}|}{M^{N_0}}\right)^{\frac{1}{n}} = M\cdot\limsup\left(\frac{|a_{N_0}|}{M^{N_0}}\right)^{\frac{1}{n}} \qquad (*)
\end{aligned}$$

Claim 15.2. For $r>0$ we have $\lim_{n\to\infty} r^{\frac{1}{n}} = 1$

Indeed, if $r \ge 1$

$$0 \le r^{\frac{1}{n}} - 1 = \frac{r-1}{r^{n-1}+r^{n-2}+\ldots+1} \le \frac{r-1}{n} \overset{n\to\infty}{\longrightarrow} 0$$

where we use the formula $a^n - b^n = (a-b)\left(a^{n-1}+a^{n-2}b+\ldots+ab^{n-2}+b^{n-1}\right)$. If $r<1$, then

$$r^{\frac{1}{n}} = \frac{1}{\left(\frac{1}{r}\right)^{\frac{1}{n}}} \overset{n\to\infty}{\longrightarrow} \frac{1}{1} = 1$$

Taking $r = \frac{|a_{N_0}|}{M^{N_0}}$ in (*) we conclude that

$$l \le M$$

□

§15.2 Series

Definition 15.3 (Convergent/Absolutely Convergent Series) — Let $\{a_n\}_{n\ge1}$ be a sequence of real numbers. For $n\ge1$, we define the partial sum

$$s_n = a_1 + \ldots + a_n$$

The series $\sum_{n=1}^{\infty} a_n$ $\left(\sum_{n\ge1} a_n\right)$ is said to converge if $\{s_n\}_{n\ge1}$ converges.
We say that the series $\sum_{n=1}^{\infty} a_n$ converges absolutely if the series $\sum_{n=1}^{\infty}|a_n|$ converges.
(Note that $\sum_{n=1}^{\infty}|a_n|$ either converges or it diverges to ∞).

Theorem 15.4 (Cauchy Criterion - Series)
A series $\sum_{n\ge1} a_n$ converges if and only if

$$\forall\epsilon>0 \quad \exists n_\epsilon\in\mathbb{N} \ni \left|\sum_{k=n+1}^{n+p} a_k\right| < \epsilon \quad \forall n\ge n_\epsilon\,\forall p\in\mathbb{N}$$

Proof. The series $\sum_{n\ge1} a_n$ converges $\iff$ the sequence $\{s_n\}_{n\ge1}$ converges $\iff$ $\{s_n\}_{n\ge1}$ is Cauchy $\iff$ $\forall\epsilon>0\ \exists n_\epsilon\in\mathbb{N}$ s.t. $|s_m - s_n| < \epsilon \quad \forall m,n\ge n_\epsilon$. Without loss of generality, we may assume $m>n$ and write $m=n+p$ for $p\in\mathbb{N}$. Note

$$|s_m - s_n| = \left|\sum_{k=1}^{n+p} a_k - \sum_{k=1}^{n} a_k\right| = \left|\sum_{k=n+1}^{n+p} a_k\right|$$

So $\sum_{n\ge1} a_n$ converges $\iff$ $\forall\epsilon>0\exists n_\epsilon\in\mathbb{N}$ s.t. $\left|\sum_{k=n+1}^{n+p} a_k\right| < \epsilon\ \forall n\ge n_\epsilon\,\forall p\in\mathbb{N}$. □

Corollary 15.5

If $\sum_{n\geq 1} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Proof. Taking $p = 1$, we find $\sum_{n\geq 1} a_n$ converges implies

$$\forall \epsilon > 0 \quad \exists n_\epsilon \in \mathbb{N} \text{ s.t. } |a_{n+1}| < \epsilon \quad \forall n \geq n_\epsilon$$

□

Corollary 15.6

If $\sum_{n\geq 1} a_n$ converges absolutely, then it converges.

Proof. $\sum_{n\geq 1} a_n$ converges absolutely $\implies \sum_{n\geq 1} |a_n|$ converges.

$$\implies \forall \epsilon > 0 \quad \exists n_\epsilon \in \mathbb{N} \text{ s.t. } \sum_{k=n+1}^{n+p} |a_k| < \epsilon \quad \forall n \geq n_\epsilon \, \forall p \in \mathbb{N}$$

Note that by $\triangle$ inequality,

$$\left| \sum_{k=n+1}^{n+p} a_k \right| \leq \sum_{k=n+1}^{n+p} |a_k| < \epsilon \quad \forall n \geq n_\epsilon \, \forall p \in \mathbb{N}$$

So $\sum_{n\geq 1} a_n$ converges by the Cauchy criterion. □

Theorem 15.7 (Comparison Test)

Let $\sum_{n\geq 1} a_n$ be a series of real numbers with $a_n \geq 0 \quad \forall n \geq 1$.

1. If $\sum_{n\geq 1} a_n$ converges and $|b_n| \leq a_n \, \forall n \geq 1$, then $\sum_{n\geq 1} b_n$ converges.
2. If $\sum_{n\geq 1} a_n$ diverges and $b_n \geq a_n \, \forall n \geq 1$, then $\sum_{n\geq 1} b_n$ diverges.

Proof.

1. $\sum_{n\geq 1} a_n$ converges $\implies \forall \epsilon > 0 \exists n_\epsilon \in \mathbb{N}$ s.t.

$$\left| \sum_{k=n+1}^{n+p} a_k \right| < \epsilon \quad \forall n \geq n_\epsilon \, \forall p \in \mathbb{N}$$

   Then $\left|\sum_{k=n+1}^{n+p} b_k\right| \leq \sum_{k=n+1}^{n+p} |b_k| \leq \sum_{k=n+1}^{n+p} a_k < \epsilon \, \forall n \geq n_\epsilon \, \forall p \in \mathbb{N}$. So by the Cauchy criterion, $\sum_{n\geq 1} b_n$ converges.

2. $b_1 + \ldots + b_n \geq a_1 + \ldots + a_n \xrightarrow{n\to\infty} \infty \implies \sum_{n\geq 1} b_n$ diverges.

□

Lemma 15.8

Let $r \in \mathbb{R}$. The series $\sum_{n\geq 0} r^n$ converges if and only if $|r| < 1$. If $|r| < 1$, then

$$\sum_{n\geq 0} r^n = \frac{1}{1-r}$$

Proof. First note that if $|r| \geq 1$, then

$$|r^n| = |r|^n \geq 1 \qquad \Longrightarrow r^n \overset{n\to\infty}{\not\longrightarrow} 0$$

By the first corollary, $\sum_{n\geq 0} r^n$ cannot converge. Assume now that $|r| < 1$. Then

$$|r^n| = |r|^n \overset{n\to\infty}{\longrightarrow} 0$$

Also

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r} \overset{n\to\infty}{\longrightarrow} \frac{1}{1-r}$$

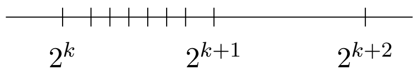
□

§16 Lec 16: Feb 10, 2021

§16.1 Series (Cont'd)

Theorem 16.1 (Dyadic Criterion)
Let $\{a_n\}_{n\geq 1}$ be a decreasing sequence of real numbers with $a_n \geq 0\,\forall n \geq 1$. Then the series $\sum_{n\geq 1} a_n$ converges if and only if the series $\sum_{n\geq 0} 2^n a_{2^n}$ converges.

Proof. For $n \geq 1$ let $s_n = \sum_{k=1}^n a_k = a_1 + \ldots + a_n$. For $n \geq 0$ let $t_n = \sum_{k=0}^n 2^k a_{2^k} = a_1 + 2a_2 + \ldots + 2^n a_{2^n}$. Note that $\{s_n\}_{n\geq 1}$ and $\{t_n\}_{n\geq 0}$ are increasing sequences.
Thus $\sum_{n\geq 1} a_n$ converges $\iff$ $\{s_n\}_{n\geq 1}$ is bounded and $\sum_{n\geq 0} 2^n a_{2^n}$ converges $\iff$ $\{t_n\}_{n\geq 0}$ is bounded. We have to prove that $\{s_n\}_{n\geq 1}$ is bounded $\iff$ $\{t_n\}_{n\geq 0}$ is bounded.



Consider:

$$\sum_{l=2^k+1}^{2^{k+1}} a_l$$

Because $\{a_n\}_{n\geq 1}$ is decreasing, we get

$$\begin{aligned}
\frac{1}{2}\left(2^{k+1} a_{2^{k+1}}\right) = 2^k a_{2^{k+1}} \leq \sum_{l=2^k+1}^{2^{k+1}} a_l \leq 2^k a_{2^k+1} \leq 2^k a_{2^k} \\
\frac{1}{2}\sum_{k=0}^{n} 2^{k+1} a_{2^{k+1}} \leq \sum_{k=0}^{n} \sum_{l=2^k+1}^{2^{k+1}} a_l \leq \sum_{k=0}^{n} 2^k a_{2^k} \\
\frac{1}{2}\sum_{l=1}^{n+1} 2^l a_{2^l} \leq \sum_{l=2}^{2^{n+1}} a_l \leq t_n \\
\frac{1}{2}\left(t_{n+1} - a_1\right) \leq s_{2^{n+1}} - a_1 \leq t_n \\
\implies \begin{cases} t_{n+1} \leq 2 s_{2^{n+1}} - a_1 \\ s_n \leq s_{2^{n+1}} \leq t_n + a_1 \text{ as } n \leq 2^{n+1}\,\forall n \geq 1 \end{cases}
\end{aligned}$$

If $\{s_n\}_{n\geq 1}$ is bounded $\implies \exists M > 0$ s.t. $|s_n| \leq M\,\forall n \geq 1$

$$\implies t_{n+1} \leq 2M + a_1 \quad \forall n \geq 1$$

If $\{t_n\}_{n\geq 0}$ is bounded $\implies \exists L > 0$ s.t. $|t_n| \leq L\,\forall n \geq 0$

$$\implies s_n \leq L + a_1 \quad \forall n \geq 1$$

□

Corollary 16.2
The series $\sum_{n\geq 1} \frac{1}{n^\alpha}$ converges if and only if $\alpha > 1$.

Proof. If $\alpha \leq 0$ then $\frac{1}{n^\alpha} = n^{-\alpha} \geq 1 \, \forall n \geq 1$. In particular, $\frac{1}{n^\alpha} \overset{n\to\infty}{\not\longrightarrow} 0$ so $\sum_{n\geq 1} \frac{1}{n^\alpha}$ cannot converge. Assume $\alpha > 0$. Then $\{\frac{1}{n^\alpha}\}_{n\geq 1}$ is a decreasing sequence of positive real numbers. By the dyadic criterion,

$$\sum_{n\geq 1} \frac{1}{n^\alpha} \text{ converges} \iff \sum_{n\geq 0} 2^n \frac{1}{(2^n)^\alpha} \text{ converges}$$

$$\sum_{n\geq 0} \frac{2^n}{(2^n)^\alpha} = \sum_{n\geq 0} \left(2^{1-\alpha}\right)^n = \sum_{n\geq 0} r^n \text{ where } r = 2^{1-\alpha}$$

This converges $\iff r < 1 \iff 2^{1-\alpha} < 1 \iff 1-\alpha < 0 \iff \alpha > 1$. □

Theorem 16.3 (Root Test)

Let $\sum_{n\geq 1} a_n$ be a series of real numbers.

1. If $\limsup |a_n|^{\frac{1}{n}} < 1$ then $\sum_{n\geq 1} a_n$ converges absolutely.
2. If $\liminf |a_n|^{\frac{1}{n}} > 1$ then $\sum_{n\geq 1} a_n$ diverges.
3. The test is inconclusive if $\liminf |a_n|^{\frac{1}{n}} \leq 1 \leq \limsup |a_n|^{\frac{1}{n}}$.

Proof. 1. Let $L = \limsup |a_n|^{\frac{1}{n}}$.

$$L < 1 \implies 1 - L > 0 \overset{\mathbb{Q} \text{ dense in } \mathbb{R}}{\implies} \exists \epsilon \in \mathbb{R} \ni 0 < \epsilon < 1 - L \implies L < L + \epsilon < 1$$

So $L + \epsilon > L = \limsup |a_n|^{\frac{1}{n}} = \inf_N \sup_{n\geq N} |a_n|^{\frac{1}{n}}$

$$\begin{aligned} &\implies \exists N_0 \in \mathbb{N} \ni \sup_{n\geq N_0} |a_n|^{\frac{1}{n}} < L + \epsilon \\ &\implies |a_n|^{\frac{1}{n}} < L + \epsilon \quad \forall n \geq N_0 \\ &\implies |a_n| < (L+\epsilon)^n \quad \forall n \geq N_0 \end{aligned}$$

As $L + \epsilon < 1$, the series

$$\begin{aligned} \sum_{n\geq N_0} (L+\epsilon)^n &= \sum_{k\geq 0} (L+\epsilon)^{N_0+k} \\ &= (L+\epsilon)^{N_0} \sum_{k\geq 0} (L+\epsilon)^k \\ &= (L+\epsilon)^{N_0} \frac{1}{1-(L+\epsilon)} \end{aligned}$$

By the Comparison Test, $\sum_{n\geq N_0} a_n$ converges absolutely and note $|a_1| + \ldots + |a_{N_0-1}| \in \mathbb{R}$.

$$\implies \sum_{n\geq 1} a_n \text{ converges absolutely}$$

2. Let $\{a_{k_n}\}_{n\geq 1}$ be a subsequence of $\{a_n\}_{n\geq 1}$ such that

$$\lim_{n\to\infty} |a_{k_n}|^{\frac{1}{k_n}} = \liminf |a_n|^{\frac{1}{n}} > 1$$

$$\begin{aligned} &\implies \exists n_0 \in \mathbb{N} \ni |a_{k_n}|^{\frac{1}{k_n}} > 1 \quad \forall n \geq n_0 \\ &\implies |a_{k_n}| > 1 \qquad \forall n \geq n_0 \\ &\implies a_{k_n} \overset{n\to\infty}{\not\longrightarrow} 0 \implies a_n \overset{n\to\infty}{\not\longrightarrow} 0 \implies \sum_{n\geq 1} a_n \text{ diverges} \end{aligned}$$

3. Consider $a_n = \frac{1}{n} \, \forall n \geq 1$. The series $\sum_{n\geq 1} a_n = \sum_{n\geq 1} \frac{1}{n}$ diverges. However,

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{n}} \overset{\text{Cesaro-Stolz}}{=} \frac{1}{\lim_{n\to\infty} \frac{n+1}{n}} = 1$$

So $\liminf \sqrt[n]{a_n} = \limsup \sqrt[n]{a_n} = 1$. Consider now $a_n = \frac{1}{n^2} \, \forall n \geq 1$. The series $\sum_{n\geq 1} a_n = \sum_{n\geq 1} \frac{1}{n^2}$ converges.

However,

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{n^2}} \overset{\text{C-S}}{=} \frac{1}{\lim_{n\to\infty} \frac{(n+1)^2}{n^2}} = 1$$

So $\liminf \sqrt[n]{a_n} = \limsup \sqrt[n]{a_n} = 1$.

□

Theorem 16.4 (Ratio Test)

Let $\sum_{n\geq 1} a_n$ be a series of non-zero real numbers.

1. If $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$ then $\sum_{n\geq 1} a_n$ converges absolutely.
2. If $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$ then $\sum_{n\geq 1} a_n$ diverges.
3. The test is conclusive if $\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$

Proof. (1) & (2) follow from the root test and the Cesaro – Stolz theorem:

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq \liminf |a_n|^{\frac{1}{n}} \leq \limsup |a_n|^{\frac{1}{n}} \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$$

For (3) consider the same examples as in the previous theorem. □

Theorem 16.5 (Abel Criterion)

Let $\{a_n\}_{n\geq 1}$ be a decreasing sequence with $\lim_{n\to\infty} a_n = 0$. Let $\{b_n\}_{n\geq 1}$ be a sequence so that $\{\sum_{k=1}^n b_k\}_{k\geq 1}$ is bounded. Then $\sum_{n\geq 1} a_n b_n$ converges.

Corollary 16.6 (Leibniz Criterion)

Let $\{a_n\}_{n\geq 1}$ be a decreasing sequence with $\lim_{n\to\infty} a_n = 0$. Then $\sum_{n\geq 1} (-1)^n a_n$ converges.

Proof. (Abel Criterion) Let $t_n = \sum_{k=1}^n b_k$ for $n \geq 1$. As $\{t_n\}_{n\geq 1}$ is bounded $\exists M > 0$ s.t. $|t_n| \leq M \, \forall n \geq 1$. We will use the Cauchy criterion to prove convergence of $\sum_{n\geq 1} a_n b_n$. Let $\epsilon > 0$.

As $\lim a_n = 0 \implies \exists n_\epsilon \in \mathbb{N}$ s.t. $|a_n| < \frac{\epsilon}{2M}\ \forall n \geq n_\epsilon$. For $n \geq n_\epsilon$ and $p \in \mathbb{N}$,

$$\begin{aligned}
\left|\sum_{k=n+1}^{n+p} a_k b_k\right| &= \left|\sum_{k=n+1}^{n+p} a_k(t_k - t_{k-1})\right| \\
&= \left|\sum_{k=n+1}^{n+p} a_k t_k - \sum_{k=n+1}^{n+p} a_k t_{k-1}\right| \\
&= \left|\sum_{k=n+1}^{n+p} a_k t_k - \sum_{k=n}^{n+p-1} a_{k+1} t_k\right| \\
&= \left|\sum_{k=n}^{n+p} t_k\left(a_k - a_{k+1}\right) - a_n t_n + a_{n+p+1} t_{n+p}\right| \\
&\leq \sum_{k=n}^{n+p} |t_k|\,|a_k - a_{k+1}| + |a_n| \cdot |t_n| + |a_{n+p+1}| \cdot |t_{n+p}| \\
&\leq \sum_{k=n}^{n+p} M(a_k - a_{k+1}) + a_n M + a_{n+p+1} M \\
&= M\left(a_n - \cancel{a_{n+p+1}}\right) + a_n M + \cancel{a_{n+p+1}} M \\
&= 2M \cdot a_n < \epsilon
\end{aligned}$$

□

§17 Lec 17: Feb 12, 2021

§17.1 Rearrangements of Series

Definition 17.1 (Rearrangement) — Let $k : \mathbb{N} \to \mathbb{N}$ be a bijective function. For a sequence $\{a_n\}_{n\geq 1}$ of real numbers, we denote

$$\tilde{a}_n = a_{k(n)} = a_{k_n}$$

Then $\sum_{n\geq 1} \tilde{a}_n$ is called a rearrangement of $\sum_{n\geq 1} a_n$

Example 17.2

Consider $a_n = \frac{(-1)^{n-1}}{n}$ $\forall n \geq 1$. The series $\sum_{n\geq 1} a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots$ Note that the sequence $\left\{\frac{1}{n}\right\}_{n\geq 1}$ is decreasing and $\lim_{n\to\infty} \frac{1}{n} = 0$. Thus, by the Leibniz criterion, $\sum_{n\geq 1} a_n$ converges. Write the series as follows:

$$\sum_{n\geq 1} a_n = 1 - \frac{1}{2} + \frac{1}{3} - \sum_{k\geq 2} \left(\frac{1}{2k} - \frac{1}{2k+1} \right)$$

Note that for $k \geq 2$

$$0 < \frac{1}{2k} - \frac{1}{2k+1} = \frac{1}{2k(2k+1)} < \frac{1}{4k^2}$$

Recall that the series $\sum_{k\geq 2} \frac{1}{4k^2}$ converges (by the dyadic criterion). By the comparison test, the series $0 < \sum_{k\geq 2} \left(\frac{1}{2k} - \frac{1}{2k+1} \right)$ converges. So $\sum_{n\geq 1} a_n < 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$. Consider next the following rearrangement:

$$\frac{1}{1} + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \dots = \sum_{k\geq 1} \left(\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} \right)$$

Then

$$\begin{aligned} 0 < \frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} &= \frac{8k^2 - 2k + 8k^2 - 6k - (16k^2 - 16k + 3)}{(4k-3)(4k-1) \cdot 2k} \\ &= \frac{8k-3}{(4k-3)(4k-1)2k} < \frac{8k}{k \cdot 3k \cdot 2k} = \frac{4}{3k^2} \end{aligned}$$

As the series $\sum_{k\geq 1} \frac{4}{3k^2}$ converges, we deduce that the series

$$\sum_{k\geq 1} \left(\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} \right)$$

converges. Moreover,

$$\begin{aligned} \sum_{k\geq 1} \left(\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} \right) &= 1 + \frac{1}{3} - \frac{1}{2} + \sum_{k\geq 2} \left(\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} \right) \\ &> 1 + \frac{1}{3} - \frac{1}{2} = \frac{5}{6} \implies \text{converge to two different numbers} \end{aligned}$$

Theorem 17.3 (Riemann)

Let $\sum_{n\geq 1} a_n$ be a series that converges, but it does not converge absolutely. Let $-\infty \leq \alpha \leq \beta \leq \infty$. Then there exists a rearrangement $\sum_{n\geq 1} \tilde{a}_n$ with partial sums $\tilde{s}_n = \sum_{k=1}^n \tilde{a}_k$ such that

$$\liminf \tilde{s}_n = \alpha \text{ and } \limsup \tilde{s}_n = \beta$$

Proof. For $n \geq 1$ let

$$b_n = \frac{|a_n| + a_n}{2} = \begin{cases} a_n, & a_n \geq 0 \\ 0, & a_n < 0 \end{cases} \implies b_n \geq 0$$

$$c_n = \frac{|a_n| - a_n}{2} = \begin{cases} 0, & a_n \geq 0 \\ -a_n, & a_n < 0 \end{cases} \implies c_n \geq 0$$

Claim 17.1. The series $\sum_{n\geq 1} b_n$ and $\sum_{n\geq 1} c_n$ both diverge.

Note $\sum_{k=1}^n b_k - \sum_{k=1}^n c_k = \sum_{k=1}^n (b_k - c_k) = \sum_{k=1}^n a_k$ which converges as $n \to \infty$.

$$\implies \sum_{k=1}^n b_k = \sum_{k=1}^n c_k + \sum_{k=1}^n a_k$$

So $\{\sum_{k=1}^n b_k\}_{n\geq 1}$ converges if and only if $\{\sum_{k=1}^n c_k\}_{n\geq 1}$ converges. On the other hand if $\sum_{n\geq 1} b_n$ and $\sum_{n\geq 1} c_n$ both converged, then

$$\underbrace{\sum_{k=1}^n b_k + \sum_{k=1}^n c_k}_{\text{converge as } n\to\infty} = \sum_{k=1}^n (b_k + c_k) = \sum_{k=1}^n |a_k|$$

which diverges as $n \to \infty$ – contradiction. Thus $\sum_{n\geq 1} b_n$ and $\sum_{n\geq 1} c_n$ diverge to infinity.

Note also that $\sum_{n\geq 1} a_n$ converges $\implies$ $\lim_{n\to\infty} a_n = 0$ and so $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 0$.

Let $B_1, B_2, B_3, \ldots$ denote the non-negative terms in $\{a_n\}_{n\geq 1}$ in the order which they appear.

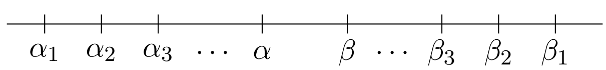
Let $C_1, C_2, C_3, \ldots$ denote the absolute values of the negative terms in $\{a_n\}_{n\geq 1}$, in the order in which they appear.

Note $\sum_{n\geq 1} B_n$ differs $\sum_{n\geq 1} b_n$ only by terms that are zero. So $\sum_{n\geq 1} B_n = \infty$. Similarly, $\sum_{n\geq 1} C_n$ differs $\sum_{n\geq 1} c_n$ only be terms that are zero. So $\sum_{n\geq 1} C_n = \infty$.

Choose sequences $\{\alpha_n\}_{n\geq 1}$ and $\{\beta_n\}_{n\geq 1}$ so that

$$\begin{cases} \alpha_n \overset{n\to\infty}{\longrightarrow} \alpha \\ \beta_n \overset{n\to\infty}{\longrightarrow} \beta \\ \alpha_n < \beta_n \quad \forall n \geq 1 \\ \beta_1 > 0 \end{cases}$$

<u>E.g.</u>



Next we construct increasing sequences $\{k_n\}_{n\geq 1}$ and $\{j_n\}_{n\geq 1}$ as follows:

1. Choose k_1 and j_1 to be the smallest natural numbers so that

   $x_1 = B_1 + B_2 + \ldots + B_{k_1} > \beta_1$ (this is possible because $\sum_{n\geq 1} B_n = \infty$)

   $y_1 = B_1 + \ldots + B_{k_1} - C_1 - C_2 - \ldots - C_{j_1} < \alpha_1$ (this is possible since $\sum_{n\geq 1} C_n = \infty$)

2. Choose k_2 and j_2 to be the smallest natural numbers so that

   $$x_2 = B_1 + \ldots + B_{k_1} - C_1 - \ldots - C_{j_1} + B_{k_1+1} + \ldots + B_{k_2} > \beta_2$$
   $$y_2 = B_1 + \ldots + B_{k_1} - C_1 - C_{j_1} + B_{k_1+1} + \ldots + B_{k_2} - C_{j_1+1} - \ldots - C_{j_2} < \alpha_2$$

   and so on.

Note that by definition,

$$\begin{aligned} x_n - B_{k_n} \leq \beta_n &\implies \beta_n - B_{k_n} < \beta_n < x_n \leq \beta_n + B_{k_n} \\ &\implies \Big| x_n - \underbrace{B_n}_{\xrightarrow{n\to\infty} \beta} \Big| \leq B_{k_n} \xrightarrow{n\to\infty} 0 \\ &\implies \lim_{n\to\infty} x_n = \beta \end{aligned}$$

Similarly,

$$\begin{aligned} y_n + C_{j_n} \geq \alpha_n &\implies \alpha_n - C_{j_n} \leq y_n < \alpha_n < \alpha_n + C_{j_n} \\ &\implies \Big| y_n - \underbrace{\alpha_n}_{\xrightarrow{n\to\infty} \alpha} \Big| \leq C_{j_n} \xrightarrow{n\to\infty} 0 \\ &\implies \lim_{n\to\infty} y_n = \alpha \end{aligned}$$

Finally, note that x_n and y_n are partial sums in the rearrangement

$$B_1 + B_2 + \ldots + B_{k_1} - C_1 - \ldots - C_{j_1} + B_{k_1+1} + \ldots + B_{k_2} - C_{j_1+1} - \ldots - C_{j_2} + \ldots$$

By construction, no number less than α or larger than β can occur as a subsequential limit of the partial sums. □

Theorem 17.4

If a series $\sum_{n\geq 1} a_n$ converges absolutely, then any rearrangement $\sum_{n\geq 1} \tilde{a}_n$ converges to $\sum_{n\geq 1} a_n$.

Proof. For $n \geq 1$ let $s_n = \sum_{k=1}^n a_k$, $\tilde{s}_n = \sum_{k=1}^n \tilde{a}_k$. As $\sum_{n\geq 1} a_n$ converges absolutely, $\forall \epsilon > 0\, \exists n_\epsilon \in \mathbb{N}$ s.t.

$$\sum_{k=n+1}^{n+p} |a_k| < \epsilon \quad \forall n \geq n_\epsilon\, \forall p \in \mathbb{N}$$

Choose N_ϵ sufficiently large so that $a_1, \ldots, a_{n_\epsilon}$ belong to the set $\{\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n\}$. Then for $n > N_\epsilon$ the terms $a_1, \ldots, a_{n_\epsilon}$ cancel in $s_n - \tilde{s}_n$

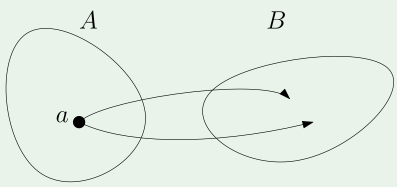
$$|s_n - \tilde{s}_n| \leq \underbrace{\sum_{k=n_\epsilon+1}^{n} |a_k| + \sum_{1\leq k\leq n} |\tilde{a}_k|}_{\text{finitely many terms and all indices are } > n_\epsilon} < \epsilon \quad (\tilde{a}_k \notin \{a_1, \ldots, a_{n_\epsilon}\})$$

As $\lim_{n\to\infty} s_n = s \in \mathbb{R}$ we deduce that $\lim_{n\to\infty} \tilde{s}_n = s$. □

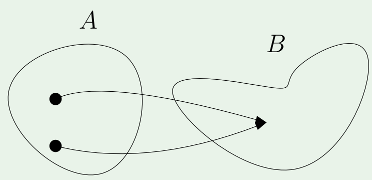
§18 Lec 18: Feb 17, 2021

§18.1 Functions

Definition 18.1 (Function) — Let A, B be two non-empty sets. A function $f : A \to B$ is a way of associating to each element $a \in A$ exactly one element in B denoted $f(a)$.



not a function



A function

A is called the domain of f.
B is called the range of f.

$f(A) = \{f(a) : a \in A\}$ is called the image of A under f. If $A' \subseteq A$ then $f(A') = \{f(a) : a \in A'\}$ is called the image of A' under f.

If $f(A) = B$ then we say that f is surjective/onto. In this case, $\forall b \in B \quad \exists a \in A$ s.t. $f(a) = b$.

We say that f is injective if it satisfies: if $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$ then $a_1 = a_2$.

We say that f is bijective if f is injective and surjective.

Remark 18.2. The injectivity and surjectivity of a function depend not only on the law f, but also on the domain and the range.

Example 18.3

$f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = 2n$ which is injective but not surjective.

$$f(n) = f(m) \implies 2n = 2m \implies n = m$$

$g : \mathbb{R} \to \mathbb{R}$, $g(x) = 2x$ bijective.

Example 18.4

$f : [0, \infty) \to [0, \infty)$, $f(x) = x^2$ bijective, $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$ not injective, not surjective.

Definition 18.5 (Composition) — Let A, B, C be non-empty sets and $f : A \to B$, $g : B \to C$ be two functions. The composition of g with f is a function $g \circ f : A \to C$, $(g \circ f)(a) = g(f(a))$.

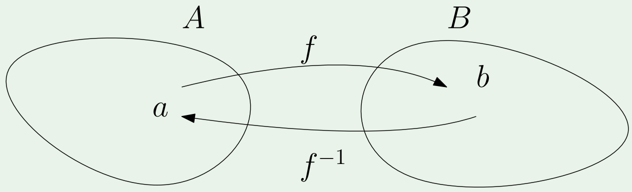
Remark 18.6. The composition of two functions need not be commutative.

$$\begin{aligned} f:\mathbb{Z}\to\mathbb{Z}, &\quad f(n)=2n \\ g:\mathbb{Z}\to\mathbb{Z}, &\quad g(n)=n+1 \\ g\circ f:\mathbb{Z}\to\mathbb{Z}, &\quad (g\circ f)(n)=g(f(n))=2n+1 \\ f\circ g:\mathbb{Z}\to\mathbb{Z}, &\quad (f\circ g)(n)=f(g(n))=2(n+1) \end{aligned}$$

Exercise 18.1. The composition of functions is associate: if $f: A\to B$, $g: B\to C$, $h: C\to D$ are three functions, then

$$(h\circ g)\circ f = h\circ (g\circ f)$$

Definition 18.7 (Inverse Function) — Let $f: A\to B$ be a bijective function. The inverse of f is a function $f^{-1}: B\to A$ defined as follows: if $b\in B$ then $f^{-1}(b)=a$ where a is the unique element in A s.t. $f(a)=b$. The existence of a is guaranteed by surjectivity and the uniqueness by injectivity.



So

$$\begin{aligned} f\circ f^{-1}&: B\to B \\ \left(f\circ f^{-1}\right)(b)&=b \end{aligned}$$

and

$$\begin{aligned} f^{-1}\circ f&: A\to A \\ \left(f^{-1}\circ f\right)(a)&=a \end{aligned}$$

Exercise 18.2. Let $f: A\to B$ and $g: B\to C$ be two bijective functions. Then $g\circ f: A\to C$ is a bijection and

$$(g\circ f)^{-1} = f^{-1}\circ g^{-1}$$

Definition 18.8 (Preimage) — Let $f: A\to B$ be a function. If $B'\subseteq B$ then the preimage of B' is $f^{-1}(B')=\{a\in A: f(a)\in B'\}$. The preimage of a set is well defined whether or not f is bijective. In fact, if $B'\subseteq B$ s.t. $B'\cap f(A)=\emptyset$ then $f^{-1}(B')=\emptyset$.

Exercise 18.3. Let $f: A\to B$ be a function and let $A_1, A_2\subseteq A$ and $B_1, B_2\subseteq B$. Then

1. $f(A_1\cup A_2)=f(A_1)\cup f(A_2)$
2. $f(A_1\cap A_2)\subseteq f(A_1)\cap f(A_2)$

3. $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$
4. $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$
5. The following are equivalent:
   i) f is injective.
   ii) $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ for all subsets $A_1, A_2 \subseteq A$.

§18.2 Cardinality

Definition 18.9 (Equipotent) — We say that two sets A and B have the same cardinality (or the same cardinal number) if there exists a bijection $f : A \to B$. In this case we write $A \sim B$.

Exercise 18.4. Show that $\sim$ is an equivalence relation on sets.

Definition 18.10 (Finite Set, Countable vs. Uncountable) — We say that a set A is finite if $A = \emptyset$ (in which case we say that it has cardinality 0) or $A \sim \{1, \dots, n\}$ for some $n \in \mathbb{N}$ (in which case we say that A has cardinality n).
We say that A is countable if $A \sim \mathbb{N}$. I this case we say that A has cardinality $\aleph_0$.
We say that A is at most countable if A is finite or countable. If A is not at most countable we say that A is uncountable.

Lemma 18.11
Let A be a finite set and let $B \subseteq A$. Then B is finite.

Proof. If $B = \emptyset$ then B is finite. Assume now that $B \neq \emptyset \implies A \neq \emptyset$. As A is finite, $\exists n \in \mathbb{N}$ and $\exists f : A \to \{1, \dots, n\}$ bijective. Then $f|_B : B \to f(B)$ is bijective.

WE merely have to relabel the elements in $f(B)$. Let $b_1 \in B$ be such that $f(b_1) = \min f(B)$.
Define $g(b_1) = 1$. If $B \setminus \{b_1\} \neq \emptyset$, let $b_2 \in B$ be such that $f(b_2) = \min f(B \setminus \{b_1\})$. Define $g(b_2) = 2$. Keep going. The process terminates in at most n steps. □

Example 18.12
$f : \mathbb{N} \cup \{0, -1, -2, \dots, -k\} \to \mathbb{N}$ where $k \in \mathbb{N}$

$$f(n) = n + k + 1 \text{ is bijective}$$

So the cardinality of $\mathbb{N} \cup \{0, -1, \dots, -k\}$ is $\aleph_0$.

Example 18.13
$f : \mathbb{Z} \to \mathbb{N}$

$$f(n) = \begin{cases} 2n + 2, & n \geq 0 \\ -2n - 1, & n < 0 \end{cases} \quad \text{is bijective}$$

So the cardinality of $\mathbb{Z}$ is $\aleph_0$.

Example 18.14

$f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

$$f(n,m) = \frac{(n+m-1)(n+m-2)}{2} + n$$

is bijective. So the cardinality of $\mathbb{N} \times \mathbb{N}$ is $\aleph_0$.

$n \backslash m$	1	2	3	4
1	$(1,1)$	$(2,2)$	$(3,3)$	$(4,4)$
2	$(2,1)$	$(2,2)$	$(2,3)$	$(2,4)$
3	$(3,1)$	$(3,2)$	$(3,3)$	$(3,4)$
4	$(4,1)$	$(4,2)$	$(4,3)$	$(4,4)$

Cont'd in Lec 19.

§19 Lec 19: Feb 19, 2021

§19.1 Functions & Cardinality (Cont'd)

From the last example of Lec 18, $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $f(n,m) = \frac{(n+m-1)(n+m-2)}{2} + n$, f is bijective.
We prove that f is surjective by induction. For $k \in \mathbb{N}$ let $P(k)$ denoted that statement

$$\exists (n,m) \in \mathbb{N} \times \mathbb{N} \text{ s.t. } f(n,m) = k$$

Base step: Note that $f(1,1) = \frac{1 \cdot 0}{2} + 1 = 1$. So $P(1)$ holds.
Inductive step: Fix $k \geq 1$ and assume that $P(k)$ holds. Then $\exists (n,m) \in \mathbb{N} \times \mathbb{N}$ s.t. $f(n,m) = k$.

$$\begin{aligned}
&\implies \frac{(n+m-1)(n+m-2)}{2} + n + 1 = k + 1 \\
&\implies \frac{[(n+1)+(m-1)-1]\,[(n+1)+(m-1)-2]}{2} + n + 1 = k + 1 \\
&\implies f(n+1, m-1) = k + 1
\end{aligned}$$

This works if $(n+1, m-1) \in \mathbb{N} \times \mathbb{N} \iff m - 1 \in \mathbb{N} \iff m \geq 2$. So if $m \geq 2$ we found $(n+1, m-1) \in \mathbb{N} \times \mathbb{N}$ s.t. $f(n+1, m-1) = k + 1$. Assume now $m = 1$. Then

$$\begin{aligned}
&\implies f(n,1) = k \iff \frac{n(n-1)}{2} + n = k \iff \frac{(n+1)n}{2} = k \\
&\implies \frac{(n+1)n}{2} + 1 = k + 1 \\
&\implies \frac{[1+(n+1)-1]\,[1+(n+1)-2]}{2} + 1 = k + 1 \\
&\implies f(1, n+1) = k + 1
\end{aligned}$$

So if $m = 1$ we found $(1, n+1) \in \mathbb{N} \times \mathbb{N}$ s.t. $f(1, n+1) = k + 1$. This proves $P(k+1)$ holds.
By induction, $\forall k \in \mathbb{N}\, \exists (n,m) \in \mathbb{N} \times \mathbb{N}$ s.t. $f(n,m) = k$, i.e. f is surjective.

Let $(n,m), (a,b) \in \mathbb{N} \times \mathbb{N}$ s.t. $f(n,m) = f(a,b)$. We want to show that $(n,m) = (a,b)$, thus proving that f is injective.

Case 1:

$$\left.\begin{aligned}
&\tfrac{(n+m-1)(n+m-2)}{2} = \tfrac{(a+b-1)(a+b-2)}{2} \\
&f(n,m) = f(a,b)
\end{aligned}\right\} \implies n = a$$

Then $(n+m-1)(n+m-2) = (n+b-1)(n+b-2)$

$$\begin{aligned}
&\implies n^2 + n(2m-3) + m^2 - 3m + 2 = n^2 + n(2b-3) + b^2 - 3b + 2 \\
&\implies 2n(m-b) + (m-b)(m+b) - 3(m-b) = 0 \\
&\implies \left.\begin{aligned} &(m-b)(2n+m+b-3) = 0 \\ &2n+m+b-3 \geq 2+1+1-3 \geq 1 \end{aligned}\right\} \implies m = b
\end{aligned}$$

Case 2: $\frac{(n+m-1)(n+m-2)}{2} = \frac{(a+b-1)(a+b-2)}{2} + r$ for some $r \in \mathbb{N}$.

Exercise 19.1. Show that this cannot occur.

Lemma 19.1

Let A be a countable set. Let B be an infinite subset of A.Then B is countable.

Proof. A is countable $\implies \exists f : \mathbb{N} \to A$ bijection. This means we can enumerate the elements of A :

$$A = \{a_1(= f(1)), a_2(= f(2)), a_3(= f(3)), \dots\}$$

Let $k_1 = \min\{n : a_n \in B\}$. Define $g(1) = a_{k_1}$. Then $B \setminus \{a_{k_1}\} \neq \emptyset$. Let $k_2 = \min\{n : a_n \in B \setminus \{a_{k_1}\}\}$. Define $g(2) = a_{k_2}$.

We proceed inductively. Assume we found $k_1 < \dots < k_j$ such that $a_{k_1}, \dots, a_{k_j} \in B$ and $g(1) = a_{k_1}, \dots, g(j) = a_{k_j}$. Then $B \setminus \left\{a_{k_1}, \dots, a_{k_j}\right\} \neq \emptyset$. Let $k_{j+1} = \min\left\{n : a_n \in B \setminus \left\{a_{k_1}, \dots, a_{k_j}\right\}\right\}$. Define $g(j+1) = a_{k_{j+1}}$.

By construction, $g : \mathbb{N} \to B$ is bijective. □

Lemma 19.2

Let A be a finite set and let B be a proper subset of A. Then A and B are not equipotent, that is, there is no bijective function $f : A \to B$.

Proof. If $B = \emptyset \implies A \neq \emptyset$. There is no function $f : A \to B$. Assume $B \neq \emptyset$. Assume towards a contradiction that there exists a bijection $f : A \to B$.

As $B \subsetneq A$, $\exists a_0 \in A \setminus B$.

For $n \geq 1$ let $a_n = \underbrace{(f \circ f \circ \dots \circ f)}_{n \text{ times}}(a_0)$. Note $a_{n+1} = f(a_n) \, \forall n \geq 0$. Note $a_n \in B \, \forall n \geq 1$.

We will show

Claim 19.1. $a_n \neq a_m$ for $n \neq m$.

If the claim holds then B (and so A) would contain countably many elements. Contradiction, since A is finite!

To prove the claim we argue by contradiction. Assume that there exists $n, k \in \mathbb{N}$ s.t. $a_{n+k} = a_n$.

Write

$$\left.\begin{aligned} a_{n+k} &= \underbrace{(f \circ f \circ \dots \circ f)}_{n \text{ times}}(a_k) \\ a_n &= \underbrace{(f \circ f \circ \dots \circ f)}_{n \text{ times}}(a_0) \\ f \text{ injective } &\implies \underbrace{f \circ f \circ \dots \circ f}_{n \text{ times}} \text{ injective} \end{aligned}\right\} \implies B \ni a_k = a_0 \in A \setminus B$$

which is a contradiction! This proves the claim and completes the proof of the lemma. □

Lemma 19.3

Every infinite set has a countable subset.

Proof. Let A be an infinite set $\implies A \neq \emptyset \implies \exists a_1 \in A$. Then $A \setminus \{a_1\} \neq \emptyset \implies \exists a_2 \in A \setminus \{a_1\}$.

We proceed inductively. Having found $a_1, \dots, a_n \in A$ distinct, $A \setminus \{a_1, \dots, a_n\} \neq \emptyset \implies \exists a_{n+1} \in A \setminus \{a_1, \dots, a_n\}$. This gives a sequence $\{a_n\}_{n \geq 1}$ of distinct elements in A. □

Theorem 19.4

A set A is infinite if and only if there is a bijection between A and a proper subset of A.

Proof. " $\Longleftarrow$ " Assume that there is a bijection $f : A \to B$ where $B \subsetneq A$. By Lemma 19.2, A must be infinite.
" $\Longrightarrow$ " Assume that A is infinite. By Lemma 19.3, there exists a countable subset B of A. Write $B = \{a_1, a_2, a_3, \ldots\}$ with $a_n \neq a_m$ if $n \neq m$. Then $A \setminus \{a_1\}$ is a proper subset of A. Define $f : A \to A \setminus \{a_1\}$ via

$$f(a) = \begin{cases} a, \text{ if } a \in A \setminus B \\ a_{j+1}, \text{ if } a = a_j \text{ for some } j \geq 1 \end{cases}$$

This is a bijective function.
Assume $f(a) = f(b)$.
<u>**Case 1:**</u> $a, b \in A \setminus B$. Then $f(a) = a$, $f(b) = b$ and so $f(a) = f(b) \implies a = b$.
<u>**Case 2:**</u> $a, b \in B \implies \exists i, j \in \mathbb{N}$ s.t. $a = a_i$, $b = a_j$

$$f(a) = f(b) \implies a_{i+1} = a_{j+1} \implies i + 1 = j + 1 \implies i = j \implies a = b$$

<u>**Case 3:**</u> $a \in A \setminus B$, $b \in B$. Then $f(a) \in A \setminus B$ and $f(b) \in B$, which cannot occur.
<u>**Case 4:**</u> $a \in B$ and $b \in A \setminus B$. Argue as for Case 3.

Exercise 19.2. f is surjective. □

Theorem 19.5 (Schröder – Bernstein)

Assume that A and B are two sets such that there exists two injective functions $f : A \to B$ and $g : B \to A$. Then A and B are equipotent.

Example 19.6

$$\begin{aligned} f &: \mathbb{N} \to \mathbb{N} \times \mathbb{N}, \quad f(n) = (1, n) \text{ injective} \\ g &: \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad g(n, m) = 2^n \cdot 3^m \text{ injective} \end{aligned}$$

By Schröder – Bernstein, $\mathbb{N} \sim \mathbb{N} \times \mathbb{N}$.

§20 Lec 20: Feb 22, 2021

§20.1 Countable vs. Uncountable Sets

Proof. (Schröder – Bernstein) We will decompose each of the sets A and B into disjoint subsets:

$$A = A_1 \cup A_2 \cup A_3 \text{ with } A_i \cap A_j = \emptyset \text{ if } i \neq j$$
$$B = B_1 \cup B_2 \cup B_3 \text{ with } B_i \cap B_j = \emptyset \text{ if } i \neq j$$

and we will show that $f : A_1 \to B_1, f : A_2 \to B_2, g : B_3 \to A_3$ are bijections.

Then $h : A \to B$ given by

$$h(a) = \begin{cases} f(a), & \text{if } a \in A_1 \cup A_2 \\ \left(g|_{B_3}\right)^{-1}(a), & \text{if } a \in A_3 \end{cases}$$

is a bijection. Exc!

For $a \in A$ consider the set

$$S_a = \left\{ \underbrace{a}_{\in A}, \underbrace{g^{-1}(a)}_{\in B}, \underbrace{f^{-1} \circ g^{-1}(a)}_{\in A}, \underbrace{g^{-1} \circ f^{-1} \circ g^{-1}(a)}_{\in B}, \ldots \right\}$$

Note that the preimage under f or g is either $\emptyset$ or it contains exactly one point (because f and g are injective).

There are three possibilities:

1. The process defining S_a does not terminate. We can always find a preimage.

2. The process defining S_a terminates in A, that is, the last element $x \in S_a$ is $x = a$ or $x = f^{-1} \circ g^{-1} \circ \ldots \circ g^{-1}(a)$ and $g^{-1}(x) = \emptyset$.

3. The process defining S_a terminates in B, that is, the last element $x \in S_a$ is $x = g^{-1}(a)$ or $x = g^{-1} \circ f^{-1} \circ \ldots \circ g^{-1}(a)$ and $f^{-1}(x) = \emptyset$.

We define

$$A_1 = \{a \in A : \text{ the process defining } S_a \text{ does not terminate}\}$$
$$A_2 = \{a \in A : \text{ the process defining } S_a \text{ terminates in } A\}$$
$$A_3 = \{a \in A : \text{ the process defining } S_a \text{ terminates in } B\}$$

Similarly, for $b \in B$ we define the set

$$T_b = \left\{ \underbrace{b}_{\in B}, \underbrace{f^{-1}(b)}_{\in A}, \underbrace{g^{-1} \circ f^{-1}(b)}_{\in B}, \underbrace{f^{-1} \circ g^{-1} \circ f^{-1}(b)}_{\in A}, \ldots \right\}$$

As before we define

$$B_1 = \{b \in B : \text{ the process defining } T_b \text{ does not terminate}\}$$
$$B_2 = \{b \in B : \text{ the process defining } T_b \text{ ends in } A\}$$
$$B_3 = \{b \in B : \text{ the process defining } T_b \text{ ends in } B\}$$

Let's show $f : A_1 \to B_1$ is a bijection. Injectivity is inherited from $f : A \to B$ is injective. Let $b \in B_1$. Then the process defining

$$T_b = \left\{b, f^{-1}(b), g^{-1} \circ f^{-1}(b), \dots\right\} \text{ does not terminate}$$

In particular, $\exists a \in A$ s.t. $f^{-1}(b) = a$. Note that

$$S_a = \left\{a, g^{-1}(a), f^{-1} \circ g^{-1}(a), \dots\right\} \quad = \left\{f^{-1}(b), g^{-1} \circ f^{-1}(b), f^{-1} \circ g^{-1} \circ f^{-1}(b), \dots\right\}$$

does not terminate. So $a \in A_1$.

This proves $f : A_1 \to B_1$ is surjective.

Let's show $f : A_2 \to B_2$ is a bijection. Again, injectivity is inherited from $f : A \to B$ is injective.

Let $b \in B_2$. Then the process defining

$$T_b = \left\{b, f^{-1}(b), g^{-1} \circ f^{-1}(b), \dots\right\} \text{ terminates in } A$$

In particular, $\exists a \in A$ s.t. $f^{-1}(b) = a$. Note that

$$S_a = \left\{a, g^{-1}(a), \dots\right\} = \left\{f^{-1}(b), g^{-1} \circ f^{-1}(b), \dots\right\}$$

terminates in $A \implies a \in A_2$. So $f : A_2 \to B_2$ is surjective.

Exercise 20.1. $g : B_3 \to A_3$ is bijective.

□

Theorem 20.1

Let $\{A_n\}_{n\geq 1}$ be a sequence of countable sets. Then

$$\bigcup_{n\geq 1} A_n = \{a : a \in A_n \text{ for some } n \geq 1\}$$

is countable.

Proof. We define

$$B_1 = A_1$$
$$B_{n+1} = A_{n+1} \setminus \bigcup_{k=1}^{n} A_k \quad \forall n \geq 1$$

By construction,

$$\begin{cases} B_n \cap B_m = \emptyset, \forall n \neq m \\ \bigcup_{n\geq 1} B_n = \bigcup_{n\geq 1} A_n \end{cases}$$

Note that each B_n is at most countable.

Let $I = \{n \in \mathbb{N} : B_n \neq \emptyset\}$. Then $\bigcup_{n\geq 1} B_n = \bigcup_{n\in I} B_n$. For $n \in I$, let $f_n : B_n \to I_n$ bijection where I_n is an at most countable subset of $\mathbb{N}$.

In particular, $f_1 : B_1 \to \mathbb{N}$ bijective $\implies f_1^{-1} : \mathbb{N} \to B_1$ bijective. To show $\bigcup_{n\in I} B_n$ is countable, we will use the Schröder – Bernstein theorem.

Let $g : \mathbb{N} \to \bigcup_{n\in I} B_n$, $g(n) = f_1^{-1}(n) \in B_1 \subseteq \bigcup_{n\in I} B_n$ is injective.

Let $h : \bigcup_{n\in I} B_n \to \mathbb{N} \times \mathbb{N}$ defined as follows: if $b \in \bigcup_{n\in I} B_n \implies \exists n \in I$ s.t. $b \in B_n$.

Define $h(b) = (n, f_n(b))$. Note that h is injective. Indeed, if $h(b_1) = h(b_2)$ then $(n_1, f_{n_1}(b_1)) = (n_2, f_{n_2}(b_2))$

$$\implies \left.\begin{cases} n_1 = n_2 \\ f_{n_1}(b_1) = f_{n_2}(b_2) \end{cases}, f_{n_1} \text{ is injective}\right\} \implies b_1 = b_2$$

Recall there exists a bijection $\phi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. So $\phi \circ h : \bigcup_{n\in I} B_n \to \mathbb{N}$ is injective. By Schröder – Bernstein, $\bigcup_{n\in I} B_n = \bigcup_{n\geq 1} A_n \sim \mathbb{N}$. □

Proposition 20.2

Let $\{A_n\}_{n\geq 1}$ be a sequence of sets such that for each $n \geq 1$, A_n has at least two elements. Then $\prod_{n\geq 1} A_n = \left\{\{a_n\}_{n\geq 1} : a_n \in A_n \, \forall n \geq 1\right\}$ is uncountable.

Proof. We argue by contradiction. Assume that $\prod_{n\geq 1} A_n$ is countable. Thus we may enumerate the elements of $\prod_{n\geq 1} A_n$:

$$\begin{aligned} a_1 &= (a_{11}, a_{12}, a_{13}, \ldots) \\ a_2 &= (a_{21}, a_{22}, a_{23}, \ldots) \\ &\ldots \\ a_n &= (a_{n1}, a_{n2}, a_{n3}, \ldots) \\ &\ldots \end{aligned}$$

Let $x = \{x_n\}_{n\geq 1} \in \prod_{n\geq 1} A_n$ such that $x_n \in A_n \setminus \{a_{nn}\}$. Then $x \neq a_n \, \forall n \geq 1$ since $x_n \neq a_{nn}$. This gives a contradiction. □

Remark 20.3. The same argument using binary expansion shows that the set $(0, 1)$ is uncountable.

§21 Lec 21: Feb 24, 2021

§21.1 Countable vs. Uncountable Sets (Cont'd)

Proposition 21.1

Let $\{A_n\}_{n\geq 1}$ be a sequence of sets s.t. $\forall n \geq 1$, the set A_n has at least two elements. Then $\prod_{n\geq 1} A_n$ is uncountable.

Remark 21.2.

1. The Cantor diagonal argument can be used to show that the set $(0,1)$ is uncountable (using binary expansion).
2. We can identify

$$\begin{aligned}\left\{\{a_n\}_{n\geq 1} : a_n \in \{0,1\}\ \forall n \geq 1\right\} &= \{f : \mathbb{N} \to \{0,1\} : f \text{ function}\}\\ &= \{0,1\} \times \{0,1\} \times \dots\\ &= \{0,1\}^{\mathbb{N}}\end{aligned}$$

By the proposition, this set is uncountable. We say it has cardinality $2^{\aleph_0}$.

Theorem 21.3

Let A be any set. Then there exists no bijection between A and the power set of A, $\mathcal{P}(A) = \{B : B \subseteq A\}$.

Proof. If $A = \emptyset$ then $\mathcal{P}(A) = \{\emptyset\}$. So the cardinality of A is 0, but the cardinality of $\mathcal{P}(A)$ is 1. Thus A is not equipotent with $\mathcal{P}(A)$.

Assume $A \neq \emptyset$. We argue by contradiction. Assume that there exists $f : A \to \mathcal{P}(A)$ a bijection.

Let $B = \{a \in A : a \notin f(a)\} \subseteq A$. f is surjective $\implies \exists b \in A$ s.t. $f(b) = B$

We distinguish two cases:

<u>**Case 1:**</u> $b \in B = f(b) \implies b \notin B$ – Contradiction.
<u>**Case 2:**</u> $b \notin B = f(b) \implies b \in B$ – Contradiction.

So A is not equipotent to $\mathcal{P}(A)$ □

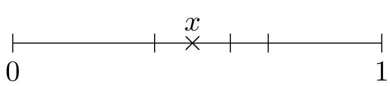
Theorem 21.4

The set $[0,1)$ has cardinality $2^{\aleph_0}$.

Proof. We write $x \in [0,1)$ using the binary expansion.

$$\begin{aligned}x &= 0.x_1x_2x_3\dots \quad \text{with } x_n \in \{0,1\}\ \forall n \geq 1\\ &= \frac{x_1}{2} + \frac{x_2}{2^2} + \frac{x_3}{2^3} + \dots = \sum_{n\geq 1} \frac{x_n}{2^n}\end{aligned}$$

with the convention that no expansion ends in all ones.



$x = 0.010\dots$

E.g.

$$\begin{aligned}x &= 0.x_1x_2x_3\dots x_n0111\dots\\ &= \frac{x_1}{2}+\dots+\frac{x_n}{2^n}+\underbrace{\frac{1}{2^{n+2}}+\frac{1}{2^{n+3}}+\dots}_{=\frac{1}{2^{n+1}}}\\ &= \frac{x_1}{2}+\dots+\frac{x_n}{2^n}+\frac{1}{2^{n+1}} = 0.x_1x_2\dots x_n1000\dots\end{aligned}$$

Note that we can identify $[0,1)$ with

$$\begin{aligned}\mathcal{F} &= \{f:\mathbb{N}\to\{0,1\} : \forall n\in\mathbb{N}\,\exists m>n \text{ s.t. } f(m)=0\}\\ &\subseteq \{f:\mathbb{N}\to\{0,1\} : f \text{ function}\}\end{aligned}$$

In particular, we have an injection $\phi : [0,1) \to \{f:\mathbb{N}\to\{0,1\}\}$. To prove the theorem, by Schröder – Bernstein, it suffices to construct an injective function $\psi : \{f:\mathbb{N}\to\{0,1\}\} \to [0,1)$. For $f:\mathbb{N}\to\{0,1\}$ we define

$$\begin{aligned}\psi(f) &= 0.0f(1)0f(2)0f(3)\dots\\ &= \frac{f(1)}{2^2}+\frac{f(2)}{2^4}+\frac{f(3)}{2^6}+\dots\\ &= \sum_{n\ge1}\frac{f(n)}{2^{2n}}\end{aligned}$$

Let's show ψ is an injective. Let $f_1, f_2 : \mathbb{N}\to\{0,1\}$ s.t. $f_1\ne f_2$. Let $n_0 = \min\{n : f_1(n)\ne f_2(n)\}$. Say, $f_1(n_0)=1$ and $f_2(n_0)=0$.

$$\begin{aligned}\psi(f_1)-\psi(f_2) = \sum_{n\ge1}\frac{f_1(n)}{2^{2n}}-\sum_{n\ge1}\frac{f_2(n)}{2^{2n}} &= \frac{f_1(n_0)-f_2(n_0)}{2^{2n_0}}+\sum_{n\ge n_0+1}\frac{f_1(n)-f_2(n)}{2^{2n}}\\ &\ge \frac{1}{2^{2n_0}}-\sum_{n\ge n_0+1}\frac{1}{2^{2n}}\\ &= \frac{1}{2^{2n_0}}-\frac{1}{2^{2(n_0+1)}}\cdot\frac{1}{1-\frac{1}{2}}\\ &= \frac{1}{2^{2n_0+1}}>0\end{aligned}$$

$\implies \psi(f_1)>\psi(f_2)$
So ψ is injective.
By Schröder – Bernstein, $[0,1)\sim\{f:\mathbb{N}\to\{0,1\}\}$ and so it has cardinality $2^{\aleph_0}$. □

§21.2 Metric Spaces

Definition 21.5 (Metric Space) — Let X be a non-empty set. A metric on X is a map $d : X \times X \to \mathbb{R}$ such that

1. $d(x, y) \geq 0 \, \forall x, y \in X$
2. $d(x, y) = 0 \iff x = y$
3. $d(x, y) = d(y, x) \, \forall x, y \in X$
4. $d(x, y) \leq d(x, z) + d(z, y) \, \forall x, y, z \in X$

Then we say (X, d) is a metric space.

Example 21.6
1. $X = \mathbb{R}$, $d(x, y) = |x - y|$ is a metric.
2. $X = \mathbb{R}^n$, $d_2(x, y) = \sqrt{\sum_{k=1}^n |x_k - y_k|^2}$ is a metric.
3. X is any non-empty set. The discrete metric

$$d(x, y) = \begin{cases} 1, \ x \neq y \\ 0, \ x = y \end{cases}$$

4. Let (X, d) be a metric space. Then $\tilde{d} : X \times X \to \mathbb{R}$, $\tilde{d}(x, y) = \frac{d(x,y)}{1+d(x,y)}$ is a metric.

   Let's see it satisfies (4). Fix $x, y, z \in X$. As d is a metric,

$$d(x, y) \leq d(x, z) + d(z, y)$$

   Note $a \mapsto \frac{a}{1+a} = 1 - \frac{1}{1+a}$ is increasing on $[0, \infty)$. Then,

$$\begin{aligned} \tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)} \leq \frac{d(x, z) + d(z, y)}{1 + d(x, z) + d(z, y)} &\leq \frac{d(x, z)}{1 + d(x, z)} + \frac{d(z, y)}{1 + d(z, y)} \\ &= \tilde{d}(x, z) + \tilde{d}(z, y) \end{aligned}$$

Definition 21.7 ((Un)Bounded Metric Space) — We say that a metric space (X, d) is bounded if $\exists M > 0$ s.t. $d(x, y) \leq M \, \forall x, y \in X$. If (X, d) is not bounded, we say that it is bounded.

Remark 21.8. If (X, d) is an unbounded metric space then $(X, \tilde{d})$ is a bounded metric space where $\tilde{d}(x, y) = \frac{d(x,y)}{1+d(x,y)}$.

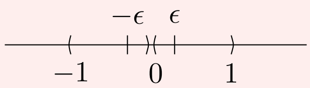
Definition 21.9 (Distance Between Sets) — Let (X, d) be a metric space and let $A, B \subseteq X$. The distance between A and B is

$$d(A, B) = \inf \{d(x, y) : x \in A, y \in B\}$$

Caution: This does not define a metric on subset of X.
In fact, $d(A, B) = 0$ does not even imply $A \cap B \neq \emptyset$.

Example 21.10

$(X, d) = (\mathbb{R}, |\cdot|)$, $A = (0, 1), B = (-1, 0)$, $d(A, b) = 0$ but $A \cap B = \emptyset$



Definition 21.11 (Distance Between Point and Set) — Let (X, d) be a metric space, $A \subseteq X, x \in X$. The distance from x to A is

$$d(x, A) = \inf \{d(x, a) : a \in A\}$$

Again, $d(x, A) = 0 \not\Longrightarrow x \in A$

§22 Lec 22: Feb 26, 2021

§22.1 Hölder & Minkowski Inequalities

Proposition 22.1 (Hölder's Inequality)
Fix $1 \le p \le \infty$ and let q denote the dual of p, that is, $\frac{1}{p} + \frac{1}{q} = 1$. Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and let $y = (y_1, \dots, y_n) \in \mathbb{R}^n$. Then

$$\sum_{k=1}^n |x_k y_k| \le \left(\sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^n |y_k|^q \right)^{\frac{1}{q}}$$

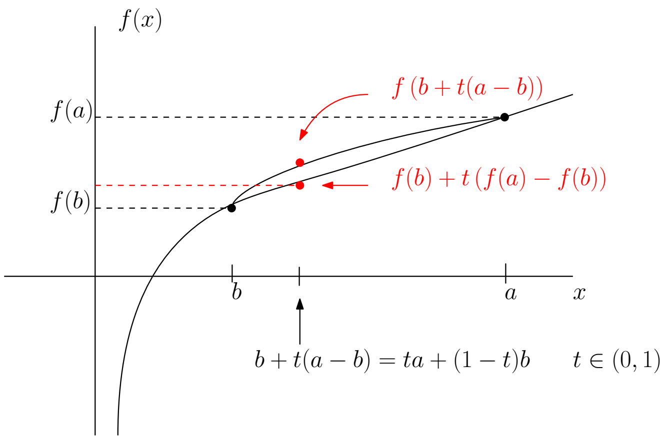
with the convention that if $p = \infty$, then $(\sum_{k=1}^n |x_k|^p)^{\frac{1}{p}} = sup_{1 \le k \le n} |x_k|$

Remark 22.2. If $p = 2 (\implies q = 2)$ we call this the Cauchy – Schwarz inequality.

Proof. If $p = 1$, then $q = \infty$.

$$\sum_{k=1}^n |x_k y_k| \le \sum_{k=1}^n |x_k| \cdot \sup_{1 \le l \le n} |y_l| \le \left(\sum_{k=1}^n |x_k| \right) \cdot \sup_{1 \le l \le n} |y_l|$$

If $p = \infty \implies (q = 1)$ a similar argument yields the claim.
Assume $1 < p < \infty$. We will use the fact that $f : (0, \infty) \to \mathbb{R}$, $f(x) = \log(x)$ is a concave function.



$$\begin{aligned} tf(a) + (1-t)f(b) &\le f(ta + (1-t)b) \quad \forall (a,b) \in (0,\infty)\, \forall t \in (0,1) \\ t\log(a) + (1-t)\log(b) &\le \log(ta + (1-t)b) \\ \log(a^t) + \log(b^{1-t}) &\le \log(ta + (1-t)b) \\ \log\left(a^t b^{1-t}\right) &\le \log(ta + (1-t)b) \\ a^t b^{1-t} &\le ta + (1-t)b \end{aligned}$$

We will apply this inequality with $a = \frac{|x_k|^p}{\sum_{l=1}^n |x_l|^p}$, $b = \frac{|y_k|^q}{\sum_{l=1}^n |y_l|^q}$.

$$t = \frac{1}{p} \implies 1 - t = 1 - \frac{1}{p} = \frac{1}{q}$$

We get

$$\frac{|x_k|}{\left(\sum_{l=1}^n |x_l|^p\right)^{\frac{1}{p}}} \cdot \frac{|y_k|}{\left(\sum_{l=1}^n |y_l|^q\right)^{\frac{1}{q}}} \le \frac{1}{p} \frac{|x_k|^p}{\sum_{l=1}^n |x_l|^p} + \frac{1}{q} \frac{|y_k|^q}{\sum_{l=1}^n |y_l|^q}$$

Sum over $1 \le k \le n$

$$\sum_{k=1}^n \frac{|x_k| \cdot |y_k|}{\left(\sum_{l=1}^n |x_l|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{l=1}^n |y_l|^q\right)^{\frac{1}{q}}} \le \frac{1}{p} \sum_{k=1}^n \frac{|x_k|^p}{\sum_{l=1}^n |x_l|^p} + \frac{1}{q} \sum_{k=1}^n \frac{|y_k|^q}{\sum_{l=1}^n |y_l|^q} \quad = \frac{1}{p} + \frac{1}{q} = 1$$

$\implies \sum_{k=1}^n |x_k y_k| \le \left(\sum_{l=1}^n |x_l|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{l=1}^n |y_l|^q\right)^{\frac{1}{q}}$. □

Corollary 22.3 (Minkowski's Inequality)
Fix $1 \le p \le \infty$ and let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Then

$$\left(\sum_{k=1}^n |x_k + y_k|^p\right)^{\frac{1}{p}} \le \left(\sum_{k=1}^n |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^n |y_k|^p\right)^{\frac{1}{p}}$$

Proof. If $p = 1$, this follows from the triangle inequality:

$$\text{LHS} = \sum_{k=1}^n |x_k + y_k| \le \sum_{k=1}^n |x_k| + |y_k| = \text{RHS}$$

If $p = \infty$,

$$\text{LHS} = sup_{1 \le k \le n} |x_k + y_k| \le \sup_{1 \le k \le n} |x_k| + \sup_{1 \le k \le n} |y_k| = \text{RHS}$$

Assume $1 < p < \infty$.

$$\begin{aligned}
\sum_{k=1}^n |x_k + y_k|^p &= \sum_{k=1}^n |x_k + y_k| \, |x_k + y_k|^{p-1} \\
&\le \sum_{k=1}^n (|x_k| + |y_k|) \, |x_k + y_k|^{p-1} \\
&= \sum_{k=1}^n |x_k| \cdot |x_k + y_k|^{p-1} + \sum_{k=1}^n |y_k| \, |x_k + y_k|^{p-1} \\
(\text{Hölder}) &\le \left(\sum_{k=1}^n |x_k|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{k=1}^n |x_k + y_k|^{(p-1)\cdot q}\right)^{\frac{1}{q}} \\
&\quad + \left(\sum_{k=1}^n |y_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^n |x_k + y_k|^{(p-1)q}\right)^{\frac{1}{q}}
\end{aligned}$$

$\frac{1}{p} + \frac{1}{q} = 1 \implies \frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p} \implies q = \frac{p}{p-1}$
Get

$$\sum_{k=1}^n |x_k + y_k|^p \le \left[\left(\sum_{k=1}^n |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^n |y_k|^p\right)^{\frac{1}{p}}\right] \cdot \left(\sum_{k=1}^n |x_k + y_k|^p\right)^{1-\frac{1}{p}}$$

$$\implies \left(\sum_{k=1}^n |x_k + y_k|^p\right)^{\frac{1}{p}} \le \left(\sum_{k=1}^n |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^n |y_k|^p\right)^{\frac{1}{p}}$$

□

Corollary 22.4
For $1 \le p < \infty$ let $d_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,

$$d_p(x,y) = \left(\sum_{k=1}^{n} |x_k - y_k|^p \right)^{\frac{1}{p}}$$

For $p = \infty$ let $d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,

$$d_\infty(x,y) = \sup_{1 \le k \le n} |x_k - y_k|$$

The d_p is a metric on $\mathbb{R}^n$ $\forall 1 \le p \le \infty$.

Proof. The triangle inequality follows from Minkowski's inequality. □

Remark 22.5. The Hölder and Minkowski inequalities generalize to sequences. For example, say $\{x_n\}_{n\ge1}$ and $\{y_n\}_{n\ge1}$ are sequences of real numbers such that $\left(\sum_{n\ge1} |x_n|^p\right)^{\frac{1}{p}} < \infty$ and $\left(\sum_{n\ge1} |y_n|^q\right)^{\frac{1}{q}} < \infty$. Then for each fixed $N \ge 1$,

$$\underbrace{\sum_{n=1}^{N} |x_k y_k|}_{\text{increasing seq indexed by N}} \le \left(\sum_{n=1}^{N} |x_n|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{n=1}^{N} |y_n|^q \right)^{\frac{1}{q}} \le \left(\sum_{n\ge1} |x_n|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{n\ge1} |y_n|^q \right)^{\frac{1}{q}} < \infty$$

So

$$\sum_{n\ge1} |x_k y_k| \le \left(\sum_{n\ge1} |x_n|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{n\ge1} |y_n|^q \right)^{\frac{1}{q}}$$

A similar argument gives Minkowski for sequences.

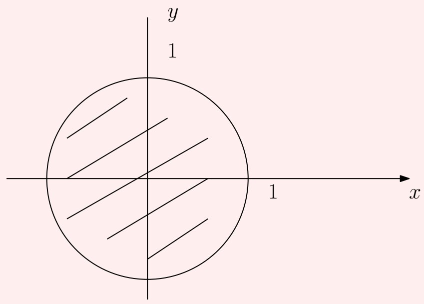
§22.2 Open Sets

Definition 22.6 (Ball/Neighborhood of a Point) — Let (X,d) be a metric space. A neighborhood of a point $a \in X$ is

$$B_r(a) = \{x \in X : d(a,x) < r\} \text{ for some } r > 0$$

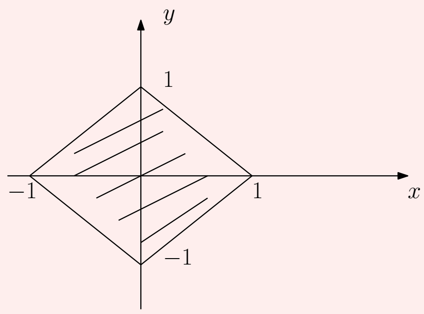
Example 22.7 1. $(\mathbb{R}^2, d_2)$

$$\begin{aligned} B_1(0) &= \left\{(x,y) \in \mathbb{R}^2 : d_2\left((x,y),(0,0)\right) < 1\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\right\} \end{aligned}$$



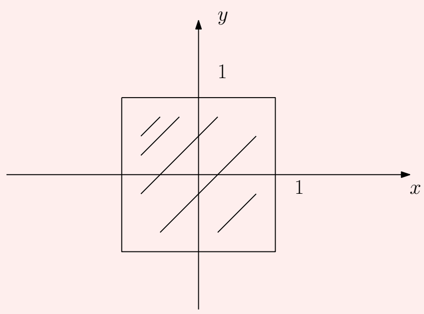
2. $(\mathbb{R}^2, d_1)$

$$B_1(0) = \left\{(x,y) \in \mathbb{R}^2 : |x| + |y| < 1\right\}$$



3. $(\mathbb{R}^2, d_\infty)$

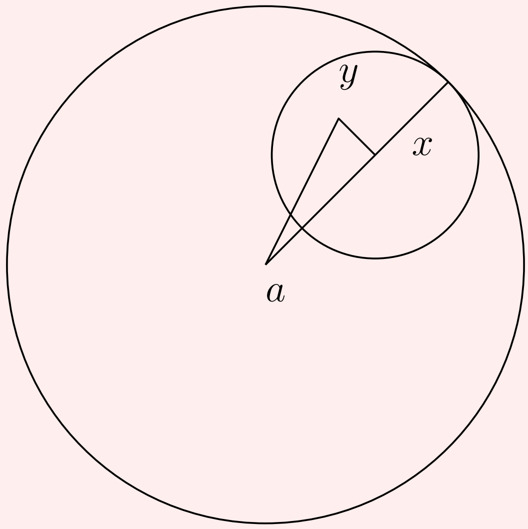
$$B_1(0) = \left\{(x,y) \in \mathbb{R}^2 : \max\left\{|x|,|y|\right\} < 1\right\}$$



Definition 22.8 (Interior Point) — Let (X, d) be a metric space and let $\emptyset \neq A \subseteq X$. We say that a point $a \in X$ is <u>an interior point</u> of A if $\exists r > 0$ s.t. $B_r(a) \subseteq A$. The set of all interior points of A is denoted $\mathring{A}$ and is called the <u>interior of A</u>. We say that A is <u>open</u> if $A = \mathring{A}$.

Example 22.9

1. $\emptyset, X$ are open sets.
2. $B_r(a)$ is an open set $\forall a \in X$, $\forall r > 0$.
   Indeed, let $x \in B_r(a) \implies d(x,a) < r \implies \rho = r - d(x,a) > 0$



Claim 22.1. $B_\rho(x) \subseteq B_r(a)$ and so $x \in \widehat{B_r(a)}^{\circ}$

Proof. Let $y \in B_\rho(x) \implies d(x,y) < \rho$

$$d(y,a) \le d(y,x) + d(x,a) < \rho + d(x,a) = r \implies y \in B_r(a)$$

□

Remark 22.10. $\mathring{A} \subseteq A$. To prove A is open, it suffices to show $A \subseteq \mathring{A}$.

§23 Lec 23: Mar 1, 2021

§23.1 Open Sets (Cont'd)

Proposition 23.1

Let (X,d) be a metric space and let $A, B \subseteq X$. Then

1. If $A \subseteq B$ then $\mathring{A} \subseteq \mathring{B}$
2. $\mathring{A} \cup \mathring{B} \subseteq \widehat{A \cup B}^{\circ}$
3. $\mathring{A} \cap \mathring{B} = \widehat{A \cap B}^{\circ}$
4. $\mathring{\mathring{A}} = \mathring{A}$. In particular, $\mathring{A}$ is an open set.
5. $\mathring{A}$ is the largest open set contained in A.
6. A finite intersection of open sets is an open set.
7. An arbitrary union of open sets is an open set.

Remark 23.2. An arbitrary intersection of open sets need not be open. E.g.

$$\bigcap_{n \geq 1} \underbrace{\left(-\frac{1}{n}, \frac{1}{n}\right)}_{B_{\frac{1}{n}}(0) \in (\mathbb{R}, |\cdot|)} = \{0\}$$

Note that $\{0\}$ is not an open set because it does not contain any neighborhood of 0.

Proof. (Of the proposition):

1. If $\mathring{A} = \emptyset$ this is clear. Assume $\mathring{A} \neq \emptyset$. Let $a \in \mathring{A} \implies \exists r > 0$ s.t.

$$\left.\begin{matrix} B_r(a) \subseteq A \\ A \subseteq B \end{matrix}\right\} \implies B_r(a) \subseteq B$$

So $a \in \mathring{B}$.

2. Consider:

$$\left.\begin{matrix} A \subseteq A \cup B \overset{(1)}{\implies} \mathring{A} \subseteq \widehat{A \cup B}^{\circ} \\ B \subseteq A \cup B \overset{(1)}{\implies} \mathring{B} \subseteq \widehat{A \cup B}^{\circ} \end{matrix}\right\} \implies \mathring{A} \cup \mathring{B} \subseteq \widehat{A \cup B}^{\circ}$$

3. Consider:

$$\left.\begin{matrix} A \cap B \subseteq A \overset{(1)}{\implies} \widehat{A \cap B}^{\circ} \subseteq \mathring{A} \\ A \cap B \subseteq B \overset{(2)}{\implies} \widehat{A \cap B}^{\circ} \subseteq \mathring{B} \end{matrix}\right\} \implies \widehat{A \cap B}^{\circ} \subseteq \mathring{A} \cap \mathring{B}$$

Now let $x \in \mathring{A} \cap \mathring{B}$

$$\implies \begin{cases} \exists r_1 > 0 \text{ s.t. } B_{r_1}(x) \subseteq A \\ \exists r_2 > 0 \text{ s.t. } B_{r_2}(x) \subseteq B \end{cases}$$

Let $r = \min\{r_1, r_2\} > 0$. Then $B_r(x) \subseteq B_{r_1}(x) \cap B_{r_2}(x) \subseteq A \cap B \implies x \in \widehat{A \cap B}^{\circ}$. So $\mathring{A} \cap \mathring{B} \subseteq \widehat{A \cap B}^{\circ}$

4. $\mathring{A} \subseteq A \overset{(1)}{\implies} \mathring{\mathring{A}} \subseteq \mathring{A}$. Let $x \in \mathring{A} \implies \exists r > 0$ s.t. $B_r(x) \subseteq A \overset{(1)}{\implies} B_r(x) = \widehat{B_r(x)}^{\circ} \subseteq \mathring{A} \implies x \in \mathring{\mathring{A}}$. So $\mathring{A} \subseteq \mathring{\mathring{A}}$.

5. By (4), $\mathring{A}$ is an open set contained in A. Let $B \subseteq A$ be an open set. Then by (1), $B = \mathring{B} \subseteq \mathring{A}$.

6. Using (3) and (4) we see that if $A = \mathring{A}$ and $B = \mathring{B}$ then $A \cap B = \widehat{A \cap B}^{\circ}$ is an open set.

   A simple inductive argument yields the claim.

7. Let $\{A_i\}_{i \in I}$ be a family of open sets. Let's show

$$\widehat{\bigcup_{i \in I} A_i}^{\circ} = \bigcup_{i \in I} A_i$$

   Let $x \in \bigcup_{i \in I} A_i \implies \exists i_0 \in I$ s.t.

$$\left.\begin{aligned} & x \in A_{i_0} \\ & A_{i_0} = \mathring{A}_{i_0} \end{aligned}\right\} \implies \exists r > 0 \text{ s.t. } B_r(x) \subseteq A_{i_0}$$

   So $B_r(x) \subseteq \bigcup_{i \in I} A_i \implies x \in \widehat{\bigcup_{i \in I} A_i}^{\circ}$. Thus, $\bigcup_{i \in I} A_i \subseteq \widehat{\bigcup_{i \in I} A_i}^{\circ}$. □

§23.2 Closed Sets

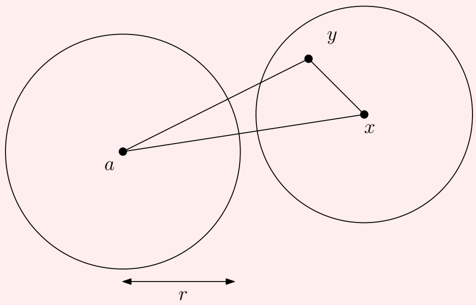
Definition 23.3 (Closed Set) — Let (X, d) be a metric space. A set $A \subseteq X$ is closed if ${}^cA$ is open.

Example 23.4 1. ϕ, X are closed.

2. If $a \in X$, $r > 0$, then ${}^cB_r(a) = \{x \in X : d(a,x) \geq r\}$ is a closed set.

3. If $a \in X$, $r > 0$, then $K_r(a) = \{x \in X : d(a,x) \leq r\}$ is a closed set.

   Let's show ${}^cK_r(a) = \{x \in X : d(a,x) > r\}$ is open. Let $x \in {}^cK_r(a) \implies d(a,x) > r$ and let $\rho = d(a,x) - r > 0$



   Claim 23.1. $B_\rho(x) \subseteq {}^cK_r(a)$

   Let $y \in B_\rho(x) \implies d(x,y) < \rho$. By the triangle inequality,

   $$d(a,y) \geq d(a,x) - d(x,y) > d(a,x) - \rho = r \implies y \in {}^cK_r(a)$$

   So $B_\rho(x) \subseteq K_r(a) \implies x \in \overset{\circ}{\widehat{{}^cK_r(a)}}$. Thus, ${}^cK_r(a)$ is an open set.

4. There are sets that are neither open nor closed. E.g. $(0,1]$ is not open and is not closed.

Definition 23.5 (Adherent Point) — Let (X,d) be a metric space and let $A \subseteq X$. A point $a \in X$ is an adherent point for A if

$$\forall r > 0 \text{ we have } B_r(a) \cap A \neq \emptyset$$

The set of all adherent points of A is denoted $\overline{A}$ and is called the closure of A.

Definition 23.6 (Isolated Point) — An adherent point a of A is called isolated if

$$\exists r > 0 \text{ s.t. } B_r(a) \cap A = \{a\} \quad (a \in A)$$

If every point in A is an isolated point of A then A is called an isolated set.

Definition 23.7 (Accumulation Point) — An adherent point a of A that is not isolated is called an accumulation point for A. The set of accumulation points of A is denoted A'. Note that

$$a \in A' \iff \forall r > 0 \quad B_r(a) \cap A \setminus \{a\} \neq \emptyset$$

Example 23.8

$(\mathbb{R}, |\cdot|)$, $\quad A = \left\{\frac{1}{n} : n \geq 1\right\}$. A is isolated. Indeed $B_{\frac{1}{n(n+1)}}\left(\frac{1}{n}\right) \cap A = \left\{\frac{1}{n}\right\}$.
$A' = \{0\}$ since $\forall r > 0\, B_r(0) = (-r, r)$ intersects $A \setminus \{0\} = A$.

Remark 23.9.

1. $A \subseteq \overline{A}$
2. $\overline{A} = A' \cup A$

Proposition 23.10

Let (X, d) be a metric space and let $A, B \subseteq X$. Then

1. ${}^c(\overline{A}) = \overset{\circ}{\widehat{{}^cA}}$
2. ${}^c(\mathring{A}) = \overline{{}^cA}$
3. A is closed set $\iff$ $A = \overline{A}$
4. If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$
5. $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$
6. $\overline{A} \cup \overline{B} = \overline{A \cup B}$
7. $\overline{\overline{A}} = \overline{A}$. In particular, $\overline{A}$ is a closed set.
8. $\overline{A}$ is the smallest closed set containing A.
9. A finite union of closed sets is a closed set.
10. An arbitrary intersection of closed sets is a closed set.

Remark 23.11. An arbitrary union of closed sets need not be a closed set. E.g.

$$\bigcup_{n\geq 1} \underbrace{\left[\frac{1}{n}, 1\right]}_{\text{closed}} = \underbrace{(0, 1]}_{\text{not closed}}$$

Proof. (of the proposition)

1. Consider

$$\begin{aligned} x \in {}^c(\overline{A}) \iff x \notin \overline{A} &\iff \exists r > 0 \text{ s.t. } B_r(x) \cap A = \emptyset \\ &\iff \exists r > 0 \text{ s.t. } B_r(x) \subseteq {}^cA \\ &\iff x \in \overset{\circ}{\widehat{{}^cA}} \end{aligned}$$

2. Apply (1) to ${}^cA$.

3. A is closed $\iff$ ${}^cA$ is open

$$\begin{aligned} &\iff {}^cA = \overset{\circ}{\widehat{{}^cA}} \\ &\overset{(1)}{\iff} {}^cA = {}^c(\overline{A}) \\ &\iff A = \overline{A} \end{aligned}$$

We continue in the next lecture. □

§24 Lec 24: Mar 3, 2021

§24.1 Closed Sets (Cont'd)

Proposition 24.1

Let (X, d) be a metric space and let $A, B \subseteq X$. Then

1. ${}^c(\overline{A}) = \mathring{\widehat{{}^cA}}$
2. ${}^c(\mathring{A}) = \overline{{}^cA}$
3. A is closed set $\iff A = \overline{A}$
4. If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$
5. $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$
6. $\overline{A} \cup \overline{B} = \overline{A \cup B}$
7. $\overline{\overline{A}} = \overline{A}$. In particular, $\overline{A}$ is a closed set.
8. $\overline{A}$ is the smallest closed set containing A.
9. A finite union of closed sets is a closed set.
10. An arbitrary intersection of closed sets is a closed set.

Proof. (Cont'd from last lecture)

4. If $\overline{A} = \emptyset$ then clearly $\overline{A} \subseteq \overline{B}$. Assume $\overline{A} \neq \emptyset$. Let $a \in \overline{A} \implies \forall r > 0$,

$$\left.\begin{array}{l} B_r(a) \cap A \neq \emptyset \\ A \subseteq B \end{array}\right\} \implies B_r(a) \cap B \neq \emptyset \, \forall r > 0$$
$$\implies a \in \overline{B}$$

So $\overline{A} \subseteq \overline{B}$

5. Have:

$$\left.\begin{array}{l} A \cap B \subseteq A \overset{(4)}{\implies} \overline{A \cap B} \subseteq \overline{A} \\ A \cap B \subseteq B \overset{(4)}{\implies} \overline{A \cap B} \subseteq \overline{B} \end{array}\right\} \implies \overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$$

6. Have

$${}^c(\overline{A \cup B}) \overset{(1)}{=} \mathring{\widehat{{}^c(A \cup B)}} = \mathring{\widehat{{}^cA \cap {}^cB}} = \mathring{\widehat{{}^cA}} \cap \mathring{\widehat{{}^cB}} \overset{(1)}{=} {}^c(\overline{A}) \cap {}^c(\overline{B})$$
$$= {}^c(\overline{A} \cup \overline{B})$$

$$\implies \overline{A \cup B} = \overline{A} \cup \overline{B}$$

7. Clearly, $A \subseteq \overline{A} \overset{(4)}{\implies} \overline{A} \subseteq \overline{\overline{A}}$. Want to show $\overline{\overline{A}} \subseteq \overline{A}$. Let $a \in \overline{\overline{A}}$. Want to prove that $\forall r > 0 \, B_r(a) \cap A \neq \emptyset$.

Fix $r > 0$. As $a \in \overline{\overline{A}} \implies B_r(a) \cap \overline{A} \neq \emptyset$. Let $x \in B_r(a) \cap \overline{A}$

$$x \in \overline{A} \implies \forall \rho > 0, \, B_\rho(x) \cap A \neq \emptyset$$

Choose $\rho = r - d(a,x) > 0$. Then

$$\left.\begin{matrix} B_\rho(x) \subseteq B_r(a) \\ B_\rho(x) \cap A \neq \emptyset \end{matrix}\right\} \implies B_r(a) \cap A \neq \emptyset$$

So $a \in \overline{A}$.

8. Note $\overline{A}$ is a closed subset containing A. Let B be a closed set containing A.

$$A \subseteq B \overset{(4)}{\implies} \overline{A} \subseteq \overline{B} \overset{(3)}{=} B$$

9. Let $\{A_n\}_{n=1}^N$ be a closed sets. Then ${}^cA_n$ is an open set $\forall 1 \leq n \leq N$. Then $\bigcap_{n=1}^N {}^cA_n$ is an open set. Now $\bigcap_{n=1}^N {}^cA_n = {}^c\left(\bigcup_{n=1}^N A_n\right)$ open $\implies \bigcup_{n=1}^N A_n$ closed.

10. Let $\{A_i\}_{i\in I}$ be a family of closed sets. Then ${}^cA_i$ is open $\forall i \in I$

$$\implies \bigcup_{i\in I} {}^cA_i = {}^c\left(\bigcap_{i\in I} A_i\right) \text{ is open}$$

$$\implies \bigcap_{i\in I} A_i \text{ is closed}$$

□

§24.2 Subspaces of Metric Spaces

Definition 24.2 (Subspace of Metric Space) — Let (X,d) be a metric space and let $\emptyset \neq Y \subseteq X$. Then $d_1 : Y \times Y \to \mathbb{R}$, $d_1(x,y) = d(x,y)\, \forall x,y \in Y$ is a metric on Y and is called the induced metric on Y. (Y,d_1) is called a subspace of (X,d).

Proposition 24.3

Let (X,d) be a metric space and let $\emptyset \neq Y \subseteq X$ equipped with the induced metric d_1.

1. A set $D \subseteq Y$ is open in (Y,d_1) if and only if there exists $O \subseteq X$ open in (X,d) s.t. $D = O \cap Y$.

2. A set $F \subseteq Y$ is closed in (Y,d_1) if and only if there exists $C \subseteq X$ closed in (X,d) s.t. $F = C \cap Y$.

Proof. 1. "$\implies$" Let $D \subseteq Y$ be open in (Y,d_1). Then $\forall a \in D\, \exists r_a > 0$ s.t. $B^y_{r_a}(a) = \{y \in Y : d(a,y) < r_a\} \subseteq D$. Note $B^y_{r_a}(a) = B^x_{r_a}(a) \cap Y$. So

$$D = \bigcup_{a\in D} B^y_{r_a}(a) = \bigcup_{a\in D} \left[B^x_{r_a}(a) \cap Y\right] = \underbrace{\left(\bigcup_{a\in D} B^x_{r_a}(a)\right)}_{\text{open in }(X,d)} \cap Y$$

"$\impliedby$" Assume that $D = O \cap Y$ for O open in (X,d). Let $a \in D \subseteq O \implies \exists r > 0$ s.t. $B^x_r(a) \subseteq O$

$$\implies B^y_r(a) = B^x_r(a) \cap Y \subseteq O \cap Y = D \implies a \text{ is an interior point of } D \text{ in the } (Y,d_1)$$

So D is open in (Y,d_1).

2. $F \subseteq Y$ is closed in $(Y, d_1) \iff Y \setminus F$ is open in $(Y, d_1) \overset{(1)}{\iff} \exists O$ open set in (X, d) s.t. $Y \setminus F = O \cap Y$. But

$$F = Y \setminus (Y \setminus F) = Y \setminus (O \cap Y) = Y \cap {}^c(O \cap Y) = Y \cap ({}^cO \cup {}^cY)$$
$$= (Y \cap {}^cO) \cup \underbrace{(Y \cap {}^cY)}_{=\emptyset} = Y \cap \underbrace{{}^cO}_{\text{closed in } (X,d)}$$

□

Example 24.4
1. $[0, 1)$ is not an open set in $(\mathbb{R}, |\cdot|)$, but it is open in $([0, 2), |\cdot|)$. Say $[0, 1) = (-1, 1) \cap [0, 2)$.
2. $(0, 1]$ is not a closed set in $(\mathbb{R}, |\cdot|)$, but it is closed in $((0, 2), |\cdot|)$. Say $(0, 1] = [-1, 1] \cap (0, 2)$.

Proposition 24.5

Let (X, d) be a metric space and let $\emptyset \neq Y \subseteq X$ equipped with the induced metric. The followings are equivalent:

1. Any $A \subseteq Y$ that is open (closed) in Y is also open(closed) in X.
2. Y is open(closed) in X.

Proof. 1) $\implies$ 2) Take $A = Y$.
2) $\implies$ 1) Assume Y is open in X. Let $A \subseteq Y$ be open in $Y \implies \exists O$ open in X s.t. $A = \underbrace{O}_{\text{open in X}} \cap \underbrace{Y}_{\text{open in X}}$ open in X. □

Proposition 24.6

Let (X, d) be a metric space and let $\emptyset \neq Y \subseteq X$ equipped with the induced metric. For a set $A \subseteq Y$,

$$\overline{A}^Y = \overline{A}^X \cap Y$$

Proof. Have:

$$\begin{aligned} a \in \overline{A}^Y &\iff \forall r > 0 \quad B_r^y(a) \cap A \neq \emptyset \\ &\iff \forall r > 0 \quad B_r^x(a) \cap \underbrace{Y \cap A}_{=A} \neq \emptyset \\ &\iff a \in \overline{A}^X \cap Y \end{aligned}$$

□

§24.3 Complete Metric Spaces

Definition 24.7 (Sequential Limit) — Let (X, d) be a metric space and let $\{x_n\}_{n\geq 1} \subseteq X$. We say $\{x_n\}_{n\geq 1}$ converges to a point $x \in X$ if

$$\forall \epsilon > 0 \quad \exists n_\epsilon \in \mathbb{N} \text{ s.t. } d(x_n, x) < \epsilon \quad \forall n \geq n_\epsilon$$

Then x is called the limit of $\{x_n\}_{n\geq 1}$ and we write $x = \lim_{n\to\infty} x_n$ or $x_n \xrightarrow[n\to\infty]{d} x$.

Exercise 24.1. The limit of a convergent sequence is unique.

Exercise 24.2. A sequence of $\{x_n\}_{n\geq 1}$ converges to $x \in X$ if and only if every subsequences of $\{x_n\}_{n\geq 1}$ converges to x.

Remark 24.8. If $x_n \xrightarrow[n\to\infty]{d} x$ and $y_n \xrightarrow[n\to\infty]{d} y$, then $d(x_n, y_n) \xrightarrow[n\to\infty]{} d(x, y)$.

Indeed,

$$\begin{aligned} |d(x_n, y_n) - d(x, y)| &\leq |d(x_n, y_n) - d(x_n, y)| + |d(x_n, y) - d(x, y)| \\ &\leq d(y_n, y) + d(x_n, x) \xrightarrow[n\to\infty]{} 0 \end{aligned}$$

Definition 24.9 (Cauchy Sequence (MS)) — Let (X, d) be a metric space. We say that $\{x_n\}_{n\geq 1} \subseteq X$ is Cauchy if

$$\forall \epsilon > 0 \quad \exists n_\epsilon \in \mathbb{N} \text{ s.t. } d(x_n, x_m) < \epsilon \quad \forall n, m \geq n_\epsilon$$

Exercise 24.3. Every convergent sequence is Cauchy.

Caution: Not every Cauchy sequence is convergent in an arbitrary metric space.

Example 24.10

1. $(X, d) = ((0, 1), |\cdot|)$, $x_n = \frac{1}{n}\, \forall n \geq 2$ is Cauchy but does not converge in X.
2. $(X, d) = (\mathbb{Q}, |\cdot|)$, $x_1 = 3$, $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}\, \forall n \geq 1$. Then $\{x_n\}_{n\geq 1}$ is Cauchy but does not converge in X.

Definition 24.11 (Complete Metric Space) — A metric space (X, d) is complete if every Cauchy sequence in X converges in X.

Example 24.12

$(\mathbb{R}, |\cdot|)$ is a complete metric space.

Exercise 24.4. Show that a Cauchy sequence with a convergent subsequence converges.

§25 Lec 25: Mar 5, 2021

§25.1 Complete Metric Spaces (Cont'd)

Lemma 25.1

Let (X,d) be a metric space and let $\emptyset \neq F \subseteq X$. The following equivalent:

1. $a \in \overline{F}$
2. There exists $\{a_n\}_{n\geq 1} \subseteq F$ s.t. $a_n \xrightarrow[n\to\infty]{d} a$

Proof. 1) $\implies$ 2) Assume $a \in \overline{F}$. Then

$$\forall r > 0, \quad B_r(a) \cap F \neq \emptyset$$

For $n \geq 1$, take $r = \frac{1}{n}$. Then $B_{\frac{1}{n}}(a) \cap F \neq \emptyset$. Let $a_n \in B_{\frac{1}{n}}(a) \cap F$. Consider $\{a_n\}_{n\geq 1} \subseteq F$. We have $\forall n \geq 1$,

$$d(a_n, a) < \frac{1}{n} \xrightarrow[n\to\infty]{} 0 \implies a_n \xrightarrow[n\to\infty]{d} a$$

2) $\implies$ 1) Assume $\exists \{a_n\}_{n\geq 1} \subseteq F$ s.t. $a_n \xrightarrow[n\to\infty]{d} a$. Fix $r > 0$. Then $\exists n_r \in \mathbb{N}$ s.t. $d(a_n, a) < r \, \forall n \geq n_r$. In particular, $\forall n \geq n_r$, $a_n \in B_r(a) \cap F \implies B_r(a) \cap F \neq \emptyset$. As r was arbitrary, we get $a \in \overline{F}$. □

Theorem 25.2

Let (X,d) be a metric space. The following are equivalent:

1. (X,d) is a complete metric space.
2. For every sequence $\{F_n\}_{n\geq 1}$ of non-empty closed subset of X, that is nested (that is, $F_{n+1} \subseteq F_n \, \forall n \geq 1$), and satisfies $\delta(F_n) \xrightarrow[n\to\infty]{} 0$, we have $\bigcap_{n\geq 1} F_n = \{a\}$ for some $a \in X$.

Proof. 1) $\implies$ 2) Assume (X,d) is complete. As $F_n \neq \emptyset \, \forall n \geq 1$, $\exists a_n \in F_n$.

Claim 25.1. $\{a_n\}_{n\geq 1}$ is Cauchy.

Let $\epsilon > 0$. As $\delta(F_n) \xrightarrow[n\to\infty]{} 0$, $\exists n_\epsilon \in \mathbb{N}$ s.t. $\delta(F_n) < \epsilon \, \forall n \geq n_\epsilon$. Let $m, n \geq n_\epsilon$. Since $\{F_n\}_{n\geq 1}$ is nested, $F_n \subseteq F_{n_\epsilon}$, $F_m \subseteq F_{n_\epsilon}$. So

$$d(a_n, a_m) \leq \delta(F_{n_\epsilon}) < \epsilon$$

So this proves the claim.

As (X,d) is complete, $\exists a \in X$ s.t. $a_n \xrightarrow[n\to\infty]{d} a$. For $\forall n \geq 1$, $\{a_m\}_{m\geq n} \subseteq F_n \implies a \in \overline{F_n} = F_n$. So $a \in \bigcap_{n\geq 1} F_n$.

It remains to show a is the only point in $\bigcap_{n\geq 1} F_n$. Assume, toward a contradiction, that $\exists y \neq a$ s.t. $y \in \bigcap_{n\geq 1} F_n$. Then $y \in F_n \, \forall n \geq 1 \implies d(y,a) \leq \delta(F_n) \xrightarrow[n\to\infty]{} 0 \implies y = a$ – Contradiction!

2) $\implies$ 1) Want to show (X,d) is complete. Let $\{x_n\}_{n\geq 1} \subseteq X$ be a Cauchy sequence. To prove that $\{x_n\}_{n\geq 1}$ converges in X, it suffices to show that $\{x_n\}_{n\geq 1}$ admits a subsequence that converges in X.

$\{x_n\}_{n\geq 1}$ is Cauchy $\implies \exists n_1 \in \mathbb{N}$ s.t. $d(x_n, x_m) < \frac{1}{2^2}\ \forall n, m \geq n_1$. Let $k_1 = n_1$ and select x_{k_1}.

$\{x_n\}_{n\geq 1}$ is Cauchy $\implies \exists n_2 \in \mathbb{N}$ s.t. $d(x_n, x_m) < \frac{1}{2^3}, \forall n, m \geq n_2$. Let $k_2 = \max\{n_2, k_1 + 1\}$ and select x_{k_2}.

Proceeding inductively, we find a strictly increasing sequence $\{k_n\}_{n\geq 1} \subseteq \mathbb{N}$ s.t.

$$d(x_l, x_m) < \frac{1}{2^{n+1}} \quad \forall l, m \geq k_n$$

For $n \geq 1$, let $F_n = K_{\frac{1}{2^n}}(X_{k_n}) = \left\{x \in X : d(x, x_{k_n}) < \frac{1}{2^n}\right\}$. Note $\emptyset \neq F_n = \overline{F_n}$ and $\delta(F_n) \leq 2 \cdot \frac{1}{2^n} \xrightarrow[n\to\infty]{} 0$.

Claim 25.2. $F_{n+1} \subseteq F_n \quad \forall n \geq 1$.

Let $y \in F_{n+1} \implies d(y, x_{k_{n+1}} \leq \frac{1}{2^{n+1}}$. By the triangle inequality,

$$d(y, x_{k_n}) \leq d(y, x_{k_{n+1}}) + d\left(x_{k_{n+1}}, x_{k_n}\right) \leq \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} = \frac{1}{2^n}$$

So $y \in F_n$. As $y \in F_{n+1}$ was arbitrary, we get $F_{n+1} \subseteq F_n$.
By hypothesis, $\bigcap_{n\geq 1} F_n = \{a\}$ for some $a \in X$. As $\forall n \geq 1$, $a \in F_n$ we have $d(a, x_{k_n}) \leq \frac{1}{2^n} \xrightarrow[n\to\infty]{} 0$

$$\left.\begin{array}{l} x_{k_n} \xrightarrow[n\to\infty]{d} a \\ \{x_n\}_{n\geq 1} \text{ is Cauchy} \end{array}\right\} \implies x_n \xrightarrow[n\to\infty]{d} a$$

□

§25.2 Examples of Complete Metric Spaces

Recall $(\mathbb{R}, |\cdot|)$ is a complete metric space.

Lemma 25.3

Assume (A, d_1) and (B, d_2) are complete metric spaces. We define $d : (A \times B) \times (A \times B) \to \mathbb{R}$ via

$$d\left((a_1, b_1), (a_2, b_2)\right) = \sqrt{d_1^2(a_1, a_2) + d_2^2(b_1, b_2)}$$

Then $(A \times B, d)$ is a complete metric space.

Exercise 25.1. Show that d is a metric on $A \times B$.

Proof. Let's show $A \times B$ is complete. Let $\{(a_n, b_n)\}_{n\geq 1} \subseteq A \times B$ be a Cauchy sequence.
Fix $\epsilon > 0$, $\exists n_\epsilon \in \mathbb{N}$ s.t. $d\left((a_n, b_n), (a_m, b_m)\right) < \epsilon\, \forall n, m \geq n_\epsilon$.

$$\implies \sqrt{d_1^2(a_n, a_m) + d_2^2(b_n, b_m)} < \epsilon \quad \forall n, m \geq n_\epsilon$$

$$\implies \begin{cases} d_1(a_n, a_m) < \epsilon & \forall n, m \geq n_\epsilon \\ d_2(b_n, b_m) < \epsilon & \forall n, m \geq n_\epsilon \end{cases}$$

So

$$\begin{cases} \{a_n\}_{n\geq 1} & \text{is Cauchy sequence in } A \\ \{b_n\}_{n\geq 1} & \text{is Cauchy sequence in } B \end{cases}$$

As A and B are complete metric spaces, $\exists a \in A$, $\exists b \in B$ s.t. $a_n \xrightarrow[n\to\infty]{d_1} a$ and $b_n \xrightarrow[n\to\infty]{d_2} b$.

Claim 25.3. $(a_n, b_n) \xrightarrow[n\to\infty]{d} (a, b)$.

Indeed,

$$\begin{aligned} d\left((a_n, b_n), (a, b)\right) &= \sqrt{d_1^2(a_n, a) + d_2^2(b_n, b)} \\ &\le d_1(a_n, a) + d_2(b_n, b) \xrightarrow[n\to\infty]{} 0 \end{aligned}$$

$\implies (a_n, b_n) \xrightarrow[n\to\infty]{d} (a, b)$. □

Corollary 25.4

For $n \ge 2$, $(\mathbb{R}^n, d_2)$ is a complete metric space.

Proof. Use induction. □

Exc!

Exercise 25.2. Show that for all $n \ge 2$, $(\mathbb{R}^n, d_p)$ is a complete metric space $\forall 1 \le p \le \infty$.

We define

$$l^2 = \left\{ \{x_n\}_{n\ge1} \subseteq \mathbb{R} : \sum_{n\ge1} |x_n|^2 < \infty \right\}$$

We define a metric on l^2 as follows: for $x = \{x_n\}_{n\ge1}$ and $y = \{y_n\}_{n\ge1} \in l^2$,

$$d_2(x, y) = \sqrt{\sum_{n\ge1} |x_n - y_n|^2}$$

The fact this is a metric follows from Minkowski's inequality.

Claim 25.4. $\left(l^2, d_2\right)$ is a complete metric space.

Proof. Let $\left\{x^{(d)}\right\}_{k\ge1}$ be a Cauchy sequence in l^2.

$$\begin{aligned} x^{(1)} &= \left\{x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \ldots\right\} \\ x^{(2)} &= \left\{x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \ldots\right\} \\ &\cdots \\ x^{(n)} &= \left\{x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \ldots\right\} \end{aligned}$$

We continue in the next lecture. □

§26 Lec 26: Mar 8, 2021

§26.1 Examples of Complete Metric Spaces (Cont'd)

Recall

$$l^2 = \left\{ \{x_n\}_{n\geq 1} \subseteq \mathbb{R} : \sum_{n\geq 1} |x_n|^2 < \infty \right\}$$

We define a metric $d_2 : l^2 \times l^2 \to \mathbb{R}$ via

$$d_2\left(\{x_n\}_{n\geq 1}, \{y_n\}_{n\geq 1}\right) = \sqrt{\sum_{n\geq 1} |x_n - y_n|^2}$$

Then (l^2, d_2) is a complete metric space. To see this, let $\{x^{(k)}\}_{k\geq 1}$ be a Cauchy sequence in l^2. Then $\forall \epsilon > 0\, \exists k_\epsilon \in \mathbb{N}$ s.t. $d_2\left(x^{(k)}, x^{(l)}\right) < \epsilon\, \forall k, l \geq k_\epsilon$. So

$$d_2\left(x^{(k)}, x^{(l)}\right) = \sqrt{\sum_{n\geq 1} \left|x_n^{(k)} - x_n^{(l)}\right|^2} < \epsilon \quad \forall k, l \geq k_\epsilon$$

$$\implies \sum_{n\geq 1} \left|x_n^{(k)} - x_n^{(l)}\right|^2 < \epsilon^2 \quad k, l \geq k_\epsilon$$

$$\implies \forall n \geq 1 \text{ we have } \left|x_n^{(k)} - x_n^{(l)}\right| < \epsilon \quad \forall k, l \geq k_\epsilon$$

So $\forall n \geq 1$, the sequence $\left\{x_n^{(k)}\right\}_{k\geq 1}$ is Cauchy in $(\mathbb{R}, |\cdot|)$. As $(\mathbb{R}, |\cdot|)$ is complete, $\exists x_n \in \mathbb{R}$ s.t. $x_n^{(k)} \xrightarrow[k\to\infty]{\mathbb{R}} x_n$.
Let $x = \{x_n\}_{n\geq 1}$

Claim 26.1. $x \in l^2$ and $x^{(k)} \xrightarrow[k\to\infty]{l^2} x$.

Note $d_2\left(x^{(k)}, x\right) = \sqrt{\sum_{n\geq 1} \left|x_n^{(k)} - x_n\right|^2}$. While $\left|x_n^{(k)} - x_n\right| \xrightarrow[k\to\infty]{} 0\, \forall n \geq 1$, the limit theorems do <u>not</u> apply to yield

$$\sum_{n\geq 1} \left|x_n^{(k)} - x_n\right|^2 \xrightarrow[k\to\infty]{} 0$$

Instead, we argue as follows:
Fix $\epsilon > 0$. As $\{x^{(k)}\}_{k\geq 1}$ is Cauchy in l^2, $\exists k_\epsilon \in \mathbb{N}$ s.t. $d_2\left(x^{(k)}, x^{(l)}\right) < \epsilon\, \forall k, l \geq k_\epsilon$. In particular, $\sum_{n\geq 1} \left|x_n^{(k)} - x_n^{(l)}\right|^2 < \epsilon^2\, \forall k, l \geq k_\epsilon$. So for each fixed $N \in \mathbb{N}$ we have

$$\sum_{n=1}^{N} \left|x_n^{(k)} - x_n^{(l)}\right|^2 < \epsilon^2 \quad \forall k, l \geq k_\epsilon$$

Note $\lim_{l\to\infty} \left|x_n^{(k)} - x_n^{(l)}\right| = \left|x_n^{(k)} - x_n\right|\ \forall n \geq 1, \forall k \geq k_\epsilon$. By the limit theorems,

$$\lim_{l\to\infty} \sum_{n=1}^{N} \left|x_n^{(k)} - x_n^{(l)}\right|^2 \leq \epsilon^2 \quad \forall k \geq k_\epsilon$$

$$\implies \sum_{n=1}^{N} \left|x_n^{(k)} - x_n\right|^2 \leq \epsilon^2 \quad \forall k \geq k_\epsilon$$

Note $\left\{\sum_{n=1}^{N}\left|x_n^{(k)}-x_n\right|^2\right\}_{N\geq 1}$ is an increasing sequence bounded above by ϵ^2. So

$$\sum_{n\geq 1}\left|x_n^{(k)}-x_n\right|^2 \leq \epsilon^2 \quad \forall k\geq k_\epsilon$$

$\implies d_2\left(x^{(k)},x\right)\leq \epsilon \quad \forall k\geq k_\epsilon$.
So $x^{(k)} \xrightarrow[k\to\infty]{l^2} x$. Finally, $x\in l^2 \iff d_2(x,0)<\infty$. But

$$d_2(x,0)\leq \underbrace{d_2(x,x^{(k)})}_{\leq\epsilon\,\forall k\geq k_\epsilon}+\underbrace{d_2\left(x^{(k)},0\right)}_{<\infty \text{ since } x^{(k)}\in l^2}<\infty$$

Exercise 26.1. 1. Fix $1\leq p<\infty$ and let

$$l^p=\left\{\{x_n\}_{n\geq 1}\subseteq\mathbb{R}:\sum_{n\geq 1}|x_n|^p<\infty\right\}$$

We define $d_p: l^p\times l^p\to\mathbb{R}$ via

$$d_p\left(\{x_n\}_{n\geq 1},\{y_n\}_{n\geq 1}\right)=\left(\sum_{n\geq 1}|x_n-y_n|^p\right)^{\frac{1}{p}}$$

Then (l^p,d_p) is a complete metric space.

2. Define $l^\infty=\left\{\{x_n\}_{n\geq 1}\subseteq\mathbb{R}: \sup_{n\geq 1}|x_n|<\infty\right\}$. We define $d_\infty: l^\infty\times l^\infty\to\mathbb{R}$ via

$$d_\infty\left(\{x_n\}_{n\geq 1},\{y_n\}_{n\geq 1}\right)=\sup_{n\geq 1}|x_n-y_n|$$

Show (l^∞,d_∞) is a complete metric space.

§26.2 Connected Sets

Definition 26.1 (Separated Set) — Let (X,d) be a metric space and let $A,B\subseteq X$. We say that A and B are separated if

$$\overline{A}\cap B=\emptyset \text{ and } A\cap\overline{B}=\emptyset$$

Remark 26.2. Separated sets are disjoint: $A\cap B\subseteq\overline{A}\cap B=\emptyset$. But disjoint sets need not be separated. For example,

$$(X,d)=(\mathbb{R},|\cdot|),\quad A=(-1,0),\quad B=[0,1)$$

Then $A\cap B=\emptyset$ but $\overline{A}\cap B=\{0\}\neq\emptyset$ so A,B are not separated.

Remark 26.3. If A and B are separated and $A_1\subseteq A$ and $B_1\subseteq B$, then A_1 and B_1 are separated.

Lemma 26.4

Let (X,d) be a metric space and let $A,B \subseteq X$. If $d(A,B) > 0$ then A and B are separated.

Proof. Assume, towards a contradiction that A and B are not separated. Then, $\overline{A} \cap B \neq \emptyset$ or $A \cap \overline{B} \neq \emptyset$. Say $\overline{A} \cap B \neq \emptyset$. Let $a \in \overline{A} \cap B$.

$$\left.\begin{aligned} &a \in B \\ &a \in \overline{A} \implies d(a,A) = 0 \end{aligned}\right\} \implies d(A,B) = 0 \quad \text{– Contradiction!}$$ □

Remark 26.5. Two sets A and B can be separated even if $d(A,B) = 0$.

Example 26.6

$A = (0,1)$ and $B = (1,2)$ separated, but $d(A,B) = 0$.

Proposition 26.7

1. Two closed sets A and B are separated $\iff A \cap B = \emptyset$.
2. Two open sets A and B are separated $\iff A \cap B = \emptyset$.

Proof. Two separated sets are disjoint. So we only have to prove " $\impliedby$ " in both cases.

1. Assume $A \cap B = \emptyset$. Then A closed $\implies A = \overline{A}$ and so $\overline{A} \cap B = A \cap B = \emptyset$. Similarly, B closed $\implies \overline{B} = B$ and so $\overline{B} \cap A = B \cap A = \emptyset$. So A and B are separated.
2. Assume $A \cap B = \emptyset \implies A \subseteq {}^cB$ where ${}^cB$ is closed since B is open.
$$\implies \overline{A} \subseteq \overline{{}^cB} = {}^cB \implies \overline{A} \cap B = \emptyset$$
A similar argument shows that $\overline{B} \cap A = \emptyset$ and so A and B are separated. □

Proposition 26.8

1. If an open set D is the union of two separated sets A and B, then A and B are both open.
2. If a closed set F is the union of two separated sets A and B, then A and B are both closed.

Proof.

1. If $A = \emptyset$, then since $D = A \cup B$ we have $B = D$ and so A and B are open.
Assume $A \neq \emptyset$. We want to show A is open $\iff A = \mathring{A}$. Let $a \in A \subseteq D$ and D open $\implies \exists r > 0$ s.t. $B_{r_1}(a) \subseteq D$. A and B are separated $\implies A \cap \overline{B} = \emptyset$. So $a \in A \subseteq {}^c\left(\overline{B}\right) = \widehat{{}^cB}^{\circ}$
$$\implies \exists r_2 > 0 \text{ s.t. } B_{r_2}(a) \subseteq {}^cB$$
Let $r = \min\{r_1, r_2\}$. Then
$$B_r(a) \subseteq D \cap {}^cB = (A \cup B) \cap {}^cB = A$$
so $a \in \mathring{A}$.
This shows A is open. A similar argument shows B is open.

2. Let's show A is closed $\iff \overline{A} = A$.

$$\left.\begin{aligned} &A \subseteq F \\ &F \text{ closed} \iff F = \overline{F} \end{aligned}\right\} \implies \overline{A} \subseteq \overline{F} = F$$

So $\overline{A} = \overline{A} \cap F = \overline{A} \cap (A \cup B) = \underbrace{(\overline{A} \cap A)}_{=A} \cup (\underbrace{\overline{A} \cap B}_{=\emptyset}) = A.$

Similarly, one can show that $\overline{B} = B$ and so B is closed. □

§27 Lec 27: Mar 10, 2021

§27.1 Connected Sets (Cont'd)

Definition 27.1 (Connected/Disconnected Set) — Let (X, d) be a metric space and let $A \subseteq X$. We say that A is disconnected if it can be written as the union of two non-empty separated sets, that is,

$$\exists B, C \subseteq X \text{ s.t. } B \neq \emptyset,\ C \neq \emptyset,\ \overline{B} \cap C = \overline{C} \cap B = \emptyset,\ A = B \cup C$$

We say that A is connected if it's not disconnected.

Lemma 27.2

Let (X, d) be a metric space and let $Y \subseteq X$ be equipped with the induced metric d_1. Then Y is connected in (Y, d_1) if and only if Y is connected in (X, d).

Proof. " $\Longrightarrow$ " Assume that Y is connected in (Y, d_1). We argue by contradiction. Assume that Y is not connected in (X, d). Then $\exists A, B \subseteq X$, $A \neq \emptyset$, $B \neq \emptyset$, $\overline{A}^X \cap B = \overline{B}^X \cap A = \emptyset$, $Y = A \cap B$.

Claim 27.1. A, B are separated in (Y, d_1). Then $Y = A \cup B$ is disconnected in (Y, d_1). Contradiction!

Indeed,

$$\overline{A}^Y \cap B = \left(\overline{A}^X \cap Y\right) \cap B = \overline{A}^X \cap \underbrace{Y \cap B}_{=B} = \overline{A}^X \cap B = \emptyset$$
$$\overline{B}^Y \cap A = \left(\overline{B}^X \cap Y\right) \cap A = \overline{B}^X \cap \underbrace{(Y \cap A)}_{=A} = \overline{B}^X \cap A = \emptyset$$

So A and B are separated in (Y, d_1).
" $\Longleftarrow$ " Assume Y is connected in (X, d). We argue by contradiction. Assume that Y is disconnected in (Y, d_1). So $\exists A, B \subseteq Y$, $A \neq \emptyset$, $B \neq \emptyset$, $\overline{A}^Y \cap B = \overline{B}^Y \cap A = \emptyset$, $Y = A \cup B$.

Claim 27.2. A, B are separated in (X, d). Then $Y = A \cup B$ is disconnected in (X, d). Contradiction!

Indeed,

$$\overline{A}^X \cap B = \overline{A}^X \cap (Y \cap B) = \left(\overline{A}^X \cap Y\right) \cap B = \overline{A}^Y \cap B = \emptyset$$
$$\overline{B}^X \cap A = \overline{B}^X \cap (Y \cap A) = \left(\overline{B}^X \cap Y\right) \cap A = \overline{B}^Y \cap A = \emptyset$$

So A and B are separated in (X, d). □

Proposition 27.3

Let (X, d) be a metric space. Then X is connected if and only if the only subsets of X that are both open and closed are $\emptyset$ and X.

Proof. " $\implies$ " Assume X is connected. We argue by contradiction. Assume $\exists \emptyset \neq A \subsetneq X$ s.t. A is both open and closed. Let

$$\begin{aligned} &B = X \setminus A \neq \emptyset \text{ (since } A \neq X) \\ &B \neq X \text{ (since } A \neq \emptyset) \\ &B \text{ is open (since A is closed)} \\ &B \text{ is closed (since A is open)} \end{aligned}$$

As A and B are closed and $A \cap B = A \cap (X \setminus A) = \emptyset$, we have that A and B are separated. So

$$\left.\begin{aligned} &X = A \cup (X \setminus A) = A \cup B \\ &A \neq \emptyset,\ B \neq \emptyset, A \text{ and } B \text{ are separated} \end{aligned}\right\} \implies X \text{ is disconnected – Contradiction!}$$

" $\impliedby$ " Assume that the only subsets of X that are both open and closed in (X, d) are $\emptyset$ and X. We argue by contradiction. Assume that X is disconnected. Then $\exists A, B \subseteq X$ s.t. $A \neq \emptyset$, $B \neq \emptyset$, $\overline{A} \cap B = \overline{B} \cap A = \emptyset$, $X = A \cup B$. As X is open (and closed) we get that A and B are both open (and closed).

$$\left.\begin{aligned} &A \text{ and } B \text{ are both open and closed} \\ &A \neq \emptyset,\ B \neq \emptyset \end{aligned}\right\} \implies A = B = X$$

But then $\overline{A} \cap B = \overline{X} \cap X = X \cap X = X \neq \emptyset$. Contradiction! ☐

Corollary 27.4

Let (X, d) be a metric space and let $\emptyset \neq A \subseteq X$. The following are equivalent:

1. A is disconnected.
2. $A \subseteq D_1 \cup D_2$ with D_1, D_2 open in (X, d), $A \cap D_1 \neq \emptyset$, $A \cap D_2 \neq \emptyset$, $A \cap D_1 \cap D_2 = \emptyset$.
3. $A \subseteq F_1 \cup F_2$ with F_1, F_2 closed in (X, d), $A \cap F_1 \neq \emptyset$, $A \cap F_2 \neq \emptyset$, $A \cap F_1 \cap F_2 = \emptyset$.

Proof. We'll show 1) $\implies$ 3) $\implies$ 2) $\implies$ 1).
1) $\implies$ 3) Assume A is disconnected. By the Proposition 27.3, there exists $\emptyset \neq B \subsetneq A$ s.t. B is both open and closed in A. Let $C = A \setminus B$. Then $C \neq \emptyset$, $C \neq A$, and C is both open and closed in A.

$$\begin{aligned} B \text{ closed in } A &\implies \exists F_1 \subseteq X \text{ closed in } (X, d) \text{ s.t. } B = A \cap F_1 \neq \emptyset \\ C \text{ closed in } A &\implies \exists F_2 \subseteq X \text{ closed in } (X, d) \text{ s.t. } C = A \cap F_2 \neq \emptyset \end{aligned}$$

Note that $A \cap F_1 \cap F_2 = (A \cap F_1) \cap (A \cap F_2) = B \cap C = B \cap (A \setminus B) = \emptyset$.
3) $\implies$ 2) Assume $A \subseteq F_1 \cup F_2$, F_1, F_2 closed in (X, d), $A \cap F_1 \neq \emptyset$, $A \cap F_2 \neq \emptyset$, $A \cap F_1 \cap F_2 = \emptyset$. Define $D_1 = {}^cF_1$ open in (X, d) and $D_2 = {}^cF_2$ open in (X, d).

$$\begin{aligned} A \subseteq F_1 \cup F_2 = {}^cD_1 \cup {}^cD_2 = {}^c(D_1 \cap D_2) &\implies A \cap (D_1 \cap D_2) = \emptyset \\ \emptyset = A \cap F_1 \cap F_2 = A \cap ({}^cD_1 \cap {}^cD_2) = A \cap {}^c(D_1 \cup D_2) &\implies A \subseteq D_1 \cup D_2 \end{aligned}$$

Let's show $A \cap D_1 \neq \emptyset$. We argue by contradiction. Assume $A \cap D_1 = \emptyset \implies A \subseteq {}^cD_1 = F_1$. But the $\emptyset = \underbrace{A \cap F_1}_{=A} \cap F_2 = A \cap F_2 \neq \emptyset$. Contradiction! This shows $A \cap D_1 \neq \emptyset$. A similar argument gives $A \cap D_2 \neq \emptyset$.

2) $\implies$ 1) Assume $A \subseteq D_1 \cup D_2$, D_1, D_2 open in (X, d), $A \cap D_1 \neq \emptyset$, $A \cap D_2 \neq \emptyset$, $A \cap D_1 \cap D_2 = \emptyset$. Let

$$\begin{aligned} B &= A \cap D_1 \neq \emptyset \text{ open in } A \text{ (since } D_1 \text{ is open in } X) \\ C &= A \cap D_2 \neq \emptyset \text{ open in } A \text{ (since } D_2 \text{ is open in } X) \\ B \cap C &= (A \cap D_1) \cap (A \cap D_2) = A \cap D_1 \cap D_2 = \emptyset \end{aligned}$$

So

$$\left.\begin{array}{l} B \text{ and } C \text{ are separated in } A \\ A \subseteq D_1 \cup D_2 \implies A = (D_1 \cup D_2) \cap A = (D_1 \cap A) \cup (D_2 \cap A) = B \cup C \\ B \neq \emptyset, \quad C \neq \emptyset \end{array}\right\} \implies$$

$\implies$ A is disconnected in A $\implies$ A is disconnected in X. □

Proposition 27.5

Let (X, d) be a metric space and let $A \subseteq X$ be disconnected. Let $F_1, F_2 \subseteq X$ be closed in (X, d) s.t. $A \subseteq F_1 \cup F_2$, $A \cap F_1 \neq \emptyset$, $A \cap F_2 \neq \emptyset$, $A \cap F_1 \cap F_2 = \emptyset$. If $B \subseteq A$ is connected then $B \subseteq F_1$ or $B \subseteq F_2$.

§28 Lec 28: Mar 12, 2021

§28.1 Connected Sets (Cont'd)

Proposition 28.1

Let (X, d) be a metric space and let $A \subseteq X$ be disconnected. Let F_1, F_2 be closed in X s.t. $A \subseteq F_1 \cup F_2$, $A \cap F_1 \neq \emptyset$, $A \cap F_2 \neq \emptyset$, $A \cap F_1 \cap F_2 = \emptyset$. Let $B \subseteq A$ be connected. Then $B \subseteq F_1$ or $B \subseteq F_2$.

Proof. We argue by contradiction. Assume $B \not\subseteq F_1$ and $B \not\subseteq F_2$.

$$\left.\begin{array}{l} \left.\begin{array}{l} B \subseteq A \subseteq F_1 \cup F_2 \\ B \not\subseteq F_1 \end{array}\right\} \implies B \cap F_2 \neq \emptyset \\ \left.\begin{array}{l} B \subseteq F_1 \cup F_2 \\ B \not\subseteq F_2 \end{array}\right\} \implies B \cap F_1 \neq \emptyset \\ B \cap F_1 \cap F_2 \subseteq A \cap F_1 \cap F_2 = \emptyset \\ B \subseteq F_1 \cup F_2 \end{array}\right\} \implies B \text{ is disconnected – Contradiction!}$$

□

Remark 28.2. One can replace the closed sets (in X) F_1 and F_2 by open sets (in X) D_1 and D_2 and the same conclusion holds.

Proposition 28.3

Let (X, d) be a metric space and let $A \subseteq X$ be connected. Then if $A \subseteq B \subseteq A^{-X}$, then B is connected.

Proof. We argue by contradiction. Assume B is disconnected. Then $\exists F_1, F_2 \subseteq X$, closed in X, s.t.

$$\begin{cases} B \subseteq F_1 \cup F_2 \\ B \cap F_1 \neq \emptyset \\ B \cap F_2 \neq \emptyset \\ B \cap F_1 \cap F_2 = \emptyset \end{cases}$$

and

$$\left.\begin{array}{l} A \subseteq B \subseteq F_1 \cup F_2 \\ A \text{ connected} \end{array}\right\} \implies A \subseteq F_1 \text{ or } A \subseteq F_2$$

Say $A \subseteq F_1 \implies B \subseteq A^{-X} \subseteq F_1^{-X} = F_1$. Then $\emptyset = \underbrace{B \cap F_1}_{=B} \cap F_2 = B \cap F_2 \neq \emptyset$.

Contradiction! □

§28.2 Connected Subsets

Proposition 28.4

Let (X,d) be a metric space and let $\{A_i\}_{i\in I}$ be a family of connected subsets of X. Assume that each two of these sets are not separated, that is, $\forall i,j\in I$, $i\neq j$, we have $\overline{A_i}\cap A_j\neq\emptyset$ or $A_i\cap\overline{A_j}\neq\emptyset$. Then $\bigcup_{i\in I}A_i$ is connected.

Proof. We argue by contradiction. Assume $\bigcup_{i\in I}A_i$ is disconnected $\implies \exists B,C$ non-empty separated sets s.t.

$$\bigcup_{i\in I}A_i = B\cup C$$

Fix $i\in I$. Then $A_i\subseteq B\cup C$.

$$\left.\begin{aligned}&\implies A_i=(B\cup C)\cap A_i=(B\cap A_i)\cup(C\cap A_i)\\ &B,C \text{ separated } \implies B\cap A_i,\ C\cap A_i \text{ separated}\\ &A_i \text{ is connected}\end{aligned}\right\} \implies \begin{cases}B\cap A_i=\emptyset\\ \quad\text{or}\\ C\cap A_i=\emptyset\end{cases}$$

Then

$$\left.\begin{aligned}A_i\subseteq B\cup C\\ A_i\cap B=\emptyset\end{aligned}\right\}\implies A_i\subseteq C$$
$$\left.\begin{aligned}A_i\subseteq B\cup C\\ A_i\cap C=\emptyset\end{aligned}\right\}\implies A_i\subseteq B$$

So for each $i\in I$, the set A_i satisfies $A_i\subseteq B$ or $A_i\subseteq C$. As $\bigcup_{i\in I}A_i=B\cup C$ $\implies \exists i,j\in I$ s.t. $A_i\cap B\neq\emptyset$ and $A_j\cap C\neq\emptyset$

$$\left.\begin{aligned}&\implies A_i\subseteq B \text{ and } A_j\subseteq C\\ &B \text{ and } C \text{ are separated}\end{aligned}\right\}\implies A_i,\ A_j \text{ are separated – Contradiction!}$$ □

Corollary 28.5

Let (X,d) be a metric space and let $\{A_i\}_{i\in I}$ be connected subsets of X. Assume $\forall i\neq j$ we have $A_i\cap A_j\neq\emptyset$. Then $\bigcup_{i\in I}A_i$ is connected.

Proposition 28.6

$\mathbb{R}$ is connected.

Proof. Assume, towards a contradiction, that $\mathbb{R}$ is disconnected. Then $\exists A,B$ non-empty subsets of $\mathbb{R}$, both open and closed in $\mathbb{R}$, disjoint, such that $\mathbb{R}\subseteq A\cup B$.

$$A\neq\emptyset \implies \exists a_1\in A$$
$$B\neq\emptyset \implies \exists b_1\in B$$

Let $\alpha_1=\frac{a_1+b_1}{2}\in\mathbb{R}=A\cup B \implies \alpha_1\in A$ or $\alpha_1\in B$. If

$$\alpha_1\in A \text{ let } (a_2,b_2):=(\alpha_1,b_1)$$
$$\alpha_1\in B \text{ let } (a_2,b_2):=(a_1,\alpha_1)$$

Let $\alpha_2 = \frac{a_2+b_2}{2} \in \mathbb{R} = A \cup B \implies \alpha_2 \in A$ or $\alpha_2 \in B$. If

$$\alpha_2 \in A \text{ let } (a_3, b_3) := (\alpha_2, b_2)$$
$$\alpha_2 \in B \text{ let } (a_3, b_3) := (a_2, \alpha_2)$$

Continuing this process, we find

- an increasing sequence $\{a_n\}_{n\geq 1} \subseteq A$ bounded above by b_1.
- a decreasing sequence $\{b_n\}_{n\geq 1} \subseteq B$ bounded below by a_1.

So $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ converge in $\mathbb{R}$. Let

$$a = \lim_{n\to\infty} a_n \in \overline{A} = A$$
$$b = \lim_{n\to\infty} b_n \in \overline{B} = B$$

Note that by contradiction, $b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2}\ \forall n \geq 1$

$$\implies |b_{n+1} - a_{n+1}| = \frac{|b_n - a_n|}{2} = \ldots = \frac{|b_1 - a_1|}{2^n} \underset{n\to\infty}{\longrightarrow} 0$$
$$\implies |b - a| = 0 \implies a = b \in A \cap B = \emptyset$$

Contradiction! □

Proposition 28.7

The only non-empty connected subsets of $\mathbb{R}$ are the intervals.

Proof. The argument in the previous proof extends easily to show that intervals are connected subset of $\mathbb{R}$.

It remains to show that if $\emptyset \neq A \subseteq \mathbb{R}$ is connected, then A is an interval. Let

$$\alpha = \inf A \quad (\alpha = -\infty \text{ if } A \text{ is unbounded below})$$
$$\beta = \sup A \quad (\beta = \infty \text{ if } A \text{ is unbounded above})$$

Claim 28.1. $(\alpha, \beta) \subseteq A$. This shows A is an interval.

We argue by contradiction. Assume $\exists c \in (\alpha, \beta) \setminus A$. Let $D_1 = (-\infty, c)$ open in $\mathbb{R}$ and $D_2 = (c, \infty)$ open in $\mathbb{R}$.

$$\left.\begin{array}{l} A \subseteq \mathbb{R} \setminus \{c\} = D_1 \cup D_2 \\ A \cap D_1 \cap D_2 = \emptyset \\ A \cap D_1 \neq \emptyset \text{ (because } \inf A = \alpha < c) \\ A \cap D_2 \neq \emptyset \text{ (because } \sup A = \beta > c) \end{array}\right\} \implies A \text{ is disconnected – Contradiction!}$$ □

Proposition 28.8

Let (X, d) be a metric space. Assume that for every pair of points in X, there exists a connected subset of X that contains them. Then X is connected.

Proof. Assume, towards a contradiction, that X is disconnected. Then there exists two non-empty separated sets $A, B \subseteq X$ s.t. $X = A \cup B$.

$$\left.\begin{aligned} A \neq \emptyset &\implies \exists a \in A \\ B \neq \emptyset &\implies \exists b \in B \end{aligned}\right\} \implies \exists C \subseteq X \text{ connected s.t. } \{a, b\} \subseteq C$$

$$\left.\begin{aligned} &C \subseteq X = A \cup B \\ &C \text{ connected} \\ &X \text{ closed} \implies A, B \text{ closed} \end{aligned}\right\} \implies \left.\begin{aligned} &\underbrace{C \subseteq A}_{b \in A \cap B} \text{ or } \underbrace{C \subseteq B}_{a \in B \cap A} \\ &A \cap B = \emptyset \end{aligned}\right\} \implies \text{Contradiction!} \qquad \square$$

Let (X, d) be a metric space. For $a, b \in X$, we write $a \sim b$ if there exists a connected subset of X, $A_{ab} \subseteq X$ s.t. $\{a, b\} \subseteq A_{ab}$.

Exercise 28.1. $\sim$ defines an equivalence relation of X.

For $a \in X$, let C_a denote the equivalence class of a.

Exercise 28.2.

1. C_a is a connected subset of X.
2. C_a is the largest connected set containing a.
3. C_a is closed in X.
4. If $a \not\sim b$ then C_a and C_b are separated.

We can decompose $X = \bigcup_{a \in X} C_a$ as a union of connected components.

III

131BH Lectures

§29 Lec 1: Mar 29, 2021

§29.1 Compactness

Definition 29.1 (Open Cover) — Let (X, d) be a metric space and let $A \subseteq X$. An open cover of A is a family $\{G_i\}_{i \in I}$ of open sets in X such that

$$A \subseteq \bigcup_{i \in I} G_i$$

The open cover is called finite if the cardinality of I is finite. If it's not finite, the open cover is called infinite.

Definition 29.2 (Compactness & Precompactness) — Let (X, d) be a metric space and let $K \subseteq X$.

1. We say that K is a compact set if every open cover $\{G_i\}_{i \in I}$ of K admits a finite subcover, that is,

$$\exists n \geq 1 \text{ and } \exists i_1, \ldots, i_n \in I \text{ s.t. } K \subseteq \bigcup_{j=1}^{n} G_{i_j}$$

2. We say that a set $A \subseteq X$ is precompact if $\overline{A}$ is compact.

Lemma 29.3

Let (X, d) be a metric space and let $\emptyset \neq Y \subseteq X$. We equip Y with the induced metric $d_1 : Y \times Y \to \mathbb{R}$, $d_1(y_1, y_2) = d(y_1, y_2)$. Let $K \subseteq Y \subseteq X$. The followings are equivalent:

1. K is compact in (X, d).
2. K is compact in (Y, d_1).

Proof. 1) $\implies$ 2) Assume K is compact in (X, d). Let $\{V_i\}_{i \in I}$ be a family of open sets in (Y, d_1) s.t.

$$K \subseteq \bigcup_{i \in I} V_i$$

For $i \in I$ fixed, V_i is open in $(Y, d_1) \implies \exists G_i \subseteq X$ open in (X, d) s.t.

$$V_i = G_i \cap Y$$

Then

$$\left.\begin{array}{l} K \subseteq \bigcup_{i \in I} V_i \subseteq \bigcup_{i \in I} G_i \\ K \text{ compact in } (X, d) \end{array}\right\} \implies \exists n \geq 1 \text{ and } \exists i_1, \ldots, i_n \in I \text{ s.t.}$$

$$\left.\begin{array}{l} K \subseteq \bigcup_{j=1}^{n} G_{i_j} \\ K \subseteq Y \end{array}\right\} \implies K \subseteq \left(\bigcup_{j=1}^{n} G_{i_j}\right) \cap Y = \bigcup_{j=1}^{n} (G_{i_j} \cap Y) = \bigcup_{j=1}^{n} V_{i_j}$$

So K is compact in (Y, d_1).
2) $\implies$ 1) Assume K is compact in (Y, d_1). Let $\{G_i\}_{i\in I}$ be a family of open sets in (X, d) s.t.

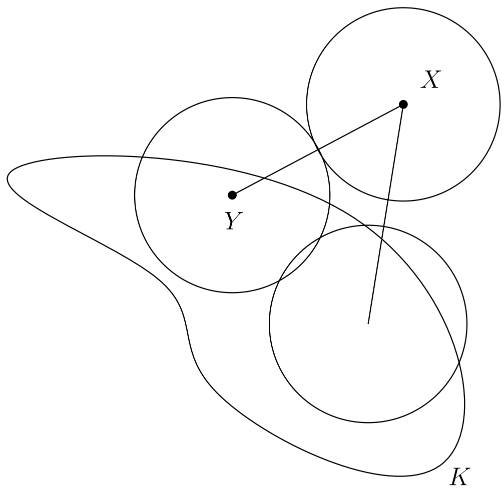
$$\left.\begin{array}{l} K \subseteq \bigcup_{i\in I} G_i \\ K \subseteq Y \end{array}\right\} \implies \left.\begin{array}{l} K \subseteq \left(\bigcup_{i\in I} G_i\right) \cap Y = \bigcup_{i\in I} \underbrace{(G_i \cap Y)}_{\text{open in } Y} \\ K \text{ is compact in } (Y, d_1) \end{array}\right\} \implies$$

$\implies \exists n \geq 1$ and $\exists i_1, \ldots, i_n \in I$ s.t. $K \subseteq \bigcup_{j=1}^n \left(G_{i_j} \cap Y\right) \subseteq \bigcup_{j=1}^n G_{i_j}$. □

Proposition 29.4

Let (X, d) be a metric space and let $K \subseteq X$ be compact. Then K is closed and bounded.

Proof. Let's prove K is closed. We'll show ${}^cK$ is open.
Case 1: ${}^cK = \emptyset$. This is open.
Case 2: ${}^cK \neq \emptyset$. Let $x \in {}^cK$
For $y \in K$ let $r_y = \frac{d(x,y)}{2}$. Note $r_y > 0$ (since $x \in {}^cK$ and $y \in K$).



Note

$$\left.\begin{array}{l} K \subseteq \bigcup_{y\in K} \underbrace{B_{r_y}(y)}_{\text{open}} \\ K \text{ is compact} \end{array}\right\} \implies \exists n \geq 1 \text{ and } \exists y_1, \ldots, y_n \in K \text{ s.t. } K \subseteq \bigcup_{j=1}^n B_{r_j}(y_j)$$

where we use the shorthand $r_j = r_{y_j}$.
Let $r = \min_{1\leq j\leq n} r_j > 0$.
By construction, $B_r(x) \cap B_{r_j}(y_j) = \emptyset \quad \forall 1 \leq j \leq n$.

$$\implies B_r(x) \subseteq {}^cB_{r_j}(y_j) \quad \forall 1 \leq j \leq n$$

$$\implies B_r(x) \subseteq \bigcap_{j=1}^n {}^cB_{r_j}(y_j) = {}^c\left(\bigcup_{j=1}^n B_{r_j}(y_j)\right) \subseteq {}^cK$$

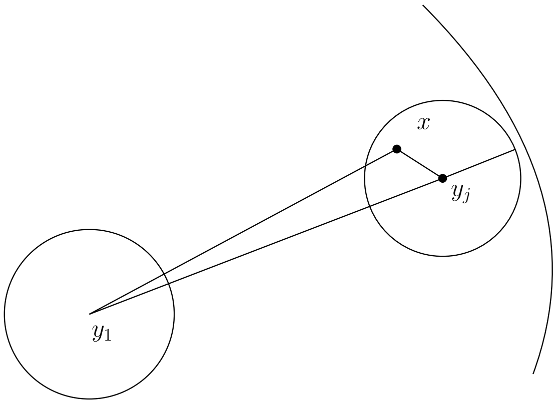
$$\implies \left.\begin{array}{l} x \in \overset{\circ}{\widehat{{}^cK}} \\ x \in {}^cK \text{ was arbitrary} \end{array}\right\} \implies {}^cK = \overset{\circ}{\widehat{{}^cK}}$$

Let's show K is bounded. Note

$$\left.\begin{array}{l} K \subseteq \bigcup_{y\in K} \underbrace{B_1(y)}_{\text{open}} \\ K \text{ compact} \end{array}\right\} \implies \exists n \geq 1 \text{ and } \exists y_1, \dots, y_n \in K \text{ s.t. } K \subseteq \bigcup_{j=1}^{n} B_1(y_j)$$

For $2 \leq j \leq n$, let $r_j = d(y_1, y_j) + 1$.

Claim 29.1. $B_1(y_j) \subseteq B_{r_j}(y_1)$



Indeed, if $x \in B_1(y_j) \implies d(x, y_j) < 1$. By the triangle inequality

$$d(y_1, x) \leq d(y_j, x) + d(y_1, y_j) < 1 + d(y_1, y_j) = r_j \implies x \in B_{r_j}(y_1)$$

So with $r = \max_{2\leq j\leq n} r_j$,

$$K \subseteq \bigcup_{j=1}^{n} B_1(y_j) \subseteq B_r(y_1)$$

□

Proposition 29.5

Let (X, d) be a metric space and let $F \subseteq K \subseteq X$ such that F is closed in X and K is compact. Then F is compact.

Proof. Let $\{G_i\}_{i\in I}$ be a family of open sets in X s.t.

$$F \subseteq \bigcup_{i\in I} G_i$$

Then

$$\left.\begin{array}{l} K \subseteq F \cup {}^cF \subseteq \bigcup_{i\in I} G_i \cup \underbrace{{}^cF}_{\text{open in } X} \\ K \text{ compact} \end{array}\right\} \implies$$

$\implies \exists n \geq 1$ and $\exists i_1, \dots, i_n \in I$ s.t.

$$\left.\begin{array}{l} K \subseteq \bigcup_{j=1}^{n} G_{i_j} \cup {}^cF \\ F \subseteq K \end{array}\right\} \implies F = \left(\bigcup_{j=1}^{n} G_{i_j} \cup {}^cF\right) \cap F \subseteq \bigcup_{j=1}^{n} G_{i_j}$$

So F is compact. □

Corollary 29.6

Let (X, d) be a metric space and let $F \subseteq X$ be closed and let $K \subseteq X$ be compact. Then $K \cap F$ is compact.

Proof. K is compact. So

$$\left.\begin{array}{l} K \text{ closed} \\ F \text{ closed} \end{array}\right\} \implies \left.\begin{array}{l} K \cap F \text{ is closed} \\ K \cap F \subseteq K \text{ compact} \end{array}\right\} \implies K \cap F \text{ is compact}$$

□

§29.2 Sequential Compactness

Definition 29.7 (Sequential Compactness) — Let (X, d) be a metric space. A set $K \subseteq X$ is called sequentially compact if every sequence $\{x_n\}_{n \geq 1} \subseteq K$ admits a subsequence that converges in K.

§30 Lec 2: Mar 31, 2021

§30.1 Sequential Compactness (Cont'd)

Theorem 30.1 (Bolzano – Weierstrass)
Let (X, d) be a metric space and let $K \subseteq X$ be infinite. The following are equivalent:

1. K is sequentially compact.
2. For every infinite $A \subseteq K$ we have $A' \cap K \neq \emptyset$.

Proof. 1) $\implies$ 2) Let $A \subseteq K$ be infinite. As every infinite set has a countable subset we can find a sequence $\{a_n\}_{n\geq 1} \subseteq A$ such that $a_n \neq a_m \, \forall n \neq m$. As K is sequentially compact, $\exists \{a_{k_n}\}_{n\geq 1}$ subsequence of $\{a_n\}_{n\geq 1}$ s.t.

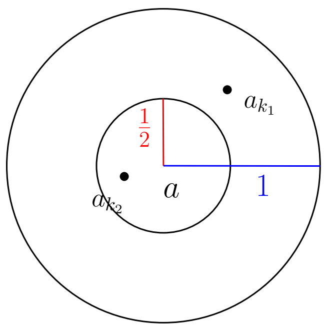
$$a_{k_n} \xrightarrow[n\to\infty]{d} a \in K$$

Claim 30.1. $a \in A' \iff \forall r > 0 \; B_r(a) \cap A \setminus \{a\} \neq \emptyset$.

Indeed, fix $r > 0$.

$$a_{k_n} \xrightarrow[n\to\infty]{d} a \implies \exists n_r \in \mathbb{N} \text{ s.t. } d(a, a_{k_n}) < r \quad \forall n \geq n_r$$

As $a_n \neq a_m \, \forall n \neq m$, $\exists n_0 \geq n_r$ s.t. $a_{k_{n_0}} \neq a$. Then $a_{k_{n_0}} \in B_r(a) \cap A \setminus \{a\}$. We get $a \in A' \cap K$.
2) $\implies$ 1) Let $\{a_n\}_{n\geq 1} \subseteq K$. We distinguish two cases:
Case 1: The sequence $\{a_n\}_{n\geq 1}$ contains a constant subsequence. That subsequence converges to an element in K.
Case 2: $\{a_n\}_{n\geq 1}$ does not contain a constant subsequence. Then $A = \{a_n : n \geq 1\}$ is infinite and $A \subseteq K$. So $A' \cap K \neq \emptyset$. Let $a \in A' \cap K$. Then $\exists \{a_{k_n}\}_{n\geq 1}$ subsequence of $\{a_n\}_{n\geq 1}$ s.t. $a_{k_n} \xrightarrow[n\to\infty]{d} a$.



$B_1(a) \cap A \setminus \{a\} \neq \emptyset$

$B_{\min\{\frac{1}{2}, d(a, a_{k_1})\}}(a) \cap A \setminus \{a\} \neq \emptyset$

$k_2 > k_1$

□

Theorem 30.2
Let (X, d) be a metric space and let $K \subseteq X$ be compact. Then K is sequentially compact.

Proof. If K is finite, then any sequence $\{x_n\}_{n\geq 1} \subseteq K$ will have a constant subsequence.
Assume now K is infinite. We will use the Bolzano – Weierstrass theorem. It suffices to prove that for any infinite $A \subseteq K$ we have $A' \cap K \neq \emptyset$.

$$\left.\begin{array}{l}\text{Note } A \subseteq K \text{ then } A' \subseteq K' \\ K \text{ compact } \implies K \text{ closed } \implies K' \subseteq K\end{array}\right\} \implies A' \subseteq K \implies A' \cap K = A'$$

We argue by contradiction. Assume $A' = \emptyset$. Then for $x \in K$ we have $x \notin A' \implies \exists r_x > 0$ s.t. $B_{r_x}(x) \cap A \setminus \{x\} = \emptyset$. So

$$\left.\begin{array}{l}K \subseteq \bigcup_{x\in K} \underbrace{B_{r_x}(x)}_{\text{open}} \\ K \text{ compact}\end{array}\right\} \implies \exists n \geq 1 \text{ and } \exists x_1, \dots, x_n \in K \text{ s.t.}$$

$$K \subseteq \bigcup_{j=1}^{n} B_{r_j}(x_j) \text{ where } r_j = r_{x_j}$$

In particular,

$$\left.\begin{array}{l}A = \left(\bigcup_{j=1}^n B_{r_j}(x_j)\right) \cap A = \bigcup_{j=1}^n \left[B_{r_j}(x_j) \cap A\right] \\ \text{By construction, } B_{r_j}(x_j) \cap A \subseteq \{x_j\}\end{array}\right\} \implies \underbrace{A}_{\text{infinite}} \subseteq \underbrace{\bigcup_{j=1}^{n} \{x_j\}}_{\text{finite}}$$

– Contradiction! So $A' \neq \emptyset$. □

Proposition 30.3

Let (X, d) be a metric space and let $K \subseteq X$ be sequentially compact. Then K is closed and bounded.

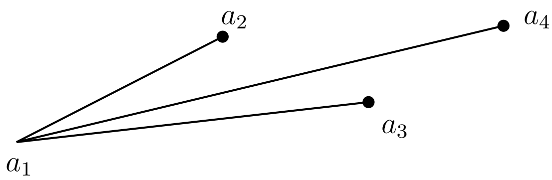
Proof. Let's show K is closed $\iff K = \overline{K}$.
We know $K \subseteq \overline{K}$. We need to show $\overline{K} \subseteq K$. Let $x \in \overline{K} \implies \exists \{x_n\}_{n\geq 1} \subseteq K$ s.t. $x_n \xrightarrow[n\to\infty]{d} x$.
K sequentially compact $\implies \exists \{x_{k_n}\}_{n\geq 1}$ subsequence of $\{x_n\}_{n\geq 1}$ s.t.

$$\left.\begin{array}{l}x_{k_n} \xrightarrow[n\to\infty]{d} y \in K \\ x_n \xrightarrow[n\to\infty]{d} x \implies x_{k_n} \xrightarrow[n\to\infty]{d} x \\ \text{Limits of convergent sequences are unique}\end{array}\right\} \implies x = y \in K$$

As $x \in \overline{K}$ was arbitrary, we get $\overline{K} \subseteq K$.
Let's show K is bounded. We argue by contradiction. Assume K is not bounded. Let $a_1 \in K$.

$$K \text{ not bounded } \implies K \not\subseteq B_1(a_1) \implies \exists a_2 \in K \text{ s.t. } d(a_1, a_2) \geq 1$$
$$K \text{ not bounded } \implies K \not\subseteq B_{1+d(a_1,a_2)}(a_1) \implies \exists a_3 \in K \text{ s.t. } d(a_1, a_3) \geq 1 + d(a_1, a_2)$$

Proceeding inductively, we find a sequence $\{a_n\}_{n\geq 1} \subseteq K$ s.t. $d(a_1, a_{n+1}) \geq 1 + d(a_1, a_n)$.



By construction,

$$|d(a_1, a_m) - d(a_1, a_n)| \geq |n - m| \quad \forall n, m \geq 1$$

By the triangle inequality,

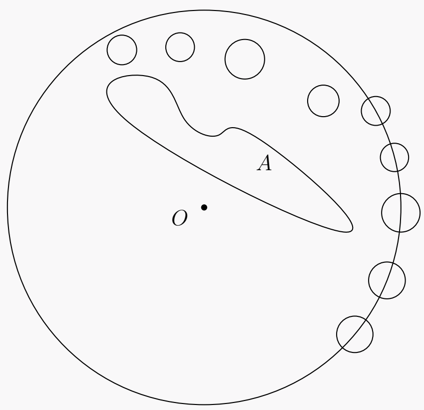
$$d(a_n, a_m) \geq |d(a_1, a_n) - d(a_1, a_m)| \geq |n - m| \quad \forall n, m \geq 1$$

This sequence cannot have a convergent (Cauchy) subsequence, thus contradiction the hypothesis that K is sequentially compact. So K is bounded. $\square$

Definition 30.4 (Totally Bounded) — Let (X, d) be a metric space. A set $A \subseteq X$ is totally bounded if for every $\varepsilon > 0$, A can be covered by finitely many balls of radius ε.

Remark 30.5.

1. A totally bounded $\implies$ A bounded.
   Indeed, taking $\varepsilon = 1$, $\exists n \geq 1$ and $\exists x_1, \ldots, x_n \in X$ s.t.
   $$A \subseteq \bigcup_{j=1}^{n} B_1(x_j) \subseteq B_r(x_1)$$
   where $r = 1 + \max_{2 \leq j \leq n} d(x_1, x_j)$.
2. A bounded $\not\Longrightarrow$ A totally bounded.
   Consider $\mathbb{N}$ equipped with the discrete metric
   $$d(n, m) = \begin{cases} 0, & n = m \\ 1, & n \neq m \end{cases}$$
   Then $\mathbb{N} = B_2(1)$, but $\mathbb{N}$ cannot be covered by finitely many balls of radius $\frac{1}{2}$ since $B_{\frac{1}{2}}(n) = \{n\}$.
3. On $(\mathbb{R}^n, d_2)$, A bounded $\implies$ A totally bounded. Indeed, A bounded $\implies$ $A \subseteq B_R(0)$ for some $R > 0$. $B_R(0)$ can be covered by $10^6 \left(\frac{R}{\varepsilon}\right)^n$ many balls of radius ε.



§31 Lec 3: Apr 2, 2021

§31.1 Heine – Borel Theorem

Theorem 31.1
Let (X, d) be a metric space and let $K \subseteq X$. The following are equivalent:

1. K is sequentially compact.
2. K is complete and totally bounded.

Proof. 1) $\implies$ 2) Let's show K is complete. Let $\{x_n\}_{n\geq 1}$ be a Cauchy sequence with $x_n \in K \quad \forall n \geq 1$.
K sequentially compact $\implies \exists \{x_{k_n}\}_{n\geq 1}$ subsequence of $\{x_n\}_{n\geq 1}$ s.t.

$$\left.\begin{array}{l} x_{k_n} \xrightarrow[n\to\infty]{d} y \in K \\ \{x_n\}_{n\geq 1} \text{ is Cauchy} \end{array}\right\} \implies x_n \xrightarrow[n\to\infty]{d} y \in K$$

As $\{x_n\}_{n\geq 1} \subseteq K$ was arbitrary, we get that K is complete.
Let's show K is totally bounded. Fix $\varepsilon > 0$ and $a_1 \in K$.

- If $K \subseteq B_\varepsilon(a_1)$, then K is totally bounded.
- If $K \not\subseteq B_\varepsilon(a_1)$, then $\exists a_2 \in K$ s.t. $d(a_1, a_2) \geq \varepsilon$
- If $K \subseteq B_\varepsilon(a_1) \cup B_\varepsilon(a_2)$, then K is totally bounded.
- If $K \not\subseteq B_\varepsilon(a_1) \cup B_\varepsilon(a_2)$, then $\exists a_3 \in K$ s.t. $d(a_1, a_3) \geq \varepsilon$ and $d(a_2, a_3) \geq \varepsilon$.

We distinguish two cases:
Case 1: The process terminates in finitely many steps $\implies$ K is totally bounded.
Case 2: The process does not terminate in finitely many steps. Then we find $\{a_n\}_{n\geq 1} \subseteq K$ s.t. $d(a_n, a_m) \geq \varepsilon \quad \forall n \neq m$. This sequence does not admit a convergent subsequence, contradicting the fact that K is sequentially compact.
2) $\implies$ 1) Let $\{a_n\}_{n\geq 1} \subseteq K$. K totally bounded $\implies$ $\mathcal{J}_1$ finite and $\left\{x_j^{(1)}\right\}_{j\in\mathcal{J}_1} \subseteq X$ s.t.

$$\left.\begin{array}{l} K \subseteq \bigcup_{j\in\mathcal{J}_1} B_1(x_j^{(1)}) \\ \{a_n\}_{n\geq 1} \subseteq K \end{array}\right\} \implies \exists j_1 \in \mathcal{J}_1 \text{ s.t. } \left|\left\{n : a_n \in B_1(x_{j_1}^{(1)})\right\}\right| = \aleph_0$$

Let $\left\{a_n^{(1)}\right\}_{n\geq 1}$ be the corresponding subsequence.
K totally bounded $\implies \exists \mathcal{J}_2$ finite and $\left\{x_j^{(2)}\right\}_{j\in\mathcal{J}_2} \subseteq X$ s.t.

$$\left.\begin{array}{l} K \subseteq \bigcup_{j\in\mathcal{J}_2} B_{\frac{1}{2}}(x_j^{(2)}) \\ \left\{a_n^{(1)}\right\}_{n\geq 1} \subseteq K \end{array}\right\} \implies \exists j_2 \in \mathcal{J}_2 \text{ s.t. } \left|\left\{n : a_n^{(1)} \in B_{\frac{1}{2}}(x_{j_2}^{(2)})\right\}\right| = \aleph_0$$

Let $\left\{a_n^{(2)}\right\}_{n\geq 1}$ denote the corresponding subsequence.
We proceed inductively. We find that $\forall k \geq 1$

- $\left\{a_n^{(k+1)}\right\}_{n\geq 1}$ subsequence of $\left\{a_n^{(k)}\right\}_{n\geq 1}$

- $\left\{a_n^{(k)}\right\}_{n\geq 1} \subseteq B_{\frac{1}{k}}\left(x_{j_k}^{(k)}\right)$ for some $x_{j_k}^{(k)} \in X$.

We consider the subsequence $\left\{a_n^{(n)}\right\}_{n\geq 1}$ of $\{a_n\}_{n\geq 1}$.

$$\begin{aligned}
\left\{a_n^{(1)}\right\}_{n\geq 1} &= \left(a_1^{(1)}, \quad a_2^{(1)}, \quad a_3^{(1)}, \quad \ldots\right) \\
\left\{a_n^{(2)}\right\}_{n\geq 1} &= \left(\quad a_1^{(2)}, \quad a_2^{(2)}, \quad a_3^{(2)}, \quad \ldots\right) \\
\left\{a_n^{(3)}\right\}_{n\geq 1} &= \left(\quad\quad a_1^{(3)}, \quad a_2^{(3)}, \quad a_3^{(3)}, \quad \ldots\right)
\end{aligned}$$

For $n, m \geq k$ the $a_n^{(n)}, a_m^{(m)}$ belong to the subsequence $\left\{a_n^{(k)}\right\}_{n\geq 1}$. In particular,

$$d(a_n^{(n)}, a_m^{(m)}) \leq d(a_n^{(n)}, x_{j_k}^{(k)}) + d(a_m^{(m)}, x_{j_k}^{(k)}) < \frac{2}{k} \quad \forall n, m \geq k$$

This shows $\left\{a_n^{(n)}\right\}_{n\geq 1}$ is Cauchy and K is complete, so $a_n^{(n)} \xrightarrow[n\to\infty]{d} a \in K$. As $\{a_n\}_{n\geq 1}$ was arbitrary, we get that K is sequentially compact. □

Lemma 31.2

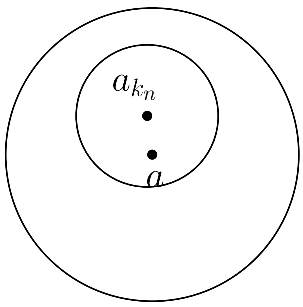
Let (X, d) be a sequentially compact metric space. Let $\{G_i\}_{i\in I}$ be an open cover of X. Then there exists $\varepsilon > 0$ such that every ball of radius ε is contained in at least one G_i.

Proof. We argue by contradiction. Then

$$\forall n \geq 1 \quad \exists a_n \in X \text{ s.t. } B_{\frac{1}{n}}(a_n) \text{ is not contained in any } G_i$$

X is sequentially compact $\implies \exists \{a_{k_n}\}_{n\geq 1}$ subsequence of $\{a_n\}_{n\geq 1}$ s.t.

$$\begin{gathered}
a_{k_n} \xrightarrow[n\to\infty]{d} a \in X = \bigcup_{i\in I} G_i \implies \exists i_0 \in I \text{ s.t. } a \in G_{i_0} \\
G_{i_0} \text{ open } \implies \exists r > 0 \text{ s.t. } B_r(a) \subseteq G_{i_0} \\
a_{k_n} \xrightarrow[n\to\infty]{d} a \implies \exists n_1(r) \in \mathbb{N} \text{ s.t. } d(a_1, a_{k_n}) < \frac{r}{2} \, \forall n \geq n_1
\end{gathered}$$



Let $n_2(r)$ s.t. $n_2 > \frac{2}{r}$.

Claim 31.1. $\forall n \geq n_r = \max\{n_1, n_2\}$ we have $B_{\frac{1}{k_n}}(a_{k_n}) \subseteq B_r(a) \subseteq G_{i_0}$ thefore giving a contradiction!

Fix $x \in B_{\frac{1}{k_n}}(a_{k_n})$. Then

$$d(a,x) \le d(x,a_{k_n}) + d(a_{k_n},a) < \frac{1}{k_n} + \frac{r}{2} < \frac{r}{2} + \frac{r}{2} = r$$

□

Theorem 31.3

A sequentially compact metric space (X,d) is compact.

Proof. Let $\{G_i\}_{i\in I}$ be an open cover of X. Let ε be given by the previous lemma. X sequentially compact $\implies$ X totally bounded $\implies$ $\exists n \ge 1$ and

$$\left.\begin{array}{l}\exists x_1,\dots,x_n \in X \text{ s.t. } X = \bigcup_{j=1}^n B_\varepsilon(x_j) \\ \forall 1 \le j \le n \quad \exists i_j \in I \text{ s.t. } B_\varepsilon(x_j) \subseteq G_{i_j}\end{array}\right\} \implies X = \bigcup_{j=1}^n G_{i_j}$$

□

Collecting our results so far we obtain

Theorem 31.4 (Heine – Borel)

Let (X,d) be a metric space and let $K \subseteq X$. The following are equivalent:

1. K is compact,
2. K is sequentially compact,
3. K is complete and totally bounded,
4. Every infinite subset of K has an accumulation point in K.

Remark 31.5. In $\mathbb{R}^n$, K is compact $\iff$ K is closed and bounded.

Definition 31.6 (Finite Intersection Property) — An infinite family $\{F_i\}_{i\in I}$ of closed sets is said to have the finite intersection property if $\forall \mathcal{J} \subseteq I$ finite we have

$$\bigcap_{j\in\mathcal{J}} F_j \ne \emptyset$$

Theorem 31.7

A metric space (X,d) is compact if and only if every infinite family $\{F_i\}_{i\in I}$ of closed sets with the finite intersection property satisfies

$$\bigcap_{i\in I} F_i \ne \emptyset$$

Proof. " $\implies$ " We argue by contradiction. Assume $\exists \{F_i\}_{i\in I}$ closed sets with the finite intersection property s.t. $\bigcap_{i\in I} F_i = \emptyset$

$$\left.\begin{array}{l} X = {}^c\left(\bigcap_{i\in I} F_i\right) = \bigcup_{i\in I} \underbrace{{}^cF_i}_{\text{open}} \\ X \text{ compact} \end{array}\right\} \implies \exists \mathcal{J} \subseteq I \text{ finite s.t. } X = \bigcup_{j\in\mathcal{J}} {}^cF_j$$

$$\implies \emptyset = {}^c\left(\bigcup_{j\in\mathcal{J}} {}^cF_j\right) = \bigcap_{j\in\mathcal{J}} F_j \text{ – Contradiction!}$$

" $\impliedby$ " We argue by contradiction. Assume $\exists \{G_i\}_{i\in I}$ open cover of X that does not admit a finite subcover.

So $\forall \mathcal{J} \subseteq I$ finite $X \neq \bigcup_{j\in\mathcal{J}} G_j \implies \emptyset \neq \bigcap_{j\in\mathcal{J}} \underbrace{{}^cG_j}_{\text{closed}}$. So $\{{}^cG_i\}_{i\in I}$ is a family of closed sets with the finite intersection property. Then

$$\bigcap_{i\in I} {}^cG_i \neq \emptyset \implies \bigcup_{i\in I} G_i \neq X$$

Contradiction! □

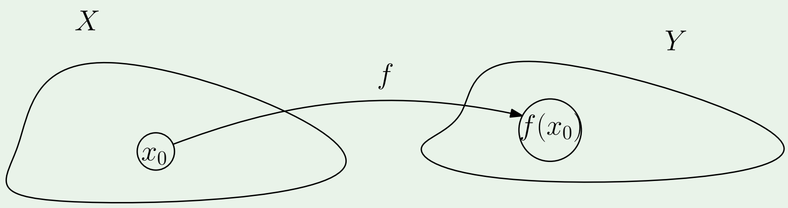
§32 Lec 4: Apr 5, 2021

§32.1 Continuity

Definition 32.1 (Continuous Function) — Let (X, d_X) and (Y, d_Y) be two metric spaces. We say that a function $f : X \to Y$ is continuous at a point $x_0 \in X$ if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \text{ s.t. } d_X(x, x_0) < \delta \text{ then } d_Y\left(f(x), f(x_0)\right) < \varepsilon$$

We say f is continuous (on X) if f is continuous at every point in X.



Remark 32.2. $f : X \to Y$ is continuous at every isolated point in X. Indeed, if $x_0 \in X$ is isolated, then $\exists \delta > 0$ s.t. $B_\delta^X(x_0) = \{x_0\}$. Then $d_X(x, x_0) < \delta \implies d_Y\left(f(x), f(x_0)\right) = 0$

Proposition 32.3

Let $(X, d_X), (Y, d_Y)$ be two metric spaces and $f : X \to Y$ be a function. The following are equivalent:

1. f is continuous at $x_0 \in X$.
2. For any $\{x_n\}_{n\geq 1} \subseteq X$ s.t. $x_n \xrightarrow[n\to\infty]{d_X} x_0$ we have $f(x_n) \xrightarrow[n\to\infty]{d_Y} f(x_0)$.

Proof. 1) $\implies$ 2) Let $\{x_n\}_{n\geq 1} \subseteq X$ s.t. $x_n \xrightarrow[n\to\infty]{d_X} x_0$.
Let $\varepsilon > 0$. f continuous at $x_0 \implies \exists \delta > 0$ s.t.

$$\left.\begin{aligned} &d_X(x, x_0) < \delta \implies d_Y\left(f(x), f(x_0)\right) < \varepsilon \\ &x_n \xrightarrow[n\to\infty]{d_X} x_0 \implies \exists n_\delta \in \mathbb{N} \text{ s.t. } d_X(x_n, x_0) < \delta \,\forall n \geq n_\delta \end{aligned}\right\} \implies d_X\left(f(x_n), f(x_0)\right) < \varepsilon$$

for each $n \geq n_\delta$.
2) $\implies$ 1) We argue by contradiction. Assume

$$\exists \varepsilon_0 > 0 \text{ s.t. } \forall \delta > 0 \quad \exists x_\delta \in X \text{ s.t. } d_X(x_\delta, x_0) < \delta \text{ but } d_Y\left(f(x_\delta), f(x_0)\right) \geq \varepsilon_0$$

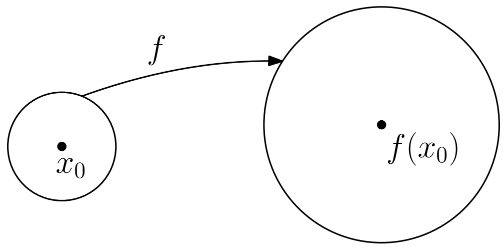
Letting $\delta = \frac{1}{n}$ we find $\{x_n\}_{n\geq 1} \subseteq X$ s.t. $d_X(x_n, x_0) < \frac{1}{n}$ but $d_Y\left(f(x_n), f(x_0)\right) \geq \varepsilon_0$ – Contradiction! □

Theorem 32.4

Let $(X, d_X), (Y, d_Y)$ be two metric spaces and let $f : X \to Y$ be a function. The following are equivalent:

1. f is continuous.
2. for any G open in Y, $f^{-1}(G) = \{x \in X : f(X) \in G\}$ is open in X.
3. for any F closed in Y, $f^{-1}(F)$ is closed in X.
4. for any $B \subseteq Y$, $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.
5. for any $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.

Proof. We will show 1) $\implies$ 2) $\implies$ 3) $\implies$ 4) $\implies$ 5) $\implies$ 1).
1) $\implies$ 2) Let $G \subseteq Y$ be open.



Let $x_0 \in f^{-1}(G)$

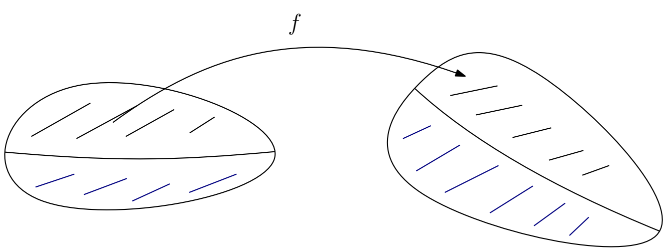
$$\implies \left.\begin{matrix} f(x_0) \in G \\ G \text{ open in } Y \end{matrix}\right\} \implies \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon^Y\left(f(x_0)\right) \subseteq G$$

f is continuous

$$\implies \exists \delta > 0 \text{ s.t. } f\left(B_\delta^X(x_0)\right) \subseteq B_\varepsilon^Y\left(f(x_0)\right) \subseteq G$$
$$\implies B_\delta^X(x_0) \subseteq f^{-1}(G) \implies x_0 \in \overset{\circ}{\widehat{f^{-1}(G)}}$$

So $f^{-1}(G)$ is open in X.
2) $\implies$ 3) Let $F \subseteq Y$ be closed $\implies$ ${}^cF = Y \setminus F$ is open in Y. By assumption,

$$\left.\begin{matrix} f^{-1}\left({}^cF\right) \text{ is open in } X \\ f^{-1}\left({}^cF\right) = {}^c\left[f^{-1}(F)\right] = X \setminus f^{-1}(F) \end{matrix}\right\} \implies f^{-1}(F) \text{ is closed in } X$$



$$f^{-1}\left(Y \setminus F\right) = f^{-1}(Y) \setminus f^{-1}(F) = X \setminus f^{-1}(F)$$

3) $\implies$ 4) Let $B \subseteq Y \implies \overline{B}$ closed in Y. By assumption,

$$\left.\begin{matrix} f^{-1}(\overline{B}) \text{ closed in } X \\ f^{-1}(\overline{B}) \supseteq f^{-1}(B) \end{matrix}\right\} \implies \overline{f^{-1}(B)} \subseteq \overline{f^{-1}(\overline{B})} = f^{-1}(\overline{B})$$

4) $\implies$ 5) Let $A \subseteq X$. Use the hypothesis with $B = f(A)$. We have

$$\overline{A} \subseteq \overline{f^{-1}(f(A))} \subseteq f^{-1}\left(\overline{f(A)}\right) \implies f(\overline{A}) \subseteq \overline{f(A)}$$

5) $\implies$ 1) We argue by contradiction. Assume $\exists x_0 \in X$ s.t. f is not continuous at x_0. Then $\exists \varepsilon_0 > 0$ and $\exists x_n \xrightarrow[n\to\infty]{d_X} x_0$ but $d_Y(f(x_n), f(x_0)) \geq \varepsilon_0$.
Let $A = \{x_n : n \geq 1\}$. Then $x_0 \in \overline{A}$ but $f(x_0) \notin \overline{\{f(x_n) : n \geq 1\}} = \overline{f(A)}$. On the other hand, we must have

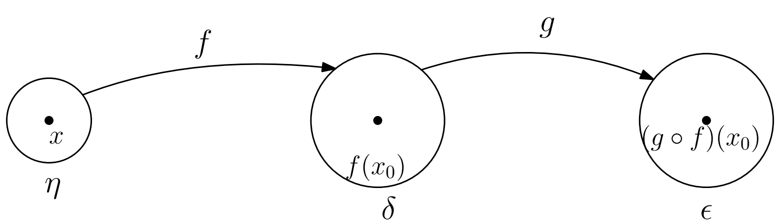
$$\left.\begin{matrix} f(\overline{A}) \subseteq \overline{f(A)} \\ x_0 \in \overline{A} \end{matrix}\right\} \implies f(x_0) \in \overline{f(A)}$$

Contradiction! □

Proposition 32.5
Let $(X, d_X), (Y, d_Y), (Z, d_Z)$ be metric spaces and assume $f : X \to Y$ is continuous at $x_0 \in X$ and $g : Y \to Z$ is continuous at $f(x_0) \in Y$. Then $g \circ f : X \to Z$ is continuous at x_0.

Proof. Fix $\varepsilon > 0$.

$$g \text{ continuous at } f(x_0) \implies \exists \delta > 0 \text{ s.t. } d_Y(y, f(x_0)) < \delta \implies d_Z(g(y), g(f(x_0))) < \varepsilon$$
$$f \text{ continuous at } x_0 \implies \exists \eta > 0 \text{ s.t. } d_X(x, x_0) < \eta \implies d_Y(f(x), f(x_0)) < \delta$$



So if $d_X(x, x_0) < \eta$ then $d_Z(g(f(x)), g(f(x_0))) < \varepsilon$. □

Exercise 32.1. Let (X, d) be a metric space and let $f, g : X \to \mathbb{R}$ be continuous at $x_0 \in X$. Then $f \pm g, f \cdot g$ are continuous at x_0. If $g(x_0) \neq 0$ then $\frac{f}{g} : X \to \mathbb{R}$ is continuous at x_0.

Exercise 32.2. Let (X, d) be a metric space and let $f_1, \ldots, f_n : X \to \mathbb{R}$. Then $f = (f_1, \ldots, f_n) : X \to \mathbb{R}^n$ is continuous at $x_0 \in X$ if and only if $f_1, \ldots, f_n$ are continuous at x_0.

Hint: $|f_i(x) - f_i(x_0)| \leq d_2(f(x), f(x_0)) = \sqrt{\sum_{j=1}^n |f_j(x) - f_j(x_0)|^2}$.

§32.2 Continuity and Compactness

Theorem 32.6

Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f : X \to Y$ be continuous. If K is compact in X, then $f(K)$ is compact in Y.

Proof. Method 1: Let $\{G_i\}_{i\in I}$ be a family of open sets in Y s.t.

$$f(K) \subseteq \bigcup_{i\in I} G_i \implies K \subseteq f^{-1}\left(\bigcup_{i\in I} G_i\right) = \bigcup_{i\in I} \underbrace{f^{-1}(G_i)}_{\text{open in } X}$$

K compact $\implies \exists n \geq 1$ and $\exists i, \ldots, i_n \in I$ s.t.

$$K \subseteq \bigcup_{j=1}^{n} f^{-1}\left(G_{i_j}\right) = f^{-1}\left(\bigcup_{j=1}^{n} G_{i_j}\right) \implies f(K) \subseteq \bigcup_{j=1}^{n} G_{i_j}$$

Method 2: Let's show $f(K)$ is sequentially compact. Let $\{y_n\}_{n\geq 1} \subseteq f(K)$.

$$y_n \in f(K) \implies \exists x_n = f^{-1}(y_n) \in K$$

As K is sequentially compact, $\exists \{x_{k_n}\}_{n\geq 1}$ subsequence of $\{x_n\}_{n\geq 1}$ s.t.

$$\left.\begin{array}{l} x_{k_n} \xrightarrow[n\to\infty]{d_X} x_0 \in K \\ f \text{ is continuous} \end{array}\right\} \implies \underbrace{f\left(x_{k_n}\right)}_{=y_{k_n}} \xrightarrow[n\to\infty]{d_Y} f(x_0) \in f(K)$$

□

§33 Lec 5: Apr 7, 2021

§33.1 Continuity and Compactness (Cont'd)

Corollary 33.1

Let (X, d_X) be a compact metric space and let $f : X \to \mathbb{R}^n$ be continuous. Then $f(X)$ is closed and bounded.

Corollary 33.2

Let (X, d_X) be a compact metric space and let $f : X \to \mathbb{R}$ be continuous. Then there exists $x_1, x_2 \in X$ s.t.

$$f(x_1) = \inf\{f(x) : x \in X\} \text{ and } f(x_2) = \sup\{f(x) : x \in X\}$$

Proof. $f(x)$ is closed and bounded.

$$\begin{aligned} \text{Boundedness} &\implies \inf f(x) \text{ and } \sup f(x) \text{ are well defined} \\ \text{Closedness} &\implies \inf f(x), \sup f(x) \in \overline{f(x)} = f(x) \end{aligned}$$

□

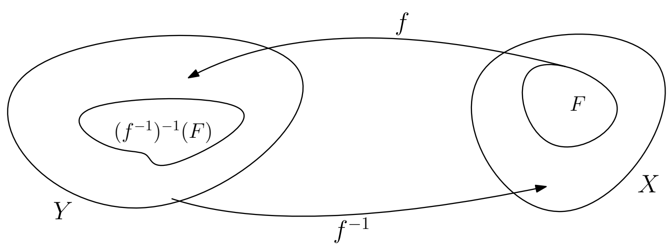
Proposition 33.3

Let $(X, d_X), (Y, d_Y)$ be metric spaces s.t. X is compact. Let $f : X \to Y$ be bijective and continuous. Then $f^{-1} : Y \to X$ is continuous.

Proof. If suffices to show that for every closed set $F \subseteq X$, we have

$$\left(f^{-1}\right)^{-1}(F) = \left\{y \in Y : f^{-1}(y) \in F\right\}$$

is closed in Y.



But $\left(f^{-1}\right)^{-1}(F) = f(F)$.

$$\left.\begin{aligned} &F \text{ closed in } X \text{ compact} \implies F \text{compact} \\ &f : X \to Y \text{ is continuous} \end{aligned}\right\} \implies f(F) \text{ is compact and closed}$$

□

Definition 33.4 (Uniform Continuity) — Let $(X, d_X), (Y, d_Y)$ be metric spaces. We say that a function $f : X \to Y$ is uniformly continuous if

$$\forall \varepsilon > 0 \quad \exists \delta = \delta(\varepsilon) \text{ s.t. } d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

Compare this with $g : X \to Y$ is continuous if

$$\forall x \in X \quad \forall \varepsilon > 0 \quad \exists \delta = \delta(\varepsilon, x) \text{ s.t. } d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

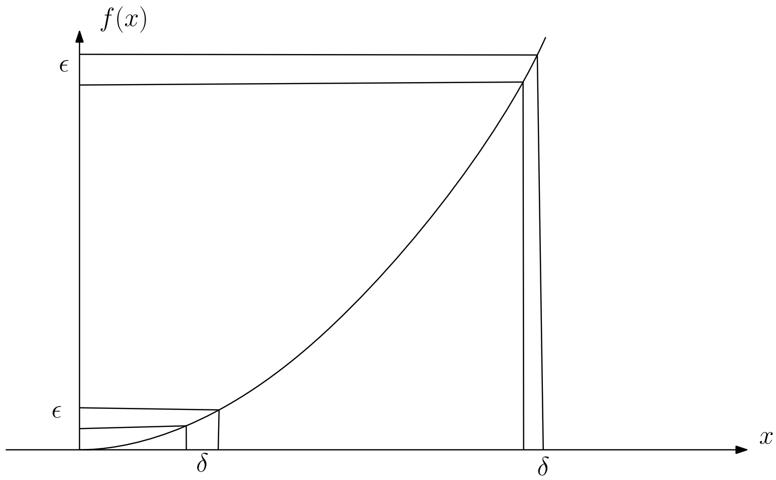
Remark 33.5.
1. Continuity is defined pointwise. Uniform continuity is a property of a function on a set.
2. Uniform continuity $\implies$ continuity.
3. There are continuous functions that are not uniformly continuous.

For example, consider

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$$

Let $x_n = n + \frac{1}{n}$, $y_n = n$

$$|x_n - y_n| = \frac{1}{n} \underset{n\to\infty}{\longrightarrow} 0$$
$$|f(x_n) - f(y_n)| = \left(n + \frac{1}{n}\right)^2 - n^2 = 2 + \frac{1}{n^2} > 2$$



Theorem 33.6

Let $(X, d_X), (Y, d_Y)$ be metric spaces with X compact. Let $f : X \to Y$ continuous. Then f is uniformly continuous.

Proof. We argue by contradiction. Assume f is not uniformly continuous $\implies \exists \varepsilon_0 > 0$ s.t. $\forall \delta > 0 \, \exists x_\delta, y_\delta \in X$ s.t. $d_X(x_\delta, y_\delta) < \delta$ but $d_Y(f(x_\delta), f(y_\delta)) \geq \varepsilon_0$.

Let $\delta = \frac{1}{n}$ to get $\{x_n\}_{n\geq 1}, \{y_n\}_{n\geq 1} \subseteq X$ s.t. $d_X(x_n, y_n) < \frac{1}{n}$ but $d_Y(f(x_n), f(y_n)) \geq \varepsilon_0$

X compact $\implies \exists \{x_{k_n}\}_{n\geq 1}$ subsequence of $\{x_n\}_{n\geq 1}$ s.t.

$$x_{k_n} \underset{n\to\infty}{\overset{d_X}{\longrightarrow}} x_0 \in X$$

By the triangle inequality,

$$d(y_{k_n}, x_0) \le \underbrace{d(x_{k_n}, y_{k_n})}_{<\frac{1}{k_n} \le \frac{1}{n} \xrightarrow[n\to\infty]{} 0} + \underbrace{d(x_{k_n}, x_0)}_{\xrightarrow[n\to\infty]{} 0} \xrightarrow[n\to\infty]{} 0 \implies y_{k_n} \xrightarrow[n\to\infty]{d_X} x_0$$

$$f \text{ continuous } \implies \begin{cases} f(x_{k_n}) \xrightarrow[n\to\infty]{d_Y} f(x_0) \\ f(y_{k_n}) \xrightarrow[n\to\infty]{d_Y} f(x_0) \end{cases}$$

But

$$\varepsilon_0 \le d_Y(f(x_{k_n}), f(y_{k_n})) \le \underbrace{d_Y(f(x_{k_n}), f(x_0))}_{\to 0} + \underbrace{d_Y(f(x_0), f(y_{k_n}))}_{\to 0} \xrightarrow[n\to\infty]{} 0$$

Contradiction! □

§33.2 Continuity and Connectedness

Theorem 33.7

Let $(X, d_X), (Y, d_Y)$ be metric spaces s.t. X is connected. Let $f : X \to Y$ be continuous. Then $f(X)$ is connected.

Proof. Method 1: Abusing notation we write $f : X \to f(x)$. It suffices to show that if $\emptyset \ne B \subseteq f(x)$ is both open and closed in $f(x)$ then $B = f(x)$.

As f is continuous, $f^{-1}(B) \ne \emptyset$ is both open and closed in X. But X is connected which implies $f^{-1}(B) = X$ and $f(x) = B$.

Method 2: Assume that $f(x)$ is not connected. Then $\exists \emptyset \ne B_1 \subseteq Y$, $\exists \emptyset \ne B_2 \subseteq Y$ s.t. $f(x) \subseteq B_1 \cup B_2$ and

$$\overline{B_1} \cap B_2 = \emptyset = B_1 \cap \overline{B_2}$$

let

$$A_1 = f^{-1}(B_1) \ne \emptyset$$
$$A_2 = f^{-1}(B_2) \ne \emptyset$$

Have

$$f(X) \subseteq B_1 \cup B_2 \implies X \subseteq f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2) = A_1 \cup A_2$$
$$\overline{A_1} \cap A_2 = \overline{f^{-1}(B_1)} \cap f^{-1}(B_2) \subseteq f^{-1}(\overline{B_1}) \cap f^{-1}(B_2) = f^{-1}(\overline{B_1} \cap B_2)$$
$$= f^{-1}(\emptyset) = \emptyset$$

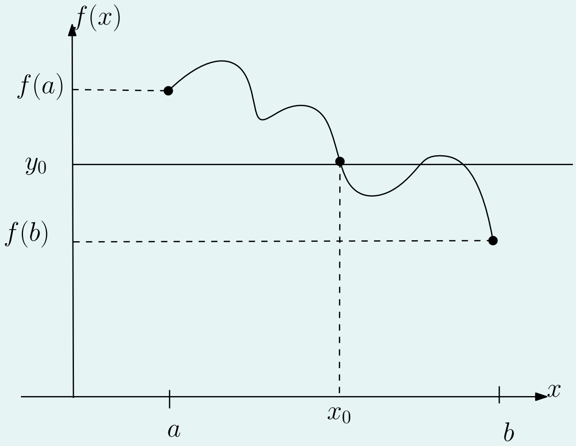
Similarly, $\overline{A_2} \cap A_1 = \emptyset$. exercise

This contradicts that X is connected. □

Corollary 33.8 (Darboux's Property)

Let (X, d_X) be a metric space and let $f : X \to \mathbb{R}$ be continuous. If $A \subseteq X$ is connected then $f(A)$ is an interval in $\mathbb{R}$.

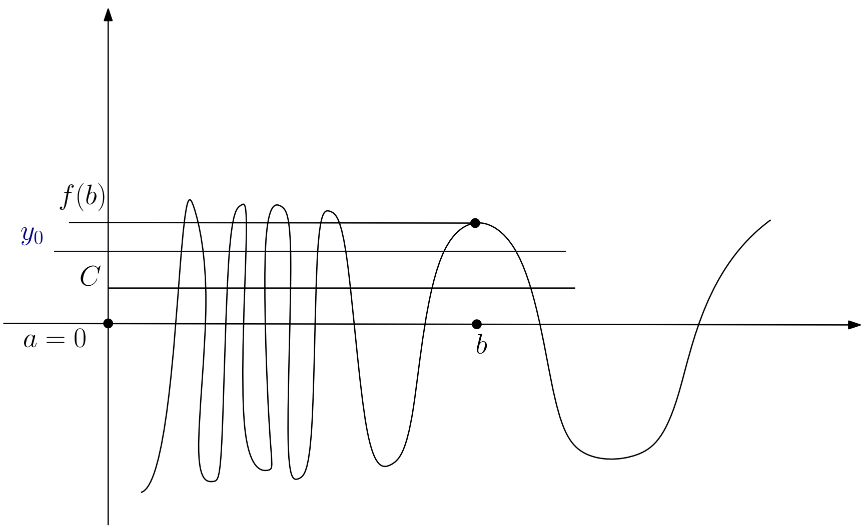
In particular, if $X = \mathbb{R}$, and $a, b \in \mathbb{R}$ s.t. $a < b$ and y_0 lies between $f(a)$ and $f(b)$, then $\exists x_0 \in (a, b)$ s.t. $f(x_0) = y_0$.



Remark 33.9. There are function that have the Darboux property, but are not continuous.

For example, consider

$$f : [0, \infty) \to \mathbb{R}, \quad f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ c, & x = 0 \end{cases} \quad \text{where } c \in [-1, 1]$$



Notice f is continuous on $(0, \infty)$ implies f has the Darboux property on $(0, \infty)$. f has the Darboux property on $[0, \infty)$, but is not continuous at $x = 0$.

§34 Lec 6: Apr 9, 2021

§34.1 Continuity and Connectedness (Cont'd)

Proposition 34.1

Let (X, d_X) and (Y, d_Y) be two connected metric spaces. Then $(X \times Y, d)$ where

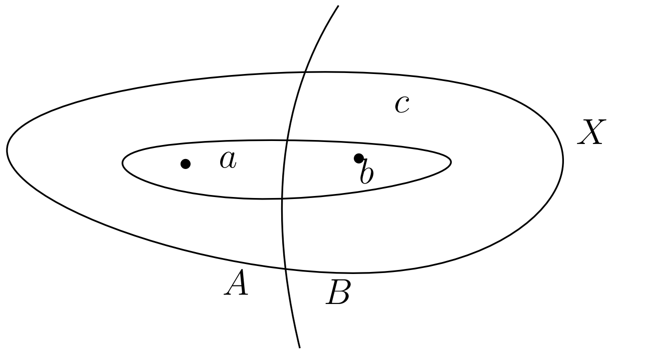
$$d((x_1, y_1), (x_2, y_2)) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}$$

is a connected metric space.

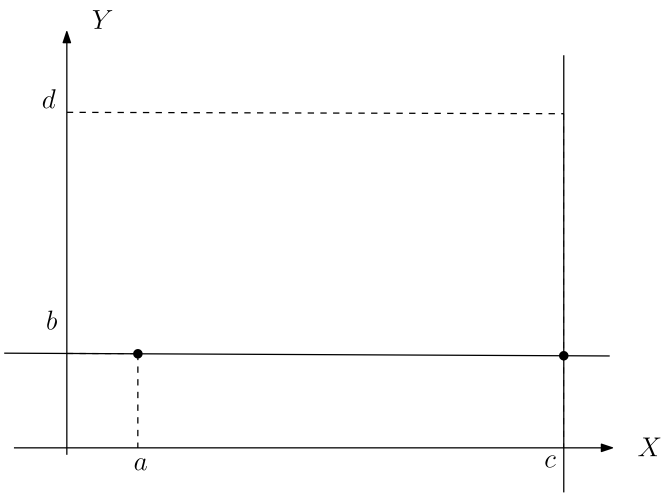
Remark 34.2. One could replace the distance d by

$$d_1((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$
$$d_\infty((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$

Proof. We will use the fact that a metric space is connected if and only if any two points are contained in a connected subset of the metric space.



So to show $X \times Y$ is connected if suffices to show that if $(a, b), (c, d) \in X \times Y$, then there exists $C \subseteq X \times Y$ connected s.t. $(a, b), (c, d) \in C$.



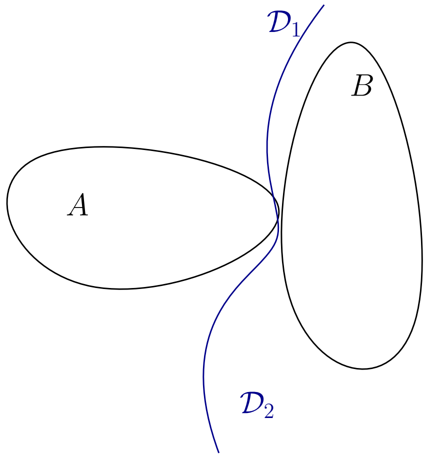
Let $f : X \to X \times Y$ where $f(x) = (x, b)$

Claim 34.1. f is continuous.

Take $\delta = \varepsilon$ in the definition of continuity. As X is connected, $f(X) = X \times \{b\}$ is connected.

Similarly, $g : Y \to X \times Y$, $g(y) = (c, y)$ is continuous and since Y is connected, $g(Y) = \{c\} \times Y$ is connected.

Finally, $f(x) \cap g(y) \ni (c, b)$ and so $f(x)$, $g(y)$ are not separated. As the union of two connected not separated sets is connected we get $f(x) \cup g(y)$ is connected.



Note $(a, b), (c, d) \in f(x) \cup g(y)$. □

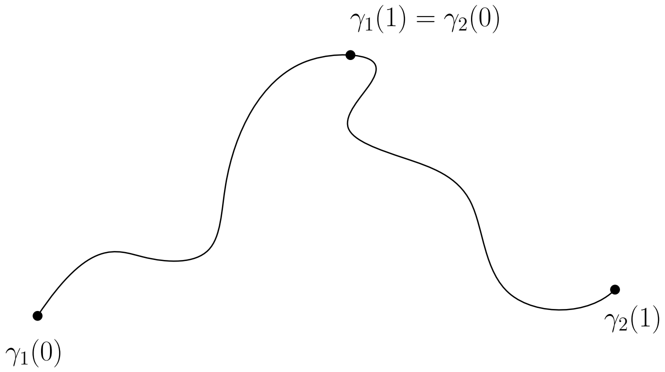
Definition 34.3 (Path) — Let (X, d) be a metric space. A path is a continuous function $\gamma : [0, 1] \to X$. $\gamma(0)$ is called the origin of the path and $\gamma(1)$ is called the end of the path.

As $[0, 1]$ is compact and connected and γ is continuous, $\gamma([0, 1])$ is compact and connected.

Given $\gamma : [0, 1] \to X$ a path, we define

$$\gamma^- : [0, 1] \to X, \qquad \gamma^-(t) = \gamma(1 - t) \text{ is a path}$$

Given $\gamma_1, \gamma_2 : [0, 1] \to X$ paths s.t. $\gamma_1(1) = \gamma_2(0)$.



We define

$$\gamma_1 \vee \gamma_2 : [0, 1] \to X$$

via

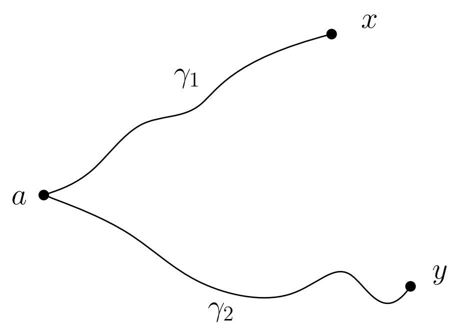
$$\gamma_1 \vee \gamma_2(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \gamma_2(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

Proposition 34.4

Let (X, d) be a metric space and let $A \subseteq X$. Then 1) $\iff$ 2) $\implies$ 3) where

1. $\exists a \in A$ s.t. $\forall x \in A\, \exists \gamma_x : [0, 1] \to A$ path s.t.
$$\gamma_x(0) = a \text{ and } \gamma_x(1) = x$$
2. $\forall x, y \in A\, \exists \gamma_{x,y} : [0, 1] \to A$ path s.t.
$$\gamma_{x,y}(0) = x \text{ and } \gamma_{x,y}(1) = y$$
3. A is connected.

Proof. 1) $\implies$ 2) Let $x, y \in A$. By hypothesis, $\exists \gamma_x, \gamma_y : [0, 1] \to A$ paths s.t.
$$\gamma_x(0) = \gamma_y(0) = a, \quad \gamma_x(1) = x, \quad \gamma_y(1) = y$$

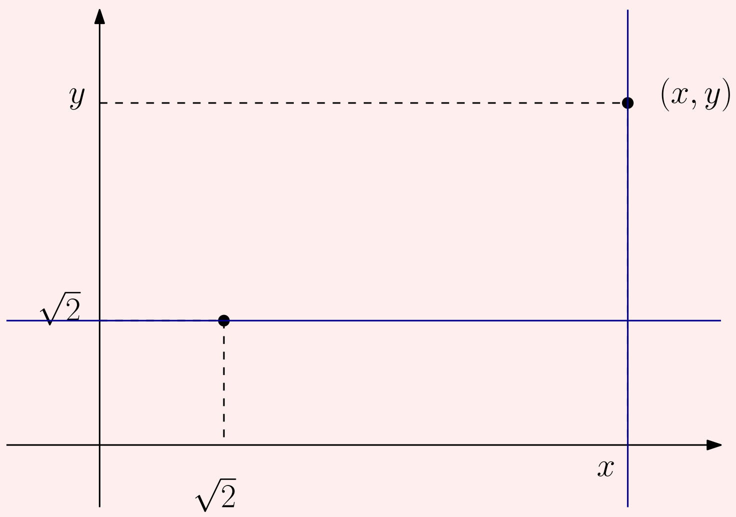


Then $\gamma_x^- \vee \gamma_y : [0, 1] \to A$ is the desired path.
2) $\implies$ 1)Choose $a \in A$ arbitrary.
1) $\implies$ 3) Given $x \in A$, let $A_x = \gamma_x([0, 1])$ connected. Note
$$a \in \bigcap_{x \in A} A_x \implies \text{ no two sets } A_x,\, A_y \text{ are separated}$$
Then $A = \bigcup_{x \in A} A_x$ is connected. □

Definition 34.5 (Path Connected) — If either 1) or 2) holds in the Proposition 34.4, we say that A is path connected. Note A is path connected implies A is connected.

Example 34.6

$\mathbb{R}^2 \setminus \mathbb{Q}^2$ is path connected.



We will show that any $(x,y) \in \mathbb{R}^2 \setminus \mathbb{Q}^2$ can be joined via path in $\mathbb{R}^2 \setminus \mathbb{Q}^2$ to $(\sqrt{2}, \sqrt{2})$.

$$(x,y) \in \mathbb{R}^2 \setminus \mathbb{Q}^2 \implies x \notin \mathbb{Q} \text{ or } y \notin \mathbb{Q}$$

Say $x \notin \mathbb{Q}$. Then $\{x\} \times \mathbb{R} \subseteq \mathbb{R}^2 \setminus \mathbb{Q}^2$. Note also that $\mathbb{R} \times \left\{\sqrt{2}\right\} \subseteq \mathbb{R}^2 \setminus \mathbb{Q}^2$. Let $\gamma : [0,1] \to \mathbb{R}^2 \setminus \mathbb{Q}^2$, $\gamma = \gamma_1 \vee \gamma_2$ where

$$\gamma_1 : [0,1] \to \mathbb{R}^2 \setminus \mathbb{Q}^2,\ \gamma_1(t) = \left(\sqrt{2} + t(x - \sqrt{2}), \sqrt{2}\right) \text{ path}$$
$$\gamma_2 : [0,1] \to \mathbb{R}^2 \setminus \mathbb{Q}^2,\ \gamma_2(t) = \left(x, \sqrt{2} + t(y - \sqrt{2})\right) \text{ path}$$

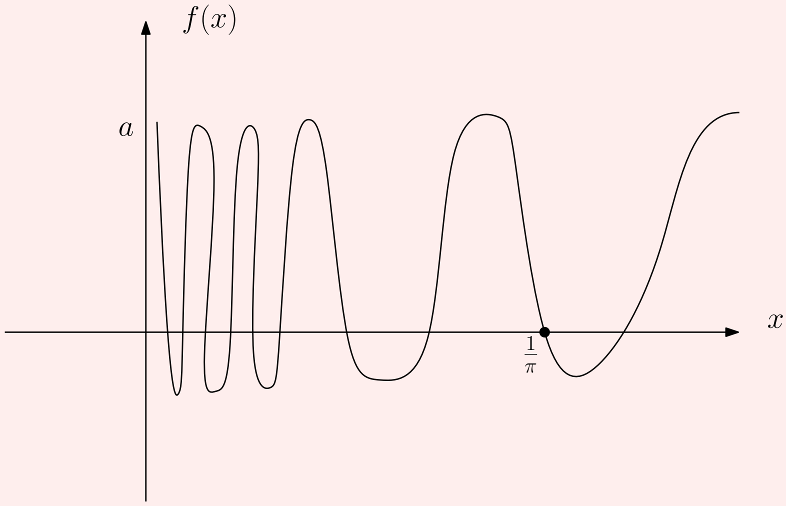
Example 34.7

A connected set which is not path connected. Let $f : [0, \infty) \to \mathbb{R}$ s.t.

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ a, & x = 0 \end{cases}$$

where $a \in [-1, 1]$ fixed.

Then $\Gamma_f = \{(x, f(x)) : x \in [0, \infty)\}$ is connected, but not path connected.



Let's show Γ_f is connected. The function $g : [0, \infty) \to \mathbb{R}^2$, $g(x) = (x, f(x))$ is continuous on $(0, \infty) \implies g((0, \infty))$ is connected.

Also, $g(\{0\}) = \{(0, a)\}$ is connected. We will show that $(0, a) \in \overline{g((0, \infty))}$ and so $\{(0, a)\}, g((0, \infty))$ are not separated. Then

$$\Gamma_f = g([0, \infty)) = g(\{0\}) \cup g((0, \infty)) \text{ is connected}$$

To see $(0, a) \in \overline{g(0, \infty)}$ we need to find $x_n \to 0$ s.t.

$$\sin\left(\frac{1}{x_n}\right) = a$$

Take $x_n = \frac{1}{\arcsin a + 2n\pi}$ where $\arcsin a \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Example 34.8 (Cont'd from above)

Now let's show Γ_f is not path connected. Assume towards a contradiction that there exists $\gamma : [0,1] \to \Gamma_f$ a path s.t.

$$\gamma(0) = (0,a), \qquad \gamma(1) = \left(\frac{1}{\Pi}, 0\right)$$

Note $\Pi_1 \circ \gamma : [0,1] \to \mathbb{R}$ is continuous

$$(\Pi_1 \circ \gamma)(0) = 0, \quad (\Pi_1 \circ \gamma)(1) = \frac{1}{\pi}$$

Let $b \in [-1,1] \setminus \{a\}$. By the Darboux property, $\exists t_n \in \left(0, \frac{1}{\pi}\right)$ s.t.

$$(\Pi_1 \circ \gamma)(t_n) = \frac{1}{\arcsin b + 2n\pi} \text{ where } \arcsin b \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

As $[0,1]$ is compact, $\exists t_{k_n} \underset{n\to\infty}{\longrightarrow} t_\infty \in [0,1]$.

$$\left.\begin{aligned} \gamma \text{ continuous } &\implies \gamma(t_{k_n}) \underset{n\to\infty}{\longrightarrow} \gamma(t_\infty) \\ \gamma(t_{k_n}) = \left(\tfrac{1}{\arcsin b + 2k_n\pi}, b\right) &\underset{n\to\infty}{\longrightarrow} (0,b) \end{aligned}\right\} \implies \gamma(t_\infty) = (0,b) \notin \Gamma_f$$

§35 Lec 7: Apr 12, 2021

§35.1 Continuity and Connectedness (Cont'd)

Example 35.1

Two connected sets $A, B \subseteq [-1,1]\times[-1,1]$ s.t. $(-1,-1),(1,1) \in A$, $(-1,1),(1,-1) \in B$, $A \cap B = \emptyset$. Let $f : [-1,1] \to [-1,1]$,

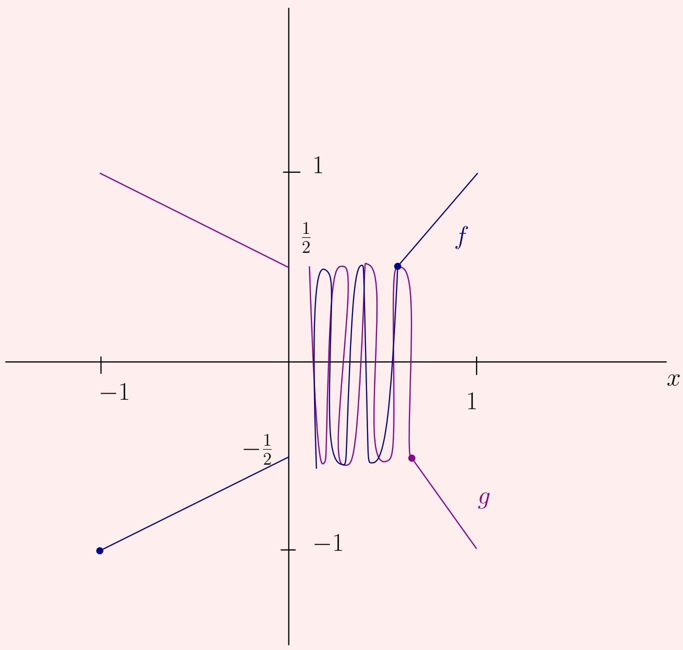
$$f(x) = \begin{cases} \frac{x-1}{2}, & -1 \le x \le 0 \\ x - \frac{1}{2}\sin\frac{\pi}{x}, & 0 < x \le \frac{1}{2} \\ x, & \frac{1}{2} \le x \le 1 \end{cases}$$

Let $g : [-1,1] \to [-1,1]$,

$$g(x) = \begin{cases} \frac{1-x}{2}, & -1 \le x \le 0 \\ -x - \frac{1}{2}\sin\frac{\pi}{x}, & 0 < x \le \frac{1}{2} \\ -x, & \frac{1}{2} \le x \le 1 \end{cases}$$

Let

$$A = \Gamma_f = \{(x_1 f(x)) : x \in [-1,1]\}$$
$$B = \Gamma_g = \{(x_1 g(x)) : x \in [-1,1]\}$$



Example 35.2 (Cont'd from above)
Let's prove $A \cap B = \emptyset$. If

$$\begin{aligned} -1 \le x \le 0, &\quad f(x) = g(x) \iff \frac{x-1}{2} = \frac{1-x}{2} \iff x = 1 \\ 0 < x \le \frac{1}{2}, &\quad f(x) = g(x) \iff x = 0 \\ \frac{1}{2} \le x \le 1, &\quad f(x) = g(x) \iff x = 0 \end{aligned}$$

Also

$$\begin{aligned} f(-1) = -1 &\implies (-1,-1) \in A \\ f(1) = 1 &\implies (1,1) \in A \\ g(-1) = 1 &\implies (-1,1) \in B \\ g(1) = -1 &\implies (1,-1) \in B \end{aligned}$$

Let's show that A is connected. A similar argument can be used to prove that B is connected.

We write $A = A_1 \cup A_2$ where $A_1 = \{(x, f(x)) : -1 \le x \le 0\}$ and $A_2 = \{(x, f(x)) : 0 < x \le 1\}$. Note that $h : [-1,1] \to \mathbb{R}^2$ where $h(x) = (x, f(x))$ is continuous on $[-1,0]$ and $(0,1]$.

Since $[-1,0]$ and $(0,1]$ are connected sets, we get that $h([-1,0]) = A_1$ and $h((0,1]) = A_2$ are connected.

To show that $A = A_1 \cup A_2$ is connected, it suffices to show that A_1 and A_2 are not separated. We will show $\left(0, -\frac{1}{2}\right) \in A_1 \cap \overline{A_2}$. It's clear that $f(0) = -\frac{1}{2} \implies \left(0, -\frac{1}{2}\right) \in A_1$. To show that $\left(0, -\frac{1}{2}\right) \in \overline{A_2}$ we need to find a decreasing sequence $x_n \to 0$ s.t.

$$f(x_n) = x_n - \frac{1}{2}\sin\frac{\pi}{x_n} \underset{n\to\infty}{\longrightarrow} -\frac{1}{2}$$

We take x_n s.t. $\sin\frac{\pi}{x_n} = 1 \iff \frac{\pi}{x_n} = \frac{\pi}{2} + 2n\pi \iff x_n = \frac{2}{4n+1} \to 0$. Notice that

$$f(x_n) = \frac{2}{4n+1} - \frac{1}{2} \underset{n\to\infty}{\longrightarrow} -\frac{1}{2}$$

§35.2 Convergent Sequences of Functions

Definition 35.3 (Pointwise Convergence) — Let $(X, d_X), (Y, d_Y)$ be two metric spaces and let $f_n : X \to Y$ be a sequence of functions. We say that $\{f_n\}_{n\ge1}$ converges pointwise if for all $x \in X$ the sequence $\{f_n(x)\}_{n\ge1}$ converges in Y. The limit $\lim_{n\to\infty} f_n(x) = f(x)$ defines a function $f : X \to Y$.

Remark 35.4. $\{f_n\}_{n\ge1}$ converges pointwise to f if

$$\forall x \in X \quad \forall \varepsilon > 0 \quad \exists n(\varepsilon, x) \in \mathbb{N} \text{ s.t. } d_Y(f_n(x), f(x)) < \varepsilon \quad \forall n \ge n(\varepsilon, x)$$

Note that for $\varepsilon > 0$ fixed, $n(\varepsilon, \cdot) : X \to \mathbb{N}$ can be bounded or unbounded. If it is bounded, we get the following

Definition 35.5 (Uniform Convergence) — Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f_n : X \to Y$ be a sequence of functions. We say that $\{f_n\}_{n\geq 1}$ <u>converges uniformly</u> to a function $f : X \to Y$ if

$$\forall \varepsilon > 0 \quad \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } d_Y(f(x), f_n(x)) < \varepsilon \quad \forall n \geq n_\varepsilon \, \forall x \in X$$

We denote $f_n \xrightarrow[n\to\infty]{u} f$.

Remark 35.6. Let $(X, d_X), (Y, d_Y)$ be metric spaces, $B(X, Y) = \{f : X \to Y;\ f \text{ is bounded}\}$, $d : B(X, Y) \times B(X, Y) \to \mathbb{R}$ via

$$d(f, g) = \sup_{x\in X} d_Y(f(x), g(x))$$

Exercise 35.1. Show that $(B(X, Y), d)$ is a metric space.

Note that $f_n \xrightarrow[n\to\infty]{u} f \iff M_n = d(f_n, f) \xrightarrow[n\to\infty]{} 0$.

" $\Longleftarrow$ " $\forall \varepsilon > 0\ \exists n_\varepsilon \in \mathbb{N}$ s.t. $M_n < \varepsilon\ \forall n \geq n_\varepsilon$

$$\implies d(f_n, f) = \sup_{x\in X} d_Y(f_n(x), f(x)) < \varepsilon \quad \forall n \geq n_\varepsilon$$
$$\implies d_Y(f_n(x), f(x)) < \varepsilon \quad \forall n \geq n_\varepsilon \quad \forall x \in X$$

" $\implies$ "

$$f_n \xrightarrow[n\to\infty]{u} f \implies \forall \varepsilon > 0 \quad \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } d_Y(f_n(x), f(x)) < \frac{\varepsilon}{2} \quad \forall n \geq n_\varepsilon \, \forall x \in X$$
$$\implies \underbrace{\sup_{x\in X} d_Y(f_n(x), f(x))}_{d(f_n, f) = M_n} \leq \frac{\varepsilon}{2} < \varepsilon \quad \forall n \geq n_\varepsilon$$

Remark 35.7.
1. Uniform convergence $\implies$ pointwise convergence
2. Pointwise convergence $\not\Longrightarrow$ uniform convergence

$f_n : [0, 1] \to \mathbb{R},\ f_n(x) = x^n$

$$\{f_n\}_{n\geq 1} \text{ converges pointwise : } \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

Let

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

Note $f_n \underset{n\to\infty}{\overset{u}{\not\to}} f$ since

$$d(f_n, f) = \sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1)} |x^n| = 1 \underset{n\to\infty}{\not\to} 0$$

Theorem 35.8 (Weierstrass)

Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f_n : X \to Y$ be a sequence of functions that converges uniformly to a function $f : X \to Y$. If $\forall n \geq 1$, f_n is continuous at $x_0 \in X$ then f is continuous at x_0.

Corollary 35.9

A uniform limit of continuous functions is a continuous function.

Proof. (of theorem) Fix $\varepsilon > 0$.

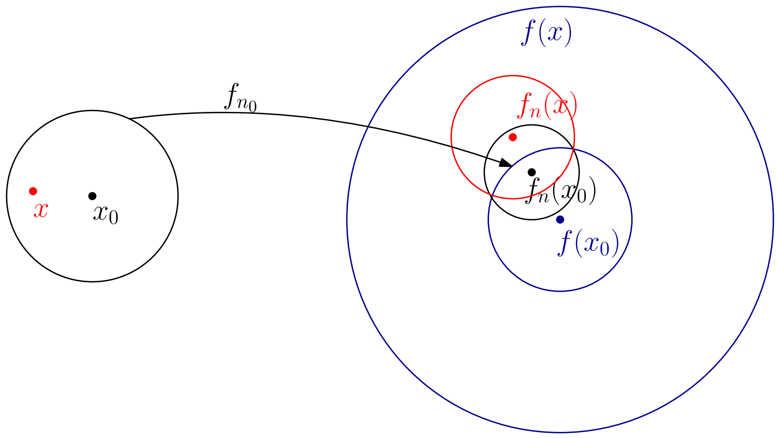
$$f_n \xrightarrow[n\to\infty]{u} f \implies \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } d_Y\left(f_n(x), f(x)\right) < \frac{\varepsilon}{3} \quad \forall n \geq n_\varepsilon \, \forall x \in X$$

Fix $n_0 \geq n_\varepsilon$. f_{n_0} is continuous at x_0

$$\implies \exists \delta > 0 \text{ s.t. if } d_X(x_0, x) < \delta$$

then

$$d_Y\left(f_{n_0}(x_0), f_{n_0}(x)\right) < \frac{\varepsilon}{3}$$



Then for $x \in B_\delta(x_0)$ we have

$$\begin{aligned} d_Y\left(f(x), f(x_0)\right) &\leq d_Y\left(f(x), f_{n_0}(x)\right) + d\left(f_{n_0}(x), f_{n_0}(x_0)\right) + d\left(f_{n_0}(x_0), f(x_0)\right) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

By definition, f is continuous at x_0. $\square$

§36 Lec 8: Apr 14, 2021

§36.1 Convergent Sequences of Functions (Cont'd)

Theorem 36.1 (Dini)
Let (X, d) be a compact metric space and let $f_n : X \to \mathbb{R}$ be a sequence of continuous functions that converges pointwise to a continuous function $f : X \to \mathbb{R}$. Assume that $\{f_n\}_{n\geq 1}$ is monotone in the sense that either $\{f_n(x)\}_{n\geq 1}$ is increasing for all $x \in X$ or $\{f_n(x)\}_{n\geq 1}$ is decreasing for all $x \in X$. Then,

$$f_n \xrightarrow[n\to\infty]{u} f \text{ i.e. } d(f_n, f) = \sup_{x\in X} |f_n(x) - f(x)| \xrightarrow[n\to\infty]{} 0$$

Proof. Assume that $\{f_n\}_{n\geq 1}$ is increasing. Then $\{f - f_n\}_{n\geq 1}$ is decreasing and for all $x \in X$ we have

$$\lim_{n\to\infty} [f(x) - f_n(x)] = \inf_{n\to\infty} [f(x) - f_n(x)] = 0$$

Then $\forall \varepsilon > 0 \quad \exists n(\varepsilon, x) \in \mathbb{N}$ s.t. $\forall n \geq n(\varepsilon, x)$ we have

$$0 \leq f(x) - f_n(x) \leq f(x) - f_{n_{\varepsilon,x}}(x) < \varepsilon$$

As $f - f_{n_{\varepsilon,x}}$ is continuous at x, $\exists \delta(\varepsilon, x) > 0$ s.t.

$$d(x, y) < \delta_{\varepsilon,x} \implies \left|\left[f(x) - f_{n_{\varepsilon,x}}(x)\right] - \left[f(y) - f_{n_{\varepsilon,x}}(y)\right]\right| < \varepsilon$$

By the triangle inequality, we get

$$\begin{aligned} 0 \leq f(y) - f_{n_{\varepsilon,x}}(y) &\leq \left|\left[f(x) - f_{n_{\varepsilon,x}}(x)\right] - \left[f(y) - f_{n_{\varepsilon,x}}(y)\right]\right| + f(x) - f_{n_{\varepsilon,x}}(x) \\ &< \varepsilon + \varepsilon = 2\varepsilon \end{aligned}$$

whenever $y \in B_{\delta_{\varepsilon,x}}(x)$. In particular,

$$0 \leq f(y) - f_n(y) \leq f(y) - f_{n_{\varepsilon,x}}(y) < 2\varepsilon \quad \forall n \geq n_{\varepsilon,x}, \forall y \in B_{\delta_{\varepsilon,x}}(x) \tag{*}$$

Note

$$\left.\begin{array}{l} X = \bigcup_{x\in X} B_{\delta_{\varepsilon,x}}(x) \\ X \text{ compact} \end{array}\right\} \implies \exists \mathcal{J} \subseteq \mathbb{N} \text{ finite and } \exists \{x_j\}_{j\in\mathcal{J}} \in X$$

s.t. $X = \bigcup_{j\in\mathcal{J}} B_{\delta_j}(x_j)$ and where $\delta_j = \delta(\varepsilon, x_j)$.
Let $n_\varepsilon = \max_{j\in\mathcal{J}} n(\varepsilon, x_j)$. Fix $n \geq n_\varepsilon$ and $x \in X$. As $x \in X = \bigcup_{j\in\mathcal{J}} B_{\delta_j}(x_j) \implies j \in \mathcal{J}$ s.t. $x \in B_{\delta_j}(x_j)$. By (*), we have

$$0 \leq f(x) - f_n(x) < 2\varepsilon$$

As $x \in X$ was arbitrary we get

$$d(f, f_n) \leq 2\varepsilon \qquad \forall n \geq n_\varepsilon$$

□

Remark 36.2. The compactness of X is necessary in Dini's theorem.

Example 36.3

$f_n : (0,1) \to \mathbb{R}$, $f_n(x) = x^n$ continuous

$$f_{n+1}(x) \le f_n(x) \quad \forall n \ge 1 \quad \forall x \in (0,1)$$
$$f_n(x) \underset{n\to\infty}{\longrightarrow} 0 \quad \forall x \in (0,1)$$

Let $f : (0,1) \to \mathbb{R}$, $f(x) = 0 \quad \forall x \in (0,1)$. It's continuous. But

$$d(f_n, f) = \sup_{x\in(0,1)} |x^n| = 1 \underset{n\to\infty}{\not\to} 0 \implies f_n \overset{u}{\underset{n\to\infty}{\not\to}} f$$

Note that $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n$ continuous, $\{f_n\}_{n\ge1}$ is decreasing and converge pointwise to $f : [0,1] \to \mathbb{R}$,

$$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases} \quad \text{which is not continuous}$$

This also shows that the continuity of the limit function is necessary in Dini's theorem.

Remark 36.4. Monotonicity is necessary in Dini's theorem.

Example 36.5

$f_n : [0,1] \to \mathbb{R}$ is continuous. $\{f_n\}_{n\ge1}$ converges pointwise to $f : [0,1] \to \mathbb{R}$, $f(x) = 0 \, \forall x \in [0,1]$ figure here f is continuous. But

$$d(f_n, f) = \sup_{x\in[0,1]} |f_n(x)| = 1 \underset{n\to\infty}{\not\to} 0 \implies f_n \overset{u}{\underset{n\to\infty}{\not\to}} f$$

Note that $\{f_n\}_{n\ge1}$ is not monotone!

§36.2 Space of Functions

Fix $a, b \in \mathbb{R}$, $a < b$. We define

$$C([a,b]) = \{f : [a,b] \to \mathbb{R};\ f \text{ is continuous}\}$$

We equip $C([a,b])$ with the metric $d : C([a,b]) \times C([a,b]) \to \mathbb{R}$, given by

$$d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|$$

Then $(C([a,b]), d)$ is a metric space.
Completeness: Let $\{f_n\}_{n\ge1} \subseteq C([a,b])$ be Cauchy. So $\forall \varepsilon > 0 \, \exists n_\varepsilon \in \mathbb{N}$ s.t. $d(f_n, f_m) < \varepsilon$ $\forall n, m \ge n_\varepsilon$

$$\implies |f_n(x) - f_m(x)| < \varepsilon \quad \forall n, m \ge n_\varepsilon \quad \forall x \in [a,b]$$

So $\{f_n(x)\}_{n\ge1}$ is Cauchy $\forall x \in [a,b]$. As $\mathbb{R}$ is complete,

$$\forall x \in [a,b] \quad f_n(x) \underset{n\to\infty}{\longrightarrow} f(x) \in \mathbb{R}$$

This defines a function $f : [a, b] \to \mathbb{R}$. Recall that for all $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ s.t.

$$\begin{aligned} |f_n(x) - f(x)| \le \varepsilon \quad \forall n \ge n_\varepsilon \quad \forall x \in [a, b] \\ \implies d(f_n, f) \le \varepsilon \quad \forall n \ge n_\varepsilon \end{aligned}$$

So $f_n \xrightarrow[n\to\infty]{u} f$. By Weierstrass, $f \in C([a, b])$. Thus $(C([a, b]), d)$ is a complete metric space.

Compactness: Note that $(C([a, b]), d)$ is not bounded and so not compact.

Example 36.6

$f_n : [a, b] \to \mathbb{R}$, $f_n(x) = n$ for all $x \in [a, b]$.

Connectedness: $(C([a, b]), d)$ is path connected and so connected.

Let $f, g \in C([a, b])$. Define $\gamma : [0, 1] \to C([a, b])$ via $\gamma(t) = f + t(g - f)$. Note $\forall t \in [0, 1]$, $\gamma(t) \in C([a, b])$ and

$$\gamma(0) = f, \quad \gamma(1) = g$$

To see that γ is a path we compute

$$\begin{aligned} d(\gamma(t), \gamma(s)) &= \sup_{x\in[a,b]} |\gamma(t; x) - \gamma(s; x)| \\ &= \sup_{x\in[a,b]} |t - s| \, |g(x) - f(x)| \\ &= |t - s| \underbrace{d(g, f)}_{\in\mathbb{R}} \xrightarrow[|t-s|\to 0]{} 0 \end{aligned}$$

So γ is a continuous function and so a path.

§37 Lec 9: Apr 16, 2021

§37.1 Arzela–Ascoli Theorem

For $a, b \in \mathbb{R}$ with $a < b$, we define

$$C([a,b]) = \{f : [a,b] \to \mathbb{R};\ f \text{ continuous}\}$$

We equip $C([a,b])$ with the uniform metric

$$d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|$$

We showed that $(C([a,b]), d)$ is a complete, connected metric space, but it's not compact.

Definition 37.1 (Equicontinuity) — We say that a set $\mathcal{F} \subseteq C([a,b])$ is equicontinuous if

$$\forall \varepsilon > 0 \quad \exists \delta(\varepsilon) > 0 \text{ s.t. } |f(x) - f(y)| < \varepsilon \quad \forall x, y \in [a,b] \text{ with } |x-y| < \delta(\varepsilon)$$

and for all $f \in \mathcal{F}$.

Note: For a fixed function $f \in \mathcal{F} \subseteq C([a,b])$, we have that f is uniformly continuous (since f is continuous on $[a,b]$ compact) which means for all $\varepsilon > 0$, there exists $\delta(\varepsilon, f) > 0$ s.t.

$$|f(x) - f(y)| < \varepsilon \quad \forall x, y \in [a,b] \text{ with } |x-y| < \delta(\varepsilon, f)$$

Note that for an equicontinuous family $\mathcal{F}$, δ_ε can be chosen uniformly for $f \in \mathcal{F}$.

Definition 37.2 (Uniformly Bounded) — We say that a set $\mathcal{F} \subseteq C([a,b])$ is uniformly bounded if $\exists M > 0$ s.t. $|f(x)| \le M\ \forall x \in [a,b]\ \forall f \in \mathcal{F}$.

Note: For a fixed $f \in \mathcal{F} \subseteq C[a,b]$ we have that $f([a,b])$ is bounded (since f continuous and $[a,b]$ compact which implies $f([a,b])$ is compact and so bounded). So $\exists M_f > 0$ s.t. $|f(x)| \le M_f\ \forall x \in [a,b]$. For a uniformly bounded family $\mathcal{F}$, we can choose the bound M uniformly for $f \in \mathcal{F}$.

Theorem 37.3 (Arzela-Ascoli)
Let $\mathcal{F} \subseteq C([a,b])$. The following are equivalent:

1. $\mathcal{F}$ is uniformly bounded and equicontinuous.
2. Every sequence in $\mathcal{F}$ admits a convergent subsequence.

Caution: We cannot guarantee that the limit of the convergent subsequence belongs to $\mathcal{F}$, unless $\mathcal{F}$ is closed in $C([a,b])$. If $\mathcal{F}$ is closed in $C([a,b])$, then the theorem becomes

$$\mathcal{F} \text{ is compact} \iff \mathcal{F} \text{ is uniformly bounded and equicontinuous}$$

Proof. 2) $\implies$ 1)

Claim 37.1. $\mathcal{F}$ is totally bounded.

Fix $\varepsilon > 0$. Let $f_1 \in \mathcal{F}$.

$$\begin{aligned}
&\text{If } \mathcal{F} \subseteq B_\varepsilon(f_1) \text{ then } \mathcal{F} \text{ is totally bounded}\\
&\text{If } \mathcal{F} \not\subseteq B_\varepsilon(f_1) \text{ then } \exists f_2 \in \mathcal{F} \text{ s.t. } d(f_1, f_2) \geq \varepsilon\\
&\text{If } \mathcal{F} \subseteq B_\varepsilon(f_1) \cup B_\varepsilon(f_2) \text{ then } \mathcal{F} \text{ is totally bounded}\\
&\text{If } \mathcal{F} \not\subseteq B_\varepsilon(f_1) \cup B_\varepsilon(f_2) \text{ then } \exists f_3 \in \mathcal{F} \text{ s.t. } \begin{cases} d(f_1, f_3) \geq \varepsilon \\ d(f_2, f_3) \geq \varepsilon \end{cases}
\end{aligned}$$

If the process terminates in finitely many steps, then $\mathcal{F}$ is totally bounded. Otherwise, we find $\{f_n\}_{n\geq 1} \subseteq \mathcal{F}$ s.t. $d(f_n, f_m) \geq \varepsilon \forall n \neq m$. This sequence does not admit a convergent subsequence, leading a contradiction.

Let's show that $\mathcal{F}$ is uniformly bounded. As $\mathcal{F}$ is totally bounded, $\exists n \geq 1$ and $\exists f_1, \ldots, f_n \in \mathcal{F}$ s.t.

$$\mathcal{F} \subseteq \bigcup_{j=1}^{n} B_1(f_j) \subseteq B_r(f_1)$$

where $r = 1 + \max_{2\leq j\leq n} d(f_1, f_j)$. In particular, for all $f \in \mathcal{F}$,

$$d(f, f_1) < r$$

f_1 is continuous on compact $[a, b] \implies \exists M_{f_1} > 0$ s.t.

$$|f_1(x)| \leq M_{f_1} \quad \forall x \in [a, b]$$

So for $f \in \mathcal{F}$

$$|f(x)| \leq |f(x) - f_1(x)| + |f_1(x)| \leq d(f, f_1) + M_{f_1} < r + M_{f_1} \quad \forall x \in [a, b]$$

So $\mathcal{F}$ is uniformly bounded.
Let's show that $\mathcal{F}$ is equicontinuous. Let $\varepsilon > 0$. As $\mathcal{F}$ is totally bounded, $\exists n \geq 1$ and $\exists f_1, \ldots, f_n \in \mathcal{F}$ s.t.

$$\mathcal{F} \subseteq \bigcup_{j=1}^{n} B_{\frac{\varepsilon}{3}}(f_j)$$

For each $1 \leq j \leq n$, f_j is uniformly continuous on $[a, b]$. So $\exists \delta_j(\varepsilon) > 0$ s.t.

$$|f_j(x) - f_j(y)| < \frac{\varepsilon}{3} \quad \forall x, y \in [a, b] \text{ with } |x - y| < \delta_j(\varepsilon)$$

Let $\delta_\varepsilon = \min_{1\leq j\leq n} \delta_j(\varepsilon) > 0$.
Fix $f \in \mathcal{F} \implies \exists 1 \leq j \leq n$ s.t. $f \in B_{\frac{\varepsilon}{3}}(f_j)$. Then for $x, y \in [a, b]$ with $|x - y| < \delta_\varepsilon$ we have

$$\begin{aligned}
|f(x) - f(y)| &\leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)|\\
&\leq 2d(f, f_j) + |f_j(x) - f_j(y)|\\
&\leq \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

This shows $\mathcal{F}$ is equicontinuous.
1) $\implies$ 2) Let $\{f_n\}_{n\geq 1} \subseteq \mathcal{F}$. As $\mathcal{F}$ is uniformly bounded,

$$\exists M > 0 \text{ s.t. } |f(x)| \leq M \quad \forall x \in [a, b]\ \forall f \in \mathcal{F}$$

In particular, $|f_n(x)| \leq M \ \forall x \in [a,b] \ \forall n \geq 1$.
Let $\{r_n\}_{n\geq 1}$ denote an enumeration of the rationals in $[a,b]$. As $\{f_n(r_1)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded by M, $\exists \left\{f_n^{(1)}\right\}_{n\geq 1}$ subsequence of $\{f_n\}_{n\geq 1}$ s.t. $\left\{f_n^{(1)}(r_1)\right\}_{n\geq 1}$ converges.
$\left\{f_n^{(1)}(r_2)\right\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded by $M \implies \exists \left\{f_n^{(2)}\right\}_{n\geq 1}$ subsequence of $\left\{f_n^{(1)}\right\}_{n\geq 1}$ s.t. $\left\{f_n^{(2)}(r_2)\right\}_{n\geq 1}$ converges.
Proceeding inductively we find $\forall k \geq 1$ $\left\{f_n^{(k+1)}\right\}_{n\geq 1}$ is a subsequence of $\left\{f_n^{(k)}\right\}_{n\geq 1}$ and $\left\{f_n^{(k)}(r_k)\right\}_{n\geq 1}$ converges.
We consider $\left\{f_n^{(n)}\right\}_{n\geq 1}$ subsequence of $\{f_n\}_{n\geq 1}$.
For $n,m \geq k$, $f_n^{(n)}$, $f_m^{(m)}$ are elements in $\left\{f_n^{(k)}\right\}_{n\geq 1}$. So $\left\{f_n^{(n)}\right\}_{n\geq 1}$ converges at r_k.
<u>Caution</u>: The convergence is not uniform in k.
Fix $\varepsilon > 0$. As $\mathcal{F}$ is equicontinuous, $\exists \delta > 0$ s.t.

$$|f(x) - f(y)| < \frac{\varepsilon}{3} \quad \forall x, y \in [a,b] \ |x-y| < \delta, \forall f \in \mathcal{F}$$

In particular,

$$|f_n(x) - f_n(y)| < \frac{\varepsilon}{3} \quad \forall x, y \in [a,b] \ |x-y| < \delta, \forall n \geq 1 \tag{*}$$

Let $r_1, \ldots, r_N \in \mathbb{Q} \cap [a,b]$ s.t. $a = r_0 < r_1 < \ldots < r_N < r_{N+1} = b$ and

$$|r_{j+1} - r_j| < \delta \qquad 0 \leq j \leq N$$

Note $N \sim \frac{|a-b|}{\delta}$. For each $1 \leq j \leq N$, $\exists n_j(\varepsilon) \in \mathbb{N}$ s.t.

$$\left|f_n^{(n)}(r_j) - f_m^{(m)}(r_j)\right| < \frac{\varepsilon}{3} \qquad \forall n, m \geq n_j(\varepsilon)$$

Let $n_\varepsilon = \max_{1\leq j\leq N} n_j(\varepsilon)$. Note

$$\left|f_n^{(n)}(r_j) - f_m^{(m)}(r_j)\right| < \frac{\varepsilon}{3} \quad \forall n, m \geq n_\varepsilon \quad \forall 1 \leq j \leq N \tag{**}$$

Let $x \in [a,b] \implies \exists 1 \leq j \leq N$ s.t. $|x - r_j| < \delta$. Then

$$\left|f_n^{(n)}(x) - f_m^{(m)}(x)\right| \leq \left|f_n^{(n)}(x) - f_n^{(n)}(r_j)\right| + \left|f_n^{(n)}(r_j) - f_m^{(m)}(r_j)\right| + \left|f_m^{(m)}(r_j) - f_m^{(m)}(x)\right|$$
$$\text{By (*) and (**)} < 2 \cdot \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \forall n, m \geq n_\varepsilon$$

So $\left\{f_n^{(n)}\right\}_{n\geq 1}$ is uniformly Cauchy and so uniformly convergent. □

> **Remark 37.4.** One can replace $[a,b]$ by any other compact metric space (X,d).

§38 Lec 10: Apr 19, 2021

§38.1 Arzela-Ascoli Theorem (Cont'd)

Remark 38.1. The compactness of the set on which the functions are defined is necessary in Arzela-Ascoli.

Example 38.2

$\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R};\ |f(x) - f(y)| \le |x - y|\ \forall x, y \in \mathbb{R}$ and $\sup_{x\in\mathbb{R}} |f(x)| \le 1\}$. Note $\mathcal{F}$ is equicontinuous and uniformly bounded. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{1+x^2}$

Claim 38.1. $f \in \mathcal{F}$.

Indeed,

$$\sup_{x\in\mathbb{R}} |f(x)| = \sup_{x\in\mathbb{R}} \frac{1}{1+x^2} = 1$$

Moreover, for $x, y \in \mathbb{R}$

$$\begin{aligned}
|f(x) - f(y)| = \left| \frac{1}{1+x^2} - \frac{1}{1+y^2} \right| &= \frac{\left|x^2 - y^2\right|}{(1+x^2)(1+y^2)} \\
&= |x - y| \cdot \frac{|x+y|}{(1+x^2)(1+y^2)} \\
&\le |x - y| \, (\underbrace{\frac{|x|}{1+x^2}}_{\le \frac{1}{2}} + \underbrace{\frac{|y|}{1+y^2}}_{\le \frac{1}{2}}) \\
&\le |x - y|
\end{aligned}$$

So $f \in \mathcal{F}$.
For $n \ge 1$, let $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = f(x - n)$. Note $f_n \in \mathcal{F}$ since $\sup_{x\in\mathbb{R}} |f_n(x)| = \sup_{x\in\mathbb{R}} \frac{1}{1+(x-n)^2} = 1$.

$$\begin{aligned}
|f_n(x) - f_n(y)| = |f(x-n) - f(y-n)| &\le |(x-n) - (y-n)| \\
&= |x - y|
\end{aligned}$$

Note that $\{f_n\}_{n\ge 1}$ converge pointwise to $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 0$ since $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{1+(x-n)^2} = 0$. However, $\{f_n\}_{n\ge 1}$ does not admit a subsequence that converges uniformly since $\forall n \ge 1$

$$d(f_n, f) = \sup_{x\in\mathbb{R}} |f_n(x)| = 1 \overset{n\to\infty}{\not\to} 0$$

Remark 38.3. Uniform boundedness is necessary in Arzela-Ascoli.

Example 38.4

$\mathcal{F} = \{f : \underbrace{[0,1]}_{\text{compact}} \to \mathbb{R};\ f \text{ is continuous and } \underbrace{\sup_{x\in[0,1]} |f(x)| \le 1}_{\text{uniformly bounded}}\}.$

Claim 38.2. $\mathcal{F}$ is not equicontinuous.

For $n \ge 1$, let $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = \sin(nx)$. Note $f_n \in \mathcal{F}$. Let $x_n = \frac{3\pi}{2n}$, $y_n = \frac{\pi}{2n}$. Then $|x_n - y_n| = \frac{\pi}{n} \xrightarrow[n\to\infty]{} 0$ but

$$|f_n(x_n) - f_n(y_n)| = 2$$

So $\{f_n\}_{n\ge1}$ is not equicontinuous $\implies \mathcal{F}$ is not equicontinuous.

Claim 38.3. $\{f_n\}_{n\ge1}$ does not admit a convergent subsequence.

Assume, towards a contradiction, that there exists a subsequence $\{f_{k_n}\}_{n\ge1}$ of $\{f_n\}_{n\ge1}$ that converges uniformly to $f : [0,1] \to \mathbb{R}$. By Weierstrass,

$$\left.\begin{array}{l} f \in C([0,1]) \\ \left.\begin{array}{l} f_{k_n}(0) = 0 \quad \forall n \ge 1 \\ f_{k_n}(0) \xrightarrow[n\to\infty]{} f(0) \end{array}\right\} \implies f(0) = 0 \end{array}\right\} \implies \forall \varepsilon > 0\, \exists \delta > 0 \text{ s.t. } |f(x)| < \varepsilon\, \forall 0 < x < \delta$$

$f_{k_n} \xrightarrow[n\to\infty]{u} f \implies \exists n_\varepsilon \in \mathbb{N}$ s.t. $d(f_{k_n}, f) < \varepsilon\ \forall n \ge n_\varepsilon$. In particular, for $0 < x < \delta$ and $n \ge n_\varepsilon$ we have

$$|f_{k_n}(x)| \le |f_{k_n}(x) - f(x)| + |f(x)| < d(f_{k_n}, f) + \varepsilon < 2\varepsilon$$

Choosing $\varepsilon \le \frac{1}{2}$ and N large so that $N \ge n_{\varepsilon=\frac{1}{2}}$ and $\frac{\pi}{2N} < \delta_{\varepsilon=\frac{1}{2}}$ we find

$$1 = \left| f_{k_N}\left(\frac{\pi}{2N}\right)\right| < 2\varepsilon \le 1 \qquad \text{Contradiction!}$$

§38.2 The oscillation of a Real Function

Definition 38.5 (Oscillation of a Function) — Let (X, d) be a metric space and let $f : X \to \mathbb{R}$ be a function. For $\emptyset \ne A \subseteq X$, the oscillation of f on A is

$$\omega(f, A) = \sup_{x\in A} f(x) - \inf_{x\in A} f(x) = \sup_{x,y\in A} [f(x) - f(y)] \ge 0$$

Note that if $A \subseteq B$ then

$$\omega(f, A) \le \omega(f, B)$$

For $x_0 \in X$, the oscillation of f at x_0 is given by

$$\omega(f, x_0) = \inf_{\delta>0} \omega(f, B_\delta(x_0))$$

Proposition 38.6

Let (X, d) be a metric space and let $f : X \to \mathbb{R}$ be a function. Then f is continuous at a point $x_0 \in X$ if and only if $\omega(f, x_0) = 0$.

Proof. " $\implies$ " Fix $\varepsilon > 0$. As f is continuous at x_0, $\exists \delta > 0$ s.t. $|f(x) - f(x_0)| < \frac{\varepsilon}{4}$ $\forall x \in B_\delta(x_0)$.

$$\begin{aligned} &\implies |f(x) - f(y)| \le |f(x) - f(x_0)| + |f(x_0) - f(y)| < \frac{\varepsilon}{2} \quad \forall x, y \in B_\delta(x_0) \\ &\implies \omega(f, B_\delta(x_0)) = \sup_{x,y \in B_\delta(x_0)} [f(x) - f(y)] \le \frac{\varepsilon}{2} < \varepsilon \\ &\implies \omega(f, x_0) \le \omega(f, B_\delta(x_0)) < \varepsilon \end{aligned}$$

As $\varepsilon > 0$ was arbitrary, $\omega(f, x_0) = 0$.
" $\impliedby$ " Fix $\varepsilon > 0$. Then $\omega(f, x_0) = 0 < \varepsilon$ implies $\exists \delta > 0$ s.t. $\omega(f, B_\delta(x_0)) < \varepsilon$

$$\begin{aligned} &\implies |f(x) - f(y)| < \varepsilon \qquad \forall x, y \in B_\delta(x_0) \\ &\implies |f(x) - f(x_0)| < \varepsilon \qquad \forall x \in B_\delta(x_0) \end{aligned}$$

So f is continuous at x_0. □

Lemma 38.7

Let (X, d) be a metric space and let $f : X \to \mathbb{R}$ be a function. Then for any $\alpha > 0$,

$$\{x \in X : \omega(f, x) < \alpha\} \text{ is open in } X$$

Proof. Fix $\alpha > 0$ and let $A = \{x \in X : \omega(f, x) < \alpha\}$. Fix $x_0 \in A \implies \omega(f, x_0) = \inf_{\delta > 0} \omega(f, B_\delta(x_0)) < \alpha$.

$$\implies \exists \delta > 0 \text{ s.t. } \omega(f, B_\delta(x_0)) < \alpha$$

Claim 38.4. $B_\delta(x_0) \subseteq A$ (which implies $x_0 \in \mathring{A}$ and so $A = \mathring{A}$).

Let $x \in B_\delta(x_0)$. Then $r = \delta - d(x, x_0) > 0$ and $B_r(x) \subseteq B_\delta(x_0)$

$$\begin{aligned} &\implies \omega(f, B_r(x)) \le \omega(f, B_\delta(x_0)) < \alpha \\ &\implies \omega(f, x) \le \omega(f, B_r(x)) < \alpha \implies x \in A \end{aligned}$$

□

Remark 38.8. Let (X, d) be a metric space and let $f : X \to \mathbb{R}$ be a function. Then

$$\begin{aligned} \{x \in X : f \text{ is continuous at } x\} &= \{x \in X : \omega(f, x) = 0\} \\ &= \bigcap_{n \ge 1} \underbrace{\left\{x \in X : \omega(f, x) < \frac{1}{n}\right\}}_{=G_n} \end{aligned}$$

By the lemma, $G_n = \mathring{G_n}$ $\forall n \ge 1$. Also, $G_{n+1} \subseteq G_n$ $\forall n \ge 1$. This observation allows us to prove that there are no functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous at every rational point and discontinuous at every irrational point.

§39 Lec 11: Apr 21, 2021

§39.1 Oscillation of a Function (Cont'd)

Recall from last lecture that there are no functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous at every rational point and discontinuous at every irrational point.

Proof. (Sketch) Assume, towards a contradiction, that $f : \mathbb{R} \to \mathbb{R}$ is such a function. Then

$$\mathbb{Q} = \{x \in \mathbb{R} : f \text{ is continuous at } x\} = \bigcap_{n\geq 1} G_n \text{ with } G_n \text{ open in } \mathbb{R}$$

Note $\forall n \geq 1$, $Q \subseteq G_n$

$$\implies \mathbb{R} = \overline{\mathbb{Q}} \subseteq \overline{G_n} \subseteq \mathbb{R}$$
$$\implies \overline{G_n} = \mathbb{R} \text{ i.e. } G_n \text{ is dense in } \mathbb{R}$$

Let $\{q_n\}_{n\geq 1}$ be an enumeration of $\mathbb{Q}$. For each $n \geq 1$, let $H_n = \mathbb{R} \setminus \{q_n\} = (-\infty, q_n) \cup (q_n, \infty)$. Note H_n is open and dense ($\overline{H_n} = \mathbb{R}$) in $\mathbb{R}$. Also

$$\bigcap_{n\geq 1} H_n = \mathbb{R} \setminus \mathbb{Q}$$

So

$$\bigcap_{n\geq 1} G_n \cap \bigcap_{n\geq 1} H_n = \mathbb{Q} \cap \mathbb{R} \setminus \mathbb{Q} = \emptyset$$

This contradicts the following property of $\mathbb{R}$:

Exercise 39.1. If $\{A_n\}_{n\geq 1}$ is a countable collection of open and dense subsets of $\mathbb{R}$, then

$$\overline{\bigcap_{n\geq 1} A_n} = \mathbb{R}$$

Apply this exercise with $\{A_n : n \geq 1\} = \{G_n : n \geq 1\} \cup \{H_n : n \geq 1\}$. □

§39.2 Weierstrass Approximation Theorem

Theorem 39.1 (Weierstrass Approximation)
Fix $a, b \in \mathbb{R}$ with $a < b$. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then, there exists a sequence of polynomials $\{P_n\}_{n\geq 1}$ with $\deg P_n \leq n \ \forall n \geq 1$ s.t.

$$P_n \xrightarrow[n\to\infty]{u} f \quad \text{on} \quad [a, b]$$

Proof. First, we reduce to the case when $[a, b]$ is $[0, 1]$. Let $\phi : [0, 1] \to [a, b]$, $\phi(t) = a + t(b - a)$. Note ϕ is a continuous, bijective function with the inverse

$$\phi^{-1} : [a, b] \to [0, 1], \quad \phi^{-1}(x) = \frac{x - a}{b - a} \text{ continuous}$$

As $f : [a, b] \to \mathbb{R}$ is continuous, $f \circ \phi : [0, 1] \to \mathbb{R}$ is continuous.
If $\{P_n\}_{n\geq 1}$ is a sequence of polynomials with $\deg P_n \leq n$ s.t.

$$P_n \xrightarrow[n\to\infty]{u} f \circ \phi \text{ on } [0, 1]$$

then $P_n \circ \phi^{-1} \xrightarrow[n\to\infty]{u} f$ on $[a,b]$. Indeed,

$$\sup_{x\in[a,b]} \left|\left(P_n \circ \phi^{-1}\right)(x) - f(x)\right| \underset{x=\phi(t)}{=} \underbrace{\sup_{t\in[0,1]} \left|P_n(t) - (f\circ\phi)(t)\right|}_{\xrightarrow[n\to\infty]{} 0}$$

Therefore, we may assume $f:[0,1]\to\mathbb{R}$ is continuous. Define the Bernstein polynomials via

$$P_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k} \qquad \deg P_n \le n$$

Note that if f is a constant, say $f(x)=c\ \forall x\in[0,1]$ then

$$P_n(x) = c\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k} = c\,(x+1-x)^n = c \quad \forall x\in[0,1]\ \forall n\ge 1$$

We want to show $P_n \xrightarrow[n\to\infty]{u} f$ on $[0,1]$. Fix $x\in[0,1]$. Consider

$$\begin{aligned}
|f(x)-P_n(x)| &= \left|f(x)\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k} - \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}\right| \\
&= \left|\sum_{k=0}^{n}\left[f(x)-f\left(\frac{k}{n}\right)\right]\binom{n}{k}x^k(1-x)^{n-k}\right| \\
&\le \sum_{k=0}^{n}\left|f(x)-f\left(\frac{k}{n}\right)\right|\binom{n}{k}x^k(1-x)^{n-k}
\end{aligned}$$

To estimate the sum we use the following

- when $\frac{k}{n}$ is close to x, we use the continuity of f.
- when $\frac{k}{n}$ is far from x, we use the fact that $x \overset{g}{\mapsto} x^k(1-x)^{n-k}$ has a local maximum at $x=\frac{k}{n}$.

$$\begin{aligned}
g'(x) &= kx^{k-1}(1-x)^{n-k} - (n-k)x^k(1-x)^{n-k-1} \\
&= x^{k-1}(1-x)^{n-k-1}\left\{k(1-x)-(n-k)x\right\} \\
&= x^{k-1}(1-x)^{n-k-1}\left\{k-nx\right\} \\
&= \begin{cases} >0 & \text{if } x<\frac{k}{n} \\ =0 & \text{if } x=\frac{k}{n} \\ <0 & \text{if } x>\frac{k}{n}\end{cases}
\end{aligned}$$

$f:[0,1]\to\mathbb{R}$ is continuous $\implies f$ is uniformly continuous. Fix $\varepsilon>0$. Then $\exists\delta>0$ s.t.

$$|f(x)-f(y)|<\varepsilon \quad \text{whenever} \quad x,y\in[0,1], \quad |x-y|<\delta$$

$f:[0,1]\to\mathbb{R}$ is continuous $\implies f$ is bounded. Let $M>0$ be s.t.

$$|f(x)|\le M \qquad \forall x\in[0,1]$$

We estimate

$$\begin{aligned}
|f(x)-P_n(x)| &\le \sum_{\substack{0\le k\le n\\ \left|x-\frac{k}{n}\right|<\delta}} \underbrace{\left|f(x)-f\left(\frac{k}{n}\right)\right|}_{<\varepsilon}\binom{n}{k}x^k(1-x)^{n-k} \\
&+ \sum_{\substack{0\le k\le n\\ \left|x-\frac{k}{n}\right|\ge\delta}} \underbrace{\left|f(x)-f\left(\frac{k}{n}\right)\right|}_{\le 2M}\binom{n}{k}x^k(1-x)^{n-k} \\
&\le \varepsilon\sum_{0\le k\le n}\binom{n}{k}x^k(1-x)^{n-k} + 2M\sum_{0\le k\le n}\frac{\left(x-\frac{k}{n}\right)^2}{\delta^2}\binom{n}{k}x^k(1-x)^{n-k} \\
&\le \varepsilon + \frac{2M}{n^2\delta^2}\sum_{k=0}^{n}(nx-k)^2\binom{n}{k}x^k(1-x)^{n-k}
\end{aligned}$$

Observe that

$$\begin{aligned}
\sum_{k=0}^{n}(nx-k)^2\binom{n}{k}x^k(1-x)^{n-k} &= n^2x^2\underbrace{\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}}_{=1} \\
&- 2nx\sum_{k=0}^{n}k\cdot\frac{n!}{k!(n-k)!}x^k(1-x)^{n-k} + \sum_{k=0}^{n}k^2\frac{n!}{k!(n-k)!}x^k(1-x)^{n-k}
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{k=0}^{n}k\cdot\frac{n!}{k!(n-k)!}x^k(1-x)^{n-k} &= x\sum_{k=1}^{n}\frac{n!}{(k-1)!(n-k)!}x^{k-1}(1-x)^{n-k} \\
&= nx\underbrace{\sum_{l=0}^{n-1}\frac{(n-1)!}{l!(n-1-l)!}x^l(1-x)^{n-1-l}}_{=(x+1-x)^{n-1}} \\
&= nx
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{n}k^2\frac{n!}{k!(n-k)!}x^k(1-x)^{n-k} &= nx\sum_{k=1}^{n}\frac{k(n-1)!}{(k-1)!(n-k)!}x^{k-1}(1-x)^{n-k} \\
&= nx\sum_{k=1}^{n}\frac{(k-1+1)(n-1)!}{(k-1)!(n-k)!}x^{k-1}(1-x)^{n-k} \\
&= n(n-1)x^2\sum_{k=2}^{n}\frac{(n-2)!}{(k-2)!(n-k)!}x^{k-2}(1-x)^{n-k} \\
&+ nx\sum_{k=1}^{n}\frac{(n-1)!}{(k-1)!(n-k)!}x^{k-1}(1-x)^{n-k} \\
&= n(n-1)x^2+nx
\end{aligned}$$

So

$$\begin{aligned}
\sum_{k=0}^{n}(nx-k)^2\binom{n}{k}x^k(1-x)^{n-k} &= n^2x^2-2n^2x^2+n(n-1)x^2+nx \\
&= nx(1-x)
\end{aligned}$$

We get

$$\begin{aligned}|f(x) - P_n(x)| &\leq \varepsilon + \frac{2M}{n^2\delta^2} \cdot nx(1-x) \\ &\leq \varepsilon + \frac{2M}{n\delta^2} \sup_{x\in[0,1]} x(1-x) \\ &\leq \varepsilon + \frac{M}{2\delta^2 n} < 2\varepsilon\end{aligned}$$

provided $n > \frac{M}{2\delta^2\varepsilon}$. So $P_n \xrightarrow[n\to\infty]{u} f$ on $[0,1]$. □

§40 Lec 12: Apr 23, 2021

§40.1 Weierstrass Approximation Theorem (Cont'd)

Corollary 40.1

Let $M > 0$. Then there exists a sequence of polynomials $\{P_n\}_{n\geq 1}$ s.t.

$$\begin{cases} \deg P_n \leq n & \forall n \geq 1 \\ P_n(0) = 0 & \forall n \geq 1 \\ P_n \xrightarrow[n\to\infty]{u} |x| \text{ on } [-M, M] \end{cases}$$

Proof. Let $f : [-M, M] \to \mathbb{R}$, $f(x) = |x|$. Then f is continuous and $[-M, M]$ compact. By Weierstrass Approximation, $\exists \{Q_n\}_{n\geq 1}$ sequence of polynomials s.t.

$$\begin{cases} \deg Q_n \leq n & \forall n \geq 1 \\ Q_n \xrightarrow[n\to\infty]{u} f \text{ on } [-M, M] \end{cases}$$

Note $Q_n \xrightarrow[n\to\infty]{u} f \implies Q_n(0) \xrightarrow[n\to\infty]{} f(0) = 0$.
Let $P_n(x) = Q_n(x) - Q_n(0)$. Then

$$\begin{cases} \deg P_n \leq n & \forall n \geq 1 \\ P_n(0) = 0 & \forall n \geq 1 \end{cases}$$

For $x \in [-M, M]$,

$$\begin{aligned} |P_n(x) - f(x)| &\leq |Q_n(x) - f(x)| + |Q_n(0)| \leq d(Q_n, f) + |Q_n(0)| \\ \implies d(P_n, f) &\leq d(Q_n, f) + |Q_n(0)| \xrightarrow[n\to\infty]{} 0 \end{aligned}$$ □

§40.2 Stone-Weierstrass Theorem

Definition 40.2 (Algebra) — Let (X, d) be a metric space and let

$$\mathcal{A} \subseteq \{f : X \to \mathbb{R}(\text{or } \mathbb{C});\ f \text{ is a function}\}$$

We say that $\mathcal{A}$ is an algebra if

1. $f + g \in \mathcal{A} \quad \forall f, g \in \mathcal{A}$.
2. $fg \in \mathcal{A} \quad \forall f, g \in \mathcal{A}$
3. $\lambda f \in \mathcal{A} \quad \forall f \in \mathcal{A}\ \forall \lambda \in \mathbb{R}(\text{or } \mathbb{C})$

We say that the algebra $\mathcal{A}$ separates points if whenever $x, y \in X$ with $x \neq y$ then $\exists f \in \mathcal{A}$ s.t. $f(x) \neq f(y)$.
We say that the algebra $\mathcal{A}$ vanishes at no point in X if $\forall x \in X\ \exists f \in \mathcal{A}$ s.t. $f(x) \neq 0$.

Lemma 40.3

Let (X,d) be a compact metric space and let $\mathcal{A} \subseteq C(X)$ be an algebra. Then its closure $\overline{\mathcal{A}}$ with respect to the uniform topology is also an algebra.

Proof. Let $f, g \in \mathcal{A}$. Then

$$\begin{cases} \exists f_n \in \mathcal{A} \text{ s.t. } f_n \xrightarrow[n\to\infty]{u} f \text{ on } X \\ \exists g_n \in \mathcal{A} \text{ s.t. } g_n \xrightarrow[n\to\infty]{u} g \text{ on } X \end{cases}$$

$$\left.\begin{array}{l} d\left(f_n+g_n, f+g\right) \le d(f_n,f) + d(g_n,g) \xrightarrow[n\to\infty]{} 0 \\ f_n+g_n \in \mathcal{A} \text{ (because } \mathcal{A} \text{ is an algebra)} \end{array}\right\} \implies f+g \in \overline{\mathcal{A}}$$

Similarly, for $\lambda \in \mathbb{R}$,

$$\left.\begin{array}{l} d\left(\lambda f_n, \lambda f\right) \le |\lambda| d\left(f_n,f\right) \xrightarrow[n\to\infty]{} 0 \\ \lambda f_n \in \mathcal{A} \text{ (because } \mathcal{A} \text{ is an algebra)} \end{array}\right\} \implies \lambda f \in \overline{\mathcal{A}}$$

Then

$$\begin{aligned} d\left(f_n g_n, fg\right) &= \sup_{x\in X} |f_n(x)g_n(x) - f(x)g(x)| \\ &\le \sup_{x\in X} \left[|f_n(x)-f(x)|\,|g_n(x)| + |f(x)|\,|g_n(x)-g(x)|\right] \\ &\le d(f_n,f)\sup_{x\in X}|g_n(x)| + d(g_n,g)\sup_{x\in X}|f(x)| \end{aligned}$$

By Weierstrass,

$$\left.\begin{array}{l} f_n \xrightarrow[n\to\infty]{u} f \text{ on } X \\ f_n \in C(X) \end{array}\right\} \implies \left.\begin{array}{l} f \in C(X) \\ X \text{ compact} \end{array}\right\} \implies \exists M>0 \text{ s.t. } \sup_{x\in X}|f(x)| \le M$$

Similarly, $g \in C(X) \implies \exists M_2 > 0$ s.t. $\sup_{x\in X}|g(x)| \le M_2$

$$d(g_n,0) \le d(g_n,g) + d(g,0) \le 1 + M_2 \qquad \forall n \ge n_1$$

Let $M_3 = \max\left\{1+M_2, \underbrace{d(g_1,0)}_{<\infty}, \dots, \underbrace{d(g_{n_1},0)}_{<\infty}\right\}$. So $d(g_n,0) \le M_3\, \forall n \ge 1$. Thus

$$\left.\begin{array}{l} d\left(f_n g_n, fg\right) \le d(f_n,f)\cdot M_3 + d(g_n,g)\cdot M_1 \xrightarrow[n\to\infty]{} 0 \\ f_n g_n \in \mathcal{A} \text{ (since } \mathcal{A} \text{ is an algebra)} \end{array}\right\} \implies f\cdot g \in \overline{\mathcal{A}}$$ □

Lemma 40.4

Let (X,d) be a compact metric space and let $\mathcal{A} \subseteq C(X)$ be an algebra that separates points and vanishes at no point in X. Then

$$\forall \alpha,\beta \in \mathbb{R} \quad \forall x_1,x_2 \in X \text{ s.t. } x_1 \ne x_2 \quad \exists f \in \mathcal{A} \text{ s.t. } \begin{cases} f(x_1) = \alpha \\ f(x_2) = \beta \end{cases}$$

Proof. Fix $\alpha, \beta \in \mathbb{R}$. Fix $x_1, x_2 \in X$ s.t. $x_1 \neq x_2$. We would like

$$f(x) = \alpha \cdot \frac{u(x)}{u(x_1)} + \beta \cdot \frac{v(x)}{v(x_1)}$$

for $u, v \in \mathcal{A}$ s.t.

$$\begin{aligned} u(x_1) \neq 0 \quad &\text{and} \quad u(x_2) = 0 \\ v(x_1) = 0 \quad &\text{and} \quad v(x_2) \neq 0 \end{aligned}$$

Then $f \in \mathcal{A}$ (because $\mathcal{A}$ is an algebra) is the desired function.

As $\mathcal{A}$ separates points, $\exists g \in \mathcal{A}$ s.t. $g(x_1) \neq g(x_2)$.

As $\mathcal{A}$ vanishes at no point in X,

$$\begin{cases} \exists h \in \mathcal{A} \text{ s.t } h(x_1) \neq 0 \\ \exists k \in \mathcal{A} \text{ s.t. } k(x_2) \neq 0 \end{cases}$$

Then, we define

$$\begin{aligned} u(x) &= [g(x) - g(x_2)] \cdot h(x) \in \mathcal{A} \\ v(x) &= [g(x) - g(x_1)] \cdot k(x) \in \mathcal{A} \end{aligned}$$

□

Theorem 40.5 (Stone-Weierstrass)

Let (X, d) be a compact metric space and let $\mathcal{A} \subseteq C(X)$ be an algebra that separates points and vanishes no point in X. Then $\mathcal{A}$ is dense in $C(X)$, i.e., $\overline{\mathcal{A}} = C(X) = \{f : X \to \mathbb{R};\ f \text{ continuous}\}$.

Proof. Want to show $\forall f \in C(X)\ \forall \varepsilon > 0\ \exists g \in \mathcal{A}$ s.t. $d(f, g) < \varepsilon$.

Step 1: If $f \in \overline{\mathcal{A}}$ then $|f| \in \overline{\mathcal{A}}$. Let $f \in \overline{\mathcal{A}} \implies \exists f_n \in \mathcal{A}$ s.t.

$$\left.\begin{aligned} &f_n \xrightarrow[n\to\infty]{u} f \text{ on } X \\ &f_n \in C(X) \end{aligned}\right\} \implies f \in C(X)$$

As X is compact, $\exists M > 0$ s.t. $|f(x)| \leq M\ \forall x \in X$. By the previous Corollary 40.1, $\exists \{P_n\}_{n\geq 1}$ sequence of polynomials with $\deg P_n \leq n\ \forall n \geq 1$ s.t.

$$\begin{cases} P_n \xrightarrow[n\to\infty]{u} |x| \text{on } [-M, M] \\ P_n(0) = 0 \end{cases} \implies P_n(f) \xrightarrow[n\to\infty]{u} |f| \text{ on } X$$

If $P_n(x) = \sum_{k=1}^n c_k x^k$ then $P_n(f) = \sum_{k=1}^n c_k f^k \in \mathcal{A}$ which implies $|f| \in \overline{\mathcal{A}}$.

Step 2: If $f, g \in \overline{\mathcal{A}}$ then $\max\{f, g\}, \min\{f, g\} \in \overline{\mathcal{A}}$.

$$\begin{aligned} \max\{f, g\} &= \frac{f+g}{2} + \frac{|f-g|}{2} \in \overline{\mathcal{A}} \\ \min\{f, g\} &= \frac{f+g}{2} - \frac{|f-g|}{2} \in \overline{\mathcal{A}} \end{aligned}$$

Step 3: $\forall f \in C(X)$, $\forall x \in X$, $\forall \varepsilon > 0$, $\exists g \in \overline{\mathcal{A}}$ s.t.

$$g(x) = f(x) \quad \text{and} \quad g(y) > f(y) - \varepsilon \quad \forall y \in X$$

Continue in the next lecture. □

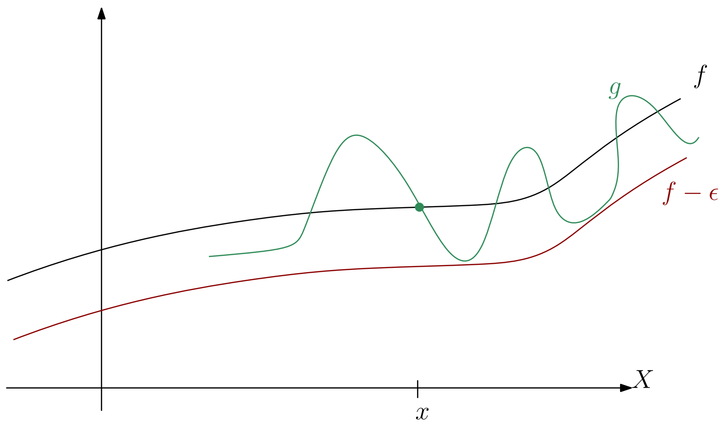
§41 Lec 13: Apr 26, 2021

§41.1 Stone-Weierstrass Theorem (Cont'd)

We continue with the proof of Stone-Weierstrass from lecture 12. Recall that we are at step 3 so far.

Proof. **Step 3**: For any $f \in C(X)$, $x \in X$, $\varepsilon > 0$, there exists $g \in \overline{\mathcal{A}}$ s.t.

$$\begin{cases} g(x) = f(x) \\ g(y) > f(y) - \varepsilon \qquad \forall y \in X \end{cases}$$



For any $y \in X$, there exists $h_y \in \overline{\mathcal{A}}$ s.t.

$$h_y(x) = f(x)$$
$$h_y(y) = f(y)$$

As $h_y \in \overline{\mathcal{A}}$, h_y is continuous. Thus, $h_y - f$ is continuous at y. So $\exists \delta_y > 0$ s.t. $|h_y(z) - f(z)| < \varepsilon$, $\forall z \in B_{\delta_y}(y)$. In particular,

$$h_y(z) > f(z) - \varepsilon \qquad \forall z \in B_{\delta_y}(y)$$

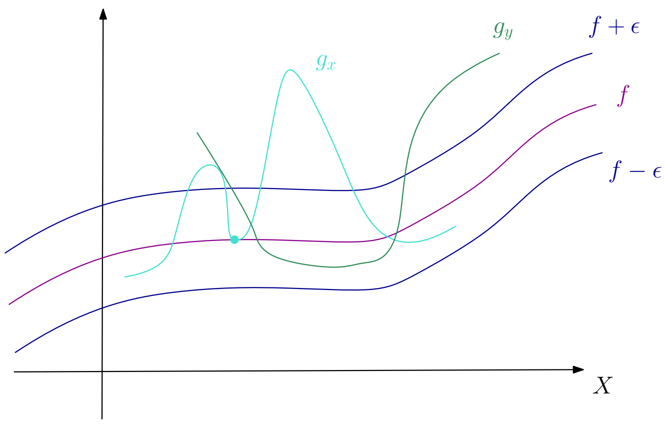
Note that

$$\left.\begin{matrix} X = \bigcup_{y \in X} B_{\delta_y}(y) \\ X \text{ compact} \end{matrix}\right\} \implies \exists N \geq 1 \text{ and } \exists y_1, \ldots, y_N \in X$$

s.t. $X = \bigcup_{n=1}^N B_{\delta_n}(y_n)$ where $\delta_n = \delta_{y_n}$.
Take $g = \max\{h_{y_1}, \ldots, h_{y_N}\}$ (by step 2). By construction, $g(x) = f(x)$. Also if $y \in X$, $\exists 1 \leq n \leq N$ s.t. $y \in B_{\delta_n}(y_n)$. So

$$g(y) \geq h_{y_n}(y) > f(y) - \varepsilon$$

Step 4: For all $f \in C(X)$ and $\varepsilon > 0$, $\exists g \in \overline{\mathcal{A}}$ s.t. $d(f,g) < \varepsilon$. Fix $f \in C(X)$, $\varepsilon > 0$



For $x \in X$, let $g_x \in \overline{\mathcal{A}}$ be the function given by step 3. In particular, $g_x(x) = f(x)$,

$$g_x(y) > f(y) - \varepsilon \qquad \forall y \in X$$

As $g_x \in \overline{\mathcal{A}}$, the function $g_x - f$ is continuous at x. So $\exists \delta_x > 0$ s.t. $|g_x(y) - f(y)| < \varepsilon$, $\forall y \in B_{\delta_x}(x)$. In particular,

$$g_x(y) < f(y) + \varepsilon \qquad \forall y \in B_{\delta_x}(x)$$

Note

$$\left.\begin{array}{l} X = \bigcup_{x\in X} B_{\delta_x}(x) \\ X \text{ compact} \end{array}\right\} \implies \exists N \geq 1 \text{ and } \exists x_1, \dots, x_N \in X \text{ s.t.}$$

$X = \bigcup_{n=1}^{N} B_{\delta_n}(x_n)$ where $\delta_n = \delta_{x_n}$.
Take $g = \min\{g_{x_1}, \dots, g_{x_N}\} \in \overline{\mathcal{A}}$ (by step 2).
For $y \in X$, $\exists 1 \leq n \leq N$ s.t. $y \in B_{\delta_n}(x_n)$ and so

$$g(y) \leq g_{x_n}(y) < f(y) + \varepsilon$$

Moreover, as $g_{x_n}(y) > f(y) - \varepsilon$, $\forall y \in X$, $\forall 1 \leq n \leq N$, we have

$$g(y) > f(y) - \varepsilon \qquad \forall y \in X$$

This shows $C(X) \subseteq \overline{\overline{\mathcal{A}}} = \overline{\mathcal{A}} \subseteq C(X)$. □

§41.2 Differentiation

Definition 41.1 (Limit) — Let $(X, d_X), (Y, d_Y)$ be metric spaces, let $\emptyset \neq A \subseteq X$, let $f : A \to Y$. For $x_0 \in A'$ and $y_0 \in Y$ we write

$$f \underset{x \to x_0}{\longrightarrow} y_0 \qquad \text{or} \qquad \lim_{x\to x_0} f(x) = y_0$$

if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $d_Y(f(x), y_0) < \varepsilon$ whenever $0 < d_X(x, x_0) < \delta$.
Equivalently, $\lim_{x\to x_0} f(x) = y_0$ if

$$\lim_{n\to\infty} f(x_n) = y_0 \text{ for every sequence } \{x_n\}_{n\geq 1} \subseteq A \setminus \{x_0\} \text{ s.t. } x_n \xrightarrow[n\to\infty]{d_X} x_0$$

Note also that if $x_0 \in A' \cap A$ then f is continuous at $x_0 \iff \lim_{x\to x_0} f(x) = f(x_0)$.

Exercise 41.1. Let (X, d) be a metric space, $\emptyset \neq A \subseteq X$, $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be functions. Assume that at a point $a \in A'$ we have

$$\lim_{x \to x_0} f(x) = \alpha \quad \text{and} \quad \lim_{x \to x_0} g(x) = \beta$$

Then

1. $\lim_{x \to x_0} (\lambda f(x)) = \lambda\alpha$, $\lambda \in \mathbb{R}$
2. $\lim_{x \to x_0} (f(x) + g(x)) = \alpha + \beta$
3. $\lim_{x \to x_0} (f(x)g(x)) = \alpha \cdot \beta$
4. If $\beta \neq 0$ then $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\alpha}{\beta}$

Definition 41.2 (Differentiability) — Let I be an open interval and let $f : I \to \mathbb{R}$ be a function. We say that f is differentiable at $a \in I$ if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \text{ exists and is finite}$$

in which case we denote it $f'(a)$.

Example 41.3

Fix $n \geq 1$ and let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^n$. For $a \in \mathbb{R}$ and $x \neq a$

$$\begin{aligned} \frac{f(x) - f(a)}{x - a} &= \frac{x^n - a^n}{x - a} \\ &= x^{n-1} + x^{n-2}a + \ldots + a^{n-1} \xrightarrow[x \to a]{} na^{n-1} \end{aligned}$$

So f is differentiable at a and $f'(a) = na^{n-1}$.

Theorem 41.4

Let I be an open interval and let $f : I \to \mathbb{R}$ be differentiable at $a \in I$. Then f is continuous at a.

Proof. For $x \in I \setminus \{a\}$, we write

$$f(x) = \underbrace{\frac{f(x) - f(a)}{x - a}}_{\xrightarrow[x \to a]{} f'(a)} \cdot \underbrace{(x - a)}_{\xrightarrow[x \to a]{} 0} + \underbrace{f(a)}_{\xrightarrow[x \to a]{} f(a)} \xrightarrow[x \to a]{} f(a)$$

□

Theorem 41.5

Let I be an open interval and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be two functions differentiable at $a \in I$. Then

1. $\forall \lambda \in \mathbb{R}$, λf is differentiable at a and
$$(\lambda f)'(a) = \lambda f'(a)$$
2. $f + g$ is differentiable at a and
$$(f+g)'(a) = f'(a) + g'(a)$$
3. $f \cdot g$ is differentiable at a and
$$(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a)$$
4. $\frac{f}{g}$ is differentiable at a if $g(a) \neq 0$ and
$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}$$

Proof. For $x \neq a$

1. Consider
$$\frac{\lambda f(x) - \lambda f(a)}{x-a} = \lambda \cdot \frac{f(x)-f(a)}{x-a} \xrightarrow[x \to a]{} \lambda f'(a)$$
2. Consider
$$\frac{(f(x)+g(x)) - (f(a)+g(a))}{x-a} = \frac{f(x)-f(a)}{x-a} + \frac{g(x)-g(a)}{x-a} \xrightarrow[x \to a]{} f'(a) + g'(a)$$
3. Consider
$$\underbrace{\frac{f(x)-f(a)}{x-a}}_{\xrightarrow[x \to a]{} f'(a)} \cdot \underbrace{g(x)}_{\xrightarrow[x \to a]{} g(a)} + \underbrace{f(a)}_{\xrightarrow[x \to a]{} f(a)} \cdot \underbrace{\frac{g(x)-g(a)}{x-a}}_{\xrightarrow[x \to a]{} g'(a)} \xrightarrow[x \to a]{} f'(a)g(a) + f(a)g'(a)$$
4. Consider
$$\frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x-a} \underbrace{\frac{f(x)-f(a)}{x-a}}_{\xrightarrow[x \to a]{} f'(a)} \cdot \underbrace{\frac{1}{g(x)}}_{\xrightarrow[x \to a]{} \frac{1}{g(a)}} + f(a) \cdot \underbrace{\frac{g(a)-g(x)}{x-a}}_{\xrightarrow[x \to a]{} -g'(a)} \cdot \underbrace{\frac{1}{g(x)}}_{\xrightarrow[x \to a]{} \frac{1}{g(a)}} \cdot \frac{1}{g(a)}$$
$$\xrightarrow[x \to a]{} \frac{f'(a)}{g(a)} - \frac{g'(a)}{g^2(a)} f(a)$$

□

§42 Lec 14: Apr 28, 2021

§42.1 Chain Rule

Theorem 42.1 (Chain Rule)
Let I and J be two open intervals and let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be two functions. Assume that f is differentiable at $a \in I$ and that g is differentiable at $f(a) \in J$. Then $g \circ f$ is well defined on a neighborhood of a, $g \circ f$ is differentiable at a, and

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$$

Proof. Consider:

$$\left.\begin{matrix} f(a) \in J \\ J \text{ is open} \end{matrix}\right\} \implies \exists \varepsilon > 0 \text{ s.t. } (f(a) - \varepsilon, f(a) + \varepsilon) \subseteq J$$

f is differentiable at $a \implies f$ is continuous at $a \implies \exists \delta > 0$ s.t. $f((a-\delta, a+\delta) \cap I) \subseteq (f(a) - \varepsilon, f(a) + \varepsilon)$. As $a \in I$ and I is open, shrinking δ if necessary, me may assume that $(a - \delta, a + \delta) \subseteq I$.

Then $g \circ f$ is well-defined on $(a - \delta, a + \delta)$.

$$\underbrace{(a-\delta, a+\delta)}_{\subseteq I} \xrightarrow{f} \underbrace{(f(a) - \varepsilon, f(a) + \varepsilon)}_{\subseteq J} \xrightarrow{g} \mathbb{R}$$

<u>Caution</u>: The following argument does not work

$$\frac{g(f(x)) - g(f(a))}{x - a} = \underbrace{\frac{g(f(x)) - g(f(a))}{f(x) - f(a)}}_{\xrightarrow{x \to a} g'(f(a))} \cdot \underbrace{\frac{f(x) - f(a)}{x - a}}_{\xrightarrow{x \to a} f'(a)}$$

because f is continuous at $a \implies f(x) \xrightarrow{x \to a} f(a)$

Instead, we argue as follows: Define $h : J \to \mathbb{R}$,

$$h(y) = \begin{cases} \frac{g(y) - g(f(a))}{y - f(a)}, & \text{if } y \in J \setminus \{f(a)\} \\ g'(f(a)), & \text{if } y = f(a) \end{cases}$$

As g is differentiable at $f(a)$, h is continuous at $f(a)$. Moreover, we can write

$$g(y) - g(f(a)) = h(y) \cdot (y - f(a)) \qquad \forall y \in J$$

For $x \in (a - \delta, a + \delta) \implies f(x) \in J$. So for $x \in (a - \delta, a + \delta) \setminus \{a\}$,

$$\frac{g(f(x)) - g(f(a))}{x - a} = \underbrace{h(f(x))}_{\xrightarrow{x \to a} h(f(a))} \cdot \underbrace{\frac{f(x) - f(a)}{x - a}}_{\xrightarrow{x \to a} f'(a)}$$

So $\lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a} = h(f(a)) f'(a) = g'(f(a)) \cdot f'(a)$. ∎

Lemma 42.2

Let $f:(a,b)\to\mathbb{R}$ be a differentiable function. If f is increasing then $f'(x)\geq 0\,\forall x\in(a,b)$ or decreasing then $f'(x)\leq 0\,\forall x\in(a,b)$.

Proof. Assume f is increasing (if f is decreasing, replace f by $-f$ in what follows). Fix $x\in(a,b)$ and let $\{x_n\}_{n\geq 1}$ be an increasing from (a,b) with $\lim_{n\to\infty} x_n = x$.
Then $f'(x)=\lim_{n\to\infty}\frac{f(x_n)-f(x)}{x_n-x}\geq 0$ where $f(x_n)-f(x)\leq 0$ and $x_n-x<0$. ☐

Theorem 42.3

Let $f:(a,b)\to\mathbb{R}$ be a function. Assume that $x_0\in(a,b)$ is a point of local maximum/minimum for f. Assume also that f is differentiable at x_0. Then $f'(x_0)=0$.

Proof. Assume that x_0 is a point of local maximum for f (if x_0 is a point of local minimum, replace f by $-f$ in what follows).

Then $\exists\delta>0$ s.t. $f(x)\leq f(x_0)\quad\forall x\in(x_0-\delta,x_0+\delta)\cap(a,b)$. For $x_n\in(x_0-\delta,x_0)\cap(a,b)$ s.t. $x_n\underset{n\to\infty}{\longrightarrow}x_0$, we have

$$f'(x_0)=\lim_{n\to\infty}\frac{f(x_n)-f(x_0)\leq 0}{x_n-x_0<0}\geq 0$$

On the other hand, for $y_n\in(x_0,x_0+\delta)\cap(a,b)$ s.t. $y_n\underset{n\to\infty}{\longrightarrow}x_0$, we have

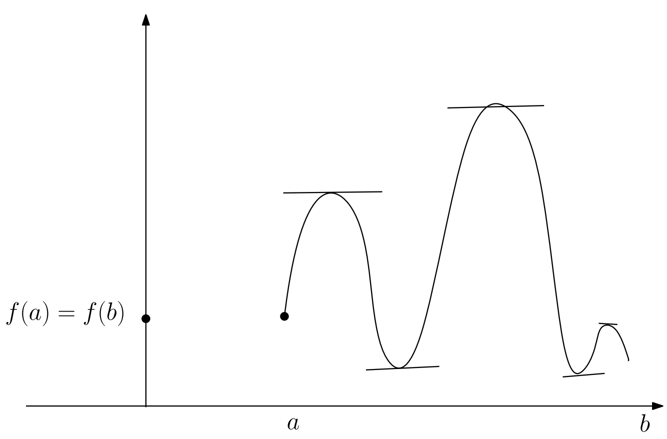
$$f'(x_0)=\lim_{n\to\infty}\frac{f(y_n)-f(x_0)\leq 0}{y_n-x_0>0}\leq 0$$

Thus, we get $f'(x_0)=0$. ☐

§42.2 Mean Value Theorem

Theorem 42.4 (Rolle)

Let $f:[a,b]\to\mathbb{R}$ be a function which is continuous on the $[a,b]$, differentiable on (a,b), and s.t. $f(a)=f(b)$. Then there exists (at least one) $x\in(a,b)$ s.t. $f'(x)=0$.



Proof. Consider:

$$\left.\begin{array}{l} f:[a,b]\to\mathbb{R} \text{ continuous} \\ [a,b] \text{ compact} \end{array}\right\} \implies \exists x_0, y_0 \in [a,b]$$

s.t.

$$f(x_0) = \sup_{x\in[a,b]} f(x) \quad \text{and} \quad f(y_0) = \inf_{x\in[a,b]} f(x)$$

So $f(y_0) \le f(x) \le f(x_0) \quad \forall x \in [a,b]$.
Case 1: We have

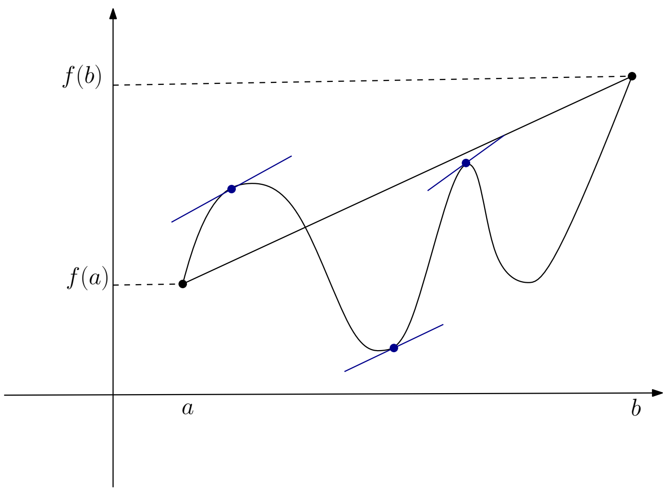
$$\left.\begin{array}{l} \{x_0, y_0\} \subseteq \{a,b\} \\ f(a) = f(b) \end{array}\right\} \implies f(x_0) = f(y_0) \implies f \text{ constant} \implies f'(x) = 0\,\forall x \in (a,b)$$

Case 2: $\{x_0, y_0\} \nsubseteq \{a,b\} \implies x_0 \notin \{a,b\}$ or $y_0 \notin \{a,b\}$. Say $x_0 \notin \{a,b\} \implies x_0 \in (a,b)$. By Theorem 42.3, we get $f'(x_0) = 0$. ☐

Theorem 42.5 (Mean Value)
Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists (at least one) $y \in (a,b)$ s.t.

$$f'(y) = \frac{f(b)-f(a)}{b-a}$$

Remark 42.6. The Mean Value Theorem implies Rolle's Theorem. We will see from the proof that Rolle's Theorem implies the Mean Value Theorem, so the two are equivalent.



Proof. We define $l:[a,b]\to\mathbb{R}$ where

$$l(x) = \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$$

Note that l is continuous on $[a,b]$, differentiable on (a,b), and

$$l'(x) = \frac{f(b)-f(a)}{b-a} \qquad \forall x \in (a,b)$$

Let $g : [a,b] \to \mathbb{R}$, $g(x) = f(x) - l(x)$. Then g is continuous on $[a,b]$, differentiable on (a,b), and $g(a) = 0 = g(b)$. Then Rolle's implies that $\exists y \in (a,b)$ s.t.

$$g'(y) = 0 \implies f'(y) - l'(y) = 0 \implies f'(y) = \frac{f(b) - f(a)}{b - a}$$ □

Corollary 42.7

If $f : (a,b) \to \mathbb{R}$ is differentiable and $f'(x) = 0 \, \forall x \in (a,b)$, then f is a constant.

Proof. Assume f is not a constant. Then $\exists a < x_1 < x_2 < b$ s.t.

$$f(x_1) \neq f(x_2)$$

Then f is continuous on $[x_1, x_2]$, differentiable on (x_1, x_2). By Mean Value, $\exists y \in (x_1, x_2)$ s.t.

$$f'(y) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} \neq 0$$

Contradiction! □

Corollary 42.8

If $f, g : (a,b) \to \mathbb{R}$ are differentiable s.t. $f'(x) = g'(x) \, \forall x \in (a,b)$, then $\exists c \in \mathbb{R}$ s.t.

$$f(x) = g(x) + c \qquad \forall x \in (a,b)$$

§43 Lec 15: Apr 30, 2021

§43.1 Mean Value Theorem (Cont'd)

Theorem 43.1

Let $f : [a, b] \to \mathbb{R}$, $g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists (at least one) $c \in (a, b)$ s.t.

$$f'(c)\,[g(b) - g(a)] = g'(c)\,[f(b) - f(a)]$$

Remark 43.2. Taking $g(x) = x$ we recover the Mean Value theorem. In fact, the two results are equivalent, as can be seen from the proof.

Proof. We define $h : [a, b] \to \mathbb{R}$

$$h(x) = f(x)\,[g(b) - g(a)] - g(x)\,[f(b) - f(a)]$$

Note that h is continuous on $[a, b]$ and differentiable on (a, b). Moreover,

$$\left.\begin{aligned} h(a) &= f(a)\,[g(b) - g(a)] - g(a)\,[f(b) - f(a)] = f(a)g(b) - g(a)f(b) \\ h(b) &= f(b)\,[g(b) - g(a)] - g(b)\,[f(b) - f(a)] = -f(b)g(a) + g(b)f(a) \end{aligned}\right\} \implies h(a) = h(b)$$

By Rolle's theorem, $\exists c \in (a, b)$ s.t $h'(c) = 0$. □

Corollary 43.3

Let $f : (a, b) \to \mathbb{R}$ be differentiable.

1. If $f'(x) > 0$ $\forall x \in (a, b)$ then f is strictly increasing.
2. If $f'(x) \geq 0$ $\forall x \in (a, b)$ then f is increasing.
3. If $f'(x) < 0$ $\forall x \in (a, b)$ then f is strictly decreasing.
4. If $f'(x) \leq 0$ $\forall x \in (a, b)$ then f is decreasing.

Proof. We only present the details for (1).
Fix $a < x_1 < x_2 < b$. f is differentiable on $(a, b) \implies f$ is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). By the Mean Value theorem, $\exists c \in (x_1, x_2)$ s.t.

$$0 < f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \implies f(x_1) < f(x_2)$$

As $a < x_1 < x_2 < b$ were arbitrary, f is strictly increasing. □

Example 43.4

The derivative of a differentiable function need not be continuous

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

f is continuous on $\mathbb{R} \setminus \{0\}$. To see that it's continuous at 0,

$$|f(x) - f(0)| = \left| x^2 \sin \frac{1}{x} \right| \leq x^2 \underset{x \to 0}{\longrightarrow} 0 \tag{*}$$

f is differentiable on $\mathbb{R} \setminus \{0\}$. To see that it's differentiable at 0, we compute

$$x \neq 0 : \quad \frac{f(x) - f(0)}{x - 0} = x \sin \frac{1}{x} \underset{x \to 0}{\longrightarrow} 0 \quad \text{(as in (*))}$$

So $f'(0) = 0$. Thus,

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} + x^2 \cos \frac{1}{x} \cdot \frac{-1}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases} = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

f' is continuous on $\mathbb{R} \setminus \{0\}$ (not continuous at 0). While $\lim_{x \to 0} 2x \sin \frac{1}{x} = 0$, for each $\lambda \in [-1, 1]$, there exists $x_n(\lambda) \underset{n \to \infty}{\longrightarrow} 0$ s.t. $\cos \frac{1}{x_n(\lambda)} = \lambda$. Nevertheless, the derivative of a differentiable function has the Darboux property.

Theorem 43.5 (Intermediate Value for Derivatives)

Let $f : (a, b) \to \mathbb{R}$ be differentiable. Then f' has the Darboux property, that is, if $a < x_1 < x_2 < b$ and λ lies between $f'(x_1)$ and $f'(x_2)$, then there exists $c \in (x_1, x_2)$ s.t.

$$f'(c) = \lambda$$

Proof. Let $g : (a, b) \to \mathbb{R}$, $g(x) = f(x) - \lambda x$. g is differentiable on $(a, b) \implies g$ is continuous on (a, b). Fix $a < x_1 < x_2 < b$ and assume without loss of generality

$$f'(x_1) < \lambda < f'(x_2)$$

Then

$$g'(x_1) = f'(x_1) - \lambda < 0$$
$$g'(x_2) = f'(x_2) - \lambda > 0$$

g is continuous on $[x_1, x_2]$

$$\implies \exists c \in [x_1, x_2] \text{ s.t. } g(c) = \inf_{x \in [x_1, x_2]} g(x)$$

If we can prove that $c \in (x_1, x_2)$ then $g'(c) = 0$. To see that $c \neq x_1$ we argue as follows:

$$0 > g'(x_1) = \lim_{x \to x_1} \frac{g(x) - g(x_1)}{x - x_1} \implies \exists \delta_1 > 0$$

s.t. if $0 < |x - x_1| < \delta_1$ then

$$\frac{g(x) - g(x_1)}{x - x_1} < 0$$

In particular, for $x \in (x_1, x_1 + \delta_1)$ we have

$$\frac{g(x) - g(x_1)}{\underbrace{x - x_1}_{>0}} < 0 \implies g(x) < g(x_1)$$

$$\implies g \text{ cannot attain its minimum at } x_1$$

Similarly,

$$0 < g'(x_2) = \lim_{x \to x_2} \frac{g(x) - g(x_2)}{x - x_2} \implies \exists \delta_2 > 0$$

s.t. if $0 < |x - x_2| < \delta_2$ then

$$\frac{g(x) - g(x_2)}{x - x_2} > 0$$

In particular, if $x \in (x_2 - \delta_2, x_2)$ then

$$\frac{g(x) - g(x_2)}{\underbrace{x - x_2}_{<0}} \implies g(x) < g(x_2)$$

$$\implies g \text{ cannot attain its minimum at } x_2$$

□

§43.2 Derivative of Inverse Functions

Theorem 43.6

Let I be an open interval and let $f : I \to \mathbb{R}$ be continuous and injective. Then $f(I) = J$ is an interval and $f : I \to J$ is bijective. If f is differentiable at $x_0 \in I$ and $f'(x_0) \neq 0$ then $f^{-1} : J \to I$ is differentiable at $y_0 = f(x_0)$ and

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'\left(f^{-1}(y_0)\right)}$$

Proof. The proof uses the following two exercises:

Exercise 43.1. Let I be an interval and let $f : I \to \mathbb{R}$ be continuous and injective. Then f is strictly monotone.

Exercise 43.2. Let I be an interval and let $f : I \to \mathbb{R}$ be strictly increasing and so that $f(I)$ is an interval. Then f is continuous.

Using exercise 1, we find that f is strictly monotone. Assume f is strictly increasing $\implies f^{-1}$ is strictly increasing.
Using exercise 2 with $g = f^{-1} : J \to I$, we find that f^{-1} is continuous.

Claim 43.1. J is an open interval.

Assume, towards a contradiction, that $\inf J \in J = f(I) \implies \exists a \in I$ s.t. $f(a) = \inf J$.

$$\left.\begin{array}{l} I \text{ open } \implies \exists \delta > 0 \text{ s.t. } (a - \delta, a + \delta) \subseteq I \\ f \text{ is strictly increasing} \end{array}\right\} \implies J = f(I) \ni f\left(a - \frac{\delta}{2}\right) < f(a) = \inf J$$

Contradiction!

Similarly, one can show that $\sup J \notin J$

$$\left.\begin{aligned} &f \text{ is diff at } x_0 \implies f'(x_0) = \lim \tfrac{f(x)-f(x_0)}{x-x_0} \\ &f'(x_0) \neq 0 \text{ and } f(x) \neq f(x_0) \quad \forall x \neq x_0 \end{aligned}\right\} \implies$$

$$\implies \lim_{x\to x_0} \frac{x-x_0}{f(x)-f(x_0)} = \frac{1}{f'(x_0)}$$

$$\implies \forall \varepsilon > 0 \quad \exists \delta > 0 \text{ s.t. } 0 < |x-x_0| < \delta \implies \left| \frac{x-x_0}{f(x)-f(x_0)} - \frac{1}{f'(x_0)} \right| < \varepsilon$$

f^{-1} is continuous at $y_0 \implies \exists \eta > 0$ s.t. $0 < |y-y_0| < \eta$ implies

$$0 < \left| f^{-1}(y) - f^{-1}(y_0) \right| < \delta$$

So for $0 < |y-y_0| < \eta$ we get

$$\left| \frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0} - \frac{1}{f'(x_0)} \right| < \varepsilon$$

which implies

$$\left(f^{-1}\right)'(y_0) = \lim_{y\to y_0} \frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0} = \frac{1}{f'(x_0)} \qquad \square$$

§44 Lec 16: May 3, 2021

§44.1 L'Hopital Rule

Definition 44.1 (Existence of Limit) — Let $-\infty \le a < b \le \infty$ and let $f : (a, b) \to \mathbb{R}$ be a function. For $c \in (a, b) \cup \{a\}$ we write

$$\lim_{x \to c^+} f(x) = L \in \mathbb{R} \cup \{\pm\infty\}$$

if for every sequence $\{x_n\}_{n \ge 1} \subseteq (c, b)$ s.t. $\lim_{n\to\infty} x_n = c$ we have

$$\lim_{n\to\infty} f(x_n) = L$$

For $c \in (a, b) \cup \{b\}$ we write

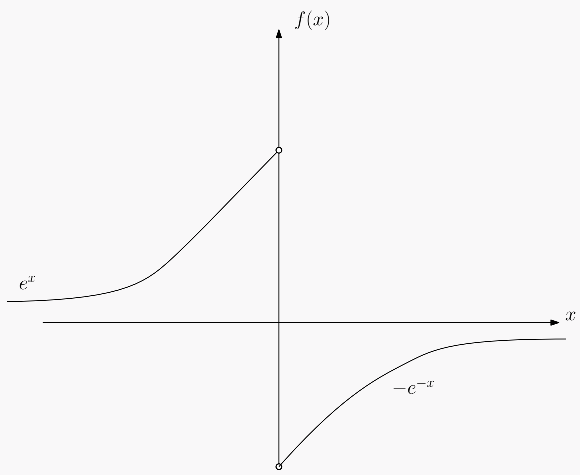
$$\lim_{x \to c^-} f(x) = M \in \mathbb{R} \cup \{\pm\infty\}$$

if for every sequence $\{x_n\}_{n \ge 1} \subseteq (a, c)$ s.t. $\lim_{n\to\infty} x_n = c$ we have

$$\lim_{n\to\infty} f(x_n) = M$$

Remark 44.2. In general, if $c \in (a, b)$ we have

$$f(c) \ne \lim_{x\to c^-} f(x) \ne \lim_{x \to c^+} f(x) \ne f(c)$$



Theorem 44.3 (L'Hopital)
Let $-\infty \leq a < b \leq \infty$ and let $f, g : (a,b) \to \mathbb{R}$ be differentiable. Assume that $g'(x) \neq 0$ $\forall x \in (a,b)$ and that

$$\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R} \cup \{\pm\infty\}$$

Assume also that either

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0 \tag{1}$$

or

$$\lim_{x\to a^+} |g(x)| = \infty \tag{2}$$

Then

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$$

Remark 44.4. $\lim_{x\to a^+}$ in the theorem can be replaced by $\lim_{x\to b^-}$ or by $\lim_{x\to c}$ for some $c \in (a,b)$.

Proof. We'll present the details for $L \in \mathbb{R}$. We'll prove

Claim 44.1. $\forall \varepsilon > 0$ $\exists \delta_1(\varepsilon) > 0$ s.t.

$$\frac{f(x)}{g(x)} < L + \varepsilon \qquad \forall x \in (a, a+\delta_1)$$

Claim 44.2. $\forall \varepsilon > 0$ $\exists \delta_2(\varepsilon) > 0$ s.t.

$$L - \varepsilon < \frac{f(x)}{g(x)} \qquad \forall x \in (a, a+\delta_2)$$

Then taking $\delta(\varepsilon) = \min\{\delta_1(\varepsilon), \delta_2(\varepsilon)\}$ we get

$$\left|\frac{f(x)}{g(x)} - L\right| < \varepsilon \qquad \forall x \in (a, a+\delta)$$

$\implies \lim_{x\to a^+} \frac{f(x)}{g(x)} = L$.
Note: If $L = -\infty$ then it suffices to prove Claim 1 with $L+\varepsilon$ replaced by $M < 0$.
If $L = \infty$ then it suffices to prove Claim 2 with $L - \varepsilon$ replaced by $M > 0$.
By assumption, $g'(x) \neq 0$ $\forall x \in (a,b)$. As g is differentiable on (a,b), g' has the Darboux property. So either $g'(x) < 0$ $\forall x \in (a,b)$ or $g'(x) > 0$ $\forall x \in (a,b)$.
Assume $g'(x) < 0$ $\forall x \in (a,b) \implies g$ strictly decreasing on (a,b). In case 1,

$$\lim_{x\to a^+} g(x) = 0$$

As g is strictly decreasing, we get

$$g(x) < 0 \qquad \forall x \in (a,b)$$

In case 2,

$$\lim_{x\to a^+} |g(x)| = \infty$$

As g is strictly decreasing, we get

$$\lim_{x\to a^+} g(x) = \infty$$

and so $\exists c \in (a,b)$ s.t. $g(x) > 0$ $\forall x \in (a,c)$ (**). In particular, in both cases $g(x) \neq 0$ $\forall x \in (a,c)$. We prove claim 1:

Fix $\varepsilon > 0$. As $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$, $\exists \delta_1(\varepsilon) > 0$ s.t.

$$\frac{f'(x)}{g'(x)} < L + \frac{\varepsilon}{2} \qquad \forall x \in (a, a+\delta_1)$$

Fix $a < x < y < \min(a+\delta_1, c)$. By (an equivalent formulation of) Mean Value theorem, $\exists z \in (x,y)$ s.t.

$$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(z)}{g'(z)} < L + \frac{\varepsilon}{2} \tag{*}$$

In case 1, take the limit $x \to a^+$ in (*) to get

$$\frac{f(y)}{g(y)} \le L + \frac{\varepsilon}{2} < L + \varepsilon \qquad \forall a < y < \min(a+\delta_1, c)$$

In case 2, we write

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(y)}{g(x)-g(y)} \cdot \frac{g(x)-g(y)}{g(x)} + \frac{f(y)}{g(x)}$$

By (**) we have $g(x) > g(y) > 0 \implies \frac{g(x)-g(y)}{g(x)} > 0$. So

$$\begin{aligned}\frac{f(x)}{g(x)} &< \left(L + \frac{\varepsilon}{2}\right) \frac{g(x)-g(y)}{g(x)} + \frac{f(y)}{g(x)} \\ &= \left(L + \frac{\varepsilon}{2}\right)\left(1 - \frac{g(y)}{g(x)}\right) + \frac{f(y)}{g(x)} \\ &= L + \frac{\varepsilon}{2} + \frac{f(y) - \left(L + \frac{\varepsilon}{2}\right) g(y)}{g(x)}\end{aligned}$$

For y fixed, $\lim_{x\to a^+} \frac{f(y)-\left(L+\frac{\varepsilon}{2}\right)g(y)}{g(x)} = 0$

$$\implies \exists \tilde{\delta_1}(\varepsilon) > 0 \text{ s.t. } \left|\frac{f(y) - \left(L + \frac{\varepsilon}{2}\right) g(y)}{g(x)}\right| < \frac{\varepsilon}{2} \qquad \forall x \in \left(a, a + \tilde{\delta_1}\right)$$

In particular,

$$\frac{f(x)}{g(x)} < L + \varepsilon \qquad \forall a < x < \min\left\{a + \delta_1, a + \tilde{\delta_1}, c\right\}$$

Exercise 44.1. Prove claim 2. □

§44.2 Taylor's Theorem

Definition 44.5 (Taylor Expansion) — Let I be an open interval and let $f : I \to \mathbb{R}$ be differentiable of any order. For $x_0 \in I$, the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

is called the Taylor expansion of f about x_0. For $n \geq 1$, we define the remainder

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

Theorem 44.6 (Taylor)

Let $n \geq 1$ and assume $f : (a, b) \to \mathbb{R}$ is n times differentiable. Let $x_0 \in (a, b)$. Then for any $x \in (a, b) \setminus \{x_0\}$ there exists y between x and x_0 s.t.

$$R_n(x) = \frac{f^{(n)}(y)}{n!}(x - x_0)^n$$

In particular,

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n)}(y)}{n!}(x - x_0)^n$$

Proof. Fix $x \in (a, b) \setminus \{x_0\}$. Define $M \in \mathbb{R}$ to be the unique solution to the equation

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + M \cdot \frac{(x - x_0)^n}{n!}$$

We want to show that there exists y between x and x_0 s.t.

$$M = f^{(n)}(y)$$

Let $g : (a, b) \to \mathbb{R}$

$$g(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(t - x_0)^k - M \cdot \frac{(t - x_0)^n}{n!}$$

Note g is n times differentiable. For $1 \leq l \leq n - 1$,

$$\begin{aligned} g^{(l)}(t) &= f^{(l)}(t) - \sum_{k \geq l}^{n-1} \frac{f^{(k)}(x_0)}{(k - l)!}(t - x_0)^{k-l} - M \frac{(t - x_0)^{n-l}}{(n - l)!} \\ g^{(n)}(t) &= f^{(n)}(t) - M \end{aligned}$$

In particular, if $0 \leq l \leq n - 1$,

$$g^{(l)}(x_0) = f^{(l)}(x_0) - f^{(l)}(x_0) = 0$$

Also $g(x) = 0$ by contradiction.

g is continuous on $[x, x_0]$, differentiable on (x, x_0) and

$$g(x) = g(x_0) = 0 \implies \exists x_1 \in (x, x_0) \text{ s.t. } g'(x_1) = 0$$

By Rolle's theorem,

$$\begin{aligned} \exists x_2 \in (x_1, x_0) \quad & \text{s.t.} \quad g''(x_2) = 0 \\ & \vdots \\ \exists x_n \in (x_{n-1}, x_0) \quad & \text{s.t.} \quad g^{(n)}(x_n) = 0 \end{aligned}$$

Set $y = x_n$. □

§45 Lec 17: May 5, 2021

§45.1 Taylor's Theorem (Cont'd)

Corollary 45.1

Fix $a > 0$ and let $f : (-a, a) \to \mathbb{R}$ be a function differentiable of any order. Assume that all derivatives of f are uniformly bounded on $(-a, a)$, that is,

$$\exists M > 0 \text{ s.t. } \left|f^{(n)}(x)\right| \leq M \quad \forall x \in (-a, a), \quad \forall n \geq 1$$

Then

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k \xrightarrow[n\to\infty]{u} 0 \text{ on } (-a, a)$$

Proof. Fix $x \in (-a, a) \setminus \{0\}$. By Taylor, there exists y between x and 0 s.t.

$$R_n(x) = \frac{f^{(n)}(y)}{n!} x^n$$

$$\implies |R_n(x)| \leq M \frac{|x|^n}{n!} \leq M \frac{a^n}{n!}$$

$$\implies \sup_{x \in (-a,a)} |R_n(x)| \leq M \cdot \frac{a^n}{n!} \xrightarrow[n\to\infty]{} 0$$

□

Example 45.2

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos x$

$$f^{(n)}(x) = \begin{cases} -\sin x, & n = 1 + 4k \\ -\cos x, & n = 2 + 4k \\ \sin x, & n = 3 + 4k \\ \cos x, & n = 4k \end{cases} \qquad \text{for } k \geq 0$$

So $\left|f^{(n)}(x)\right| \leq 1 \ \forall x \in \mathbb{R} \ \forall n \geq 0$. We get

$$f(x) = u - \lim_{N\to\infty} \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n \quad \text{on } (-a, a) \text{ for any } a > 0$$

Let $n = 2l$

$$\implies f^{(n)}(0) = \begin{cases} -1, & \text{if } l \text{ odd} \\ 1, & \text{if } l \text{ even} \end{cases} = (-1)^l$$

$$\implies f(X) = \sum_{n\geq 0} \frac{f^{(n)}(0)}{n!} x^n = \sum_{l \geq 0} \frac{(-1)^l}{(2l)!} x^{2l}$$

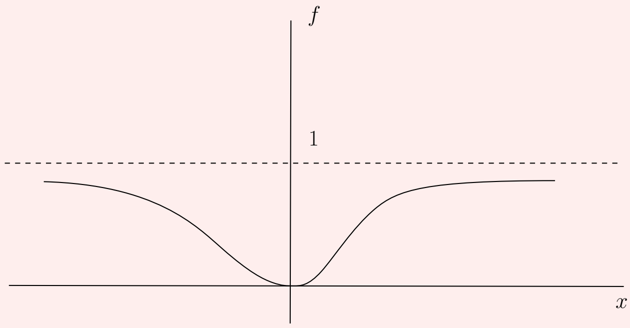
A similar argument gives

$$\sin x = \sum_{n\geq 0} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Example 45.3

$f : \mathbb{R} \to \mathbb{R}$ where

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$



Note f is differentiable of any order on $\mathbb{R}$. Clearly, this holds on $\mathbb{R} \setminus \{0\}$. In fact, for $x \in \mathbb{R} \setminus \{0\}$,

$$f^{(n)}(x) = P_n\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}$$

where

$$P_n\left(\frac{1}{x}\right) = \left(\frac{2}{x^3}\right)^n + \dots$$

To see that f is differentiable at 0 we compute

$$\lim_{x\to 0^+} \frac{f(x)}{x} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{e^{\frac{1}{x^2}}} = \lim_{t\to\infty} \frac{t}{e^{t^2}} = \lim_{t\to\infty} \frac{1}{2te^{t^2}} = 0$$

Similarly,

$$\lim_{x\to 0^-} \frac{f(x)}{x} = \lim_{t\to -\infty} \frac{t}{e^{t^2}} = 0$$

Proceeding inductively, we can prove that f is differentiable of any order at 0 and

$$f^{(n)}(0) = 0$$

We consider

$$\lim_{x\to 0^+} \frac{f^{(n)}(x)}{x} = \lim_{x\to 0^+} \frac{P_n\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}}{x} \lim_{t\to\infty} \frac{tP_n(t)}{e^{t^2}} = 0$$

and

$$\lim_{x\to 0^-} \frac{f^{(n)}(x)}{x} = 0$$

Example 45.4 (Cont'd from above)
Thus,

$$\sum_{n\geq 0} \frac{f^{(n)}(0)}{n!} x^n \equiv 0$$

At leading order as $x \to 0$,

$$f^{(n)}(x) \sim 2^n \cdot \left(\frac{1}{x^2}\right)^{\frac{3n}{2}} e^{-\frac{1}{x^2}} \sim 2^n e^{-\frac{1}{x^2}+\frac{3n}{2}\ln\frac{1}{x^2}}$$

The function $g : (0, \infty) \to \mathbb{R}$, $g(t) = -t + \frac{3n}{2} \ln t$ achieves its maximum at

$$g'(t) = 0 \iff -1 + \frac{3n}{2t} = 0 \iff t = \frac{3n}{2}$$

So $f^{(n)}\left(\sqrt{\frac{2}{3n}}\right) \sim 2^n e^{-\frac{3n}{2}+\frac{3n}{2}\ln\frac{3n}{2}} \sim 2^n e^{\frac{3n}{2}\ln\left(\frac{3n}{2e}\right)} \sim 2^n \left(\frac{3n}{2e}\right)^{\frac{3n}{2}} \underset{n\to\infty}{\longrightarrow} \infty$.

Theorem 45.5
Assume that $f_n : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b). Assume also that

1. $\{f_n'\}_{n\geq 1}$ converges uniformly on (a, b)
2. $\{f_n\}_{n\geq 1}$ converges at some x_0 in $[a, b]$

Then $\{f_n\}_{n\geq 1}$ converges uniformly on $[a, b]$ to some function f. Moreover, f is differentiable on (a, b) and

$$f'(x) = \lim_{n\to\infty} f_n'(x) \qquad \forall x \in (a, b)$$

Remark 45.6. We can restate the conclusion as follows:

$$\lim_{y\to x}\lim_{n\to\infty} \frac{f_n(y) - f_n(x)}{y - x} = \lim_{y\to x} \frac{f(y) - f(x)}{y - x} = f'(x) = \lim_{n\to\infty}\lim_{y\to x} \frac{f_n(y) - f_n(x)}{y - x}$$

Proof. Let's prove that $\{f_n\}_{n\geq 1}$ converges uniformly on $[a, b]$. Fix $\varepsilon > 0$.
$\{f_n'\}_{n\geq 1}$ converges uniformly on (a, b) which implies $\{f_n'\}_{n\geq 1}$ is uniformly Cauchy on (a, b) which also implies $\exists n_1(\varepsilon) \in \mathbb{N}$ s.t.

$$\left|f_n'(x) - f_m'(x)\right| < \varepsilon \quad \forall n, m \geq n_1(\varepsilon) \quad \forall x \in (a, b)$$

Also, we know that $\{f_n(x_0)\}_{n\geq 1}$ converges which means $\{f_n(x_0)\}$ is Cauchy which implies $\exists n_2(\varepsilon) \in \mathbb{N}$ s.t.

$$|f_n(x_0) - f_m(x_0)| < \varepsilon \quad \forall n, m \geq n_2(\varepsilon)$$

For $x \in [a, b] \setminus \{x_0\}$,

$$|f_n(x) - f_m(x)| \leq |f_n(x_0) - f_m(x_0)| + |[f_n(x) - f_m(x)] - [f_n(x_0) - f_m(x_0)]|$$

By the Mean Value theorem, there exists y between x and x_0 s.t.

$$|\,[f_n(x) - f_m(x)] - [f_n(x_0) - f_m(x_0)]\,| = \left|f_n'(y) - f_m'(y)\right| |x - x_0| < \varepsilon(b - a)$$

So for $n, m \geq n(\varepsilon) = \max\{n_1(\varepsilon), n_2(\varepsilon)\}$ we get

$$|f_n(x) - f_m(x)| \leq |f_n(x_0) - f_m(x_0)| + \varepsilon(b-a) \leq \varepsilon(1+b-a)$$
$$\implies sup_{x\in[a,b]} |f_n(x) - f_m(x)| \leq \varepsilon(1+b-a) \qquad \forall n, m \geq n(\varepsilon)$$

So $\{f_n\}_{n\geq 1}$ are uniformly Cauchy on $[a, b]$ and so converge to a function $f = \lim_{n\to\infty} f_n$. It remains to show that f is differentiable on (a, b) and

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

which we will prove in the next lecture. □

§46 Lec 18: May 7, 2021

§46.1 Taylor's Theorem (Cont'd)

Proof. (Cont'd from lecture 17) Fix $x \in (a,b)$. We want to show that f is differentiable at x and

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

We define

$$g : [a,b] \setminus \{x\} \to \mathbb{R}, \quad g(y) = \frac{f(y)-f(x)}{y-x}$$
$$g_n : [a,b] \setminus \{x\} \to \mathbb{R}, \quad g_n(y) = \frac{f_n(y)-f_n(x)}{y-x}$$

Since $f_n \xrightarrow[n\to\infty]{u} f$ we have

$$\lim_{n\to\infty} g_n(y) = g(y)$$

Since f_n is differentiable at x,

$$\lim_{y\to x} g_n(y) = f_n'(x)$$

Let $L(x) = \lim_{n\to\infty} f_n'(x)$. We want to show that

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \text{ s.t. } |g(y) - L(x)| < \varepsilon \text{ whenever } 0 < |y-x| < \delta \ y \in [a,b]$$

Fix $\varepsilon > 0$. By the triangle inequality,

$$|g(y) - L(x)| \le |g(y) - g_n(y)| + \left|g_n(y) - f_n'(x)\right| + \left|f_n'(x) - L(x)\right|$$

We have $\{f_n'\}_{n\ge1}$ converges uniformly on $(a,b) \implies \{f_n'\}_{n\ge1}$ is uniformly Cauchy on $(a,b) \implies \exists n_1(\varepsilon) \in \mathbb{N}$ s.t.

$$\left|f_n'(z) - f_m'(z)\right| < \varepsilon \qquad \forall n,m \ge n_1(\varepsilon) \quad \forall z \in (a,b) \tag{1}$$

Letting $m \to \infty$ we get

$$\left|f_n'(z) - L(z)\right| \le \varepsilon \qquad \forall n \ge n_1(\varepsilon) \quad \forall z \in (a,b)$$

For $y \in [a,b] \setminus \{x\}$, by the Mean Value theorem, we can find a point z between x and y so that

$$\begin{aligned}|g_n(y) - g_m(y)| &= \left|\frac{f_n(y)-f_n(x)}{y-x} - \frac{f_m(y)-f_m(x)}{y-x}\right| \\ &= \frac{|[f_n(y)-f_m(y)] - [f_n(x)-f_m(x)]|}{|y-x|} \\ &= \left|f_n'(z) - f_m'(z)\right| \overset{(1)}{<} \varepsilon \quad \forall n,m \ge n_1(\varepsilon)\end{aligned}$$

Letting $m \to \infty$ we find

$$|g_n(y) - g(y)| \le \varepsilon \qquad \forall n \ge n_1(\varepsilon) \quad \forall y \in [a,b] \setminus \{x\} \tag{3}$$

Fix $n \ge n_1(\varepsilon)$. As f_n is differentiable at x we find $\delta = \delta(\varepsilon, n) > 0$ s.t.

$$\left|g_n(y) - f_n'(x)\right| < \varepsilon \qquad \forall 0 < |y-x| < \delta \quad y \in [a,b] \tag{4}$$

Thus for this $n \ge n_1(\varepsilon)$ and $0 < |y-x| < \delta$ we have

$$|g(y) - L(x)| \le |g(y) - g_n(y)| + \left|g_n(y) - f_n'(x)\right| + \left|f_n'(x) - L(x)\right|$$
$$\text{by (2), (3), (4)} \le 3\varepsilon$$

□

Example 46.1

$f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{x}{1+nx^2}$, f_n is differentiable and

$$f_n'(x) = \frac{1}{1+nx^2} - \frac{x \cdot 2nx}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}$$

Now

$$f_n \xrightarrow[n\to\infty]{u} f \equiv 0$$

$$f_n'(x) \xrightarrow[n\to\infty]{} \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

Note that f_n' do not converge uniformly since their limit is not continuous.

$$\lim_{n\to\infty} \lim_{y\to 0} \frac{f_n(y) - f_n(0)}{y - 0} = \lim_{n\to\infty} f_n'(0) = 1$$

but

$$\lim_{y\to 0} \lim_{n\to\infty} \frac{f_n(y) - f_n(0)}{y - 0} = \lim_{y\to 0} 0 = 0$$

§46.2 Darboux Integral

Definition 46.2 (Partition) — Let $f : [a, b] \to \mathbb{R}$ be a <u>bounded</u> function. If $S \subseteq [a, b]$ we denote

$$M(f; S) = \sup_{x\in S} f(x) \quad \text{and} \quad m(f; S) = \inf_{x\in S} f(x)$$

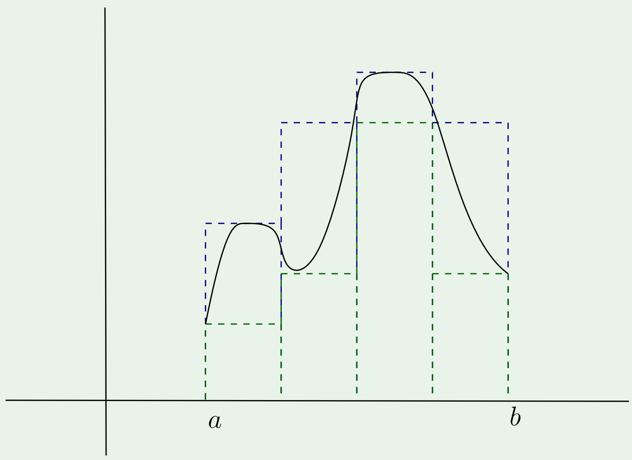
A <u>partition</u> of $[a, b]$ is a finite ordered set $P \subseteq [a, b]$. We write

$$P = \{a = t_0 < t_1 < \ldots < t_n = b\}$$

for some $n \geq 1$.

Definition 46.3 (Darboux Sum) — The upper Darboux sum of f with respect to P is

$$U(f;P) = \sum_{k=1}^{n} M\left(f;[t_{k-1},t_k]\right)(t_k - t_{k-1})$$



The lower Darboux sum of f with respect to P is

$$L(f;P) = \sum_{k=1}^{n} m\left(f;[t_{k-1},t_k]\right)(t_k - t_{k-1})$$

Note that

$$m\left(f;[a,b]\right)(b-a) \leq L(f;P) \leq U(f;P) \leq M\left(f;[a,b]\right)(b-a)$$

So

$$\{L(f;P):\ P \text{ partition of } [a,b]\} \text{ is bounded above}$$
$$\{U(f;P):\ P \text{ partition of } [a,b]\} \text{ is bounded below}$$

Definition 46.4 (Darboux Integral) — The upper Darboux integral of f on $[a,b]$ is

$$U(f) = \inf\{U(f;P):\ P \text{ partition of } [a,b]\}$$

The lower Darboux integral of f on $[a,b]$ is

$$L(f) = \sup\{L(f;P):\ P \text{ partition of } [a,b]\}$$

We say that f is Darboux integrable on $[a,b]$ if $U(f) = L(f)$. In this case we write

$$\int_a^b f(x)\,dx = U(f) = L(f)$$

Example 46.5

Let $f : [0, M] \to \mathbb{R}$, $f(x) = x^3$. Then f is Darboux integrable.
Let $P = \{0 = t_0 < \ldots < t_n = M\}$ be a partition of $[0, M]$ and

$$\begin{aligned} U(f; P) &= \sum_{k=1}^{n} M\left(f; [t_{k-1}, t_k]\right)(t_k - t_{k-1}) \\ &= \sum_{k=1}^{n} t_k^3 (t_k - t_{k-1}) \end{aligned}$$

Similarly,

$$L(f; P) = \sum_{k=1}^{n} m\left(f; [t_{k-1}, t_k]\right)(t_k - t_{k-1}) = \sum_{k=1}^{n} t_{k-1}^3 (t_k - t_{k-1})$$

Take $t_k = \frac{kM}{n}$ $0 \le k \le n$. Then

$$U(f; P) = \sum_{k=1}^{n} \left(\frac{kM}{n}\right)^3 \cdot \frac{M}{n} = \frac{M^4}{n^4} \sum_{k=1}^{n} k^3 = \frac{M^4}{n^4}\left[\frac{n(n+1)^2}{2}\right] \xrightarrow[n\to\infty]{} \frac{M^4}{4}$$

$$L(f; P) = \sum_{k=1}^{n} \left(\frac{(k-1)M}{n}\right)^3 \cdot \frac{M}{n} = \frac{M^4}{n^4} \sum_{k=0}^{n-1} k^3 = \frac{M^4}{n^4}\left[\frac{n(n-1)^2}{2}\right] \xrightarrow[n\to\infty]{} \frac{M^4}{4}$$

So, $U(f) \le \frac{M^4}{4}$ and $L(f) \ge \frac{M^4}{4}$ and we will show that $L(f) \le U(f)$ which imply $U(f) = L(f) = \frac{M^4}{4}$. So f is Darboux integrable and $\int_0^M f(x)\,dx = \frac{M^4}{4}$.

Example 46.6

Given

$$f : [0, 1] \to \mathbb{R}, \quad f(x) = \begin{cases} 1, & x \in [0, 1] \cap \mathbb{Q} \\ 0, & x \in [0, 1] \setminus \mathbb{Q} \end{cases}$$

f is not Darboux integrable. For any partition P, $U(f; P) = 1$ and $L(f; P) = 0$ which implies $U(f) = 1$ and $L(f) = 0$.

§47 Lec 19: May 10, 2021

§47.1 Darboux Integral (Cont'd)

Recall: If $f : [a, b] \to \mathbb{R}$ bounded

$$P = \{a = t_0 < \ldots < t_n = b\} \text{ partition of } [a, b]$$

then

$$U(f; P) = \sum_{k=1}^{n} M\left(f; [t_{k-1}, t_k]\right)(t_k - t_{k-1})$$
$$L(f; P) = \sum_{k=1}^{n} m\left(f; [t_{k-1}, t_k]\right)(t_k - t_{k-1})$$

are the upper and lower Darboux sum associated with P, respectively f is Darboux integrable if $U(f) = L(f)$ where

$$U(f) = \inf_P U(f; P) \quad \text{and} \quad L(f) = \sup_P L(f; P)$$

Proposition 47.1

Let $f : [a, b] \to \mathbb{R}$ be two bounded and let P and Q be partitions of $[a, b]$ s.t. $P \subseteq Q$. Then

$$L(f; p) \leq L(f; Q) \leq U(f; Q) \leq U(f; P)$$

Proof. We will prove the third inequality. The first inequality follows from a similar argument. Arguing by induction, it suffices to prove the claim when the partition Q contains exactly one extra point compared to the partition P. Let

$$P = \{a = t_0 < t_1 < \ldots < t_n = b\}$$
$$Q = \{a = t_0 < \ldots < t_{l-1} < s < t_l < \ldots < t_n = b\}$$

for some $1 \leq l \leq n$.

$$U(f; Q) = \sum_{k=1}^{l-1} M\left(f; [t_{k-1}, t_k]\right)(t_k - t_{k-1}) + M\left(f; [t_{l-1}, s]\right)(s - t_{l-1}) + M\left(f; [s, t_l]\right)(t_l - s) + \sum_{k=l+1}^{n} M\left(f; [t_{k-1}, t_k]\right)(t_k - t_{k-1})$$

Clearly,

$$M\left(f; [t_{l-1}, s]\right) \leq M\left(f; [t_{l-1}, t_l]\right)$$
$$M\left(f; [s, t_l]\right) \leq M\left(f; [t_{l-1}, t_l]\right)$$

So

$$U(f; Q) \leq \sum_{k=1}^{n} M\left(f; [t_{k-1}, t_k]\right)(t_k - t_{k-1}) = U(f; P)$$ □

Corollary 47.2

Let $f : [a, b] \to \mathbb{R}$ be bounded and let P, Q be two partitions of $[a, b]$. Then

$$L(f; P) \leq U(f; Q)$$

Consequently,

$$L(f) \leq U(f)$$

Proof. Consider the partition $P \cup Q$. We have

$$\begin{aligned} L(f; P) \leq L(f; P \cup Q) \leq U(f; P \cup Q) \leq U(f; Q) \\ \implies L(f) = \sup_P L(f; P) \leq U(f; Q) \\ \implies L(f) \leq \inf_Q U(f; Q) = U(f) \end{aligned}$$ □

Theorem 47.3

Let $f : [a, b] \to \mathbb{R}$ be bounded. Then f is Darboux integrable if and only if

$$\forall \varepsilon > 0 \quad \exists P \text{ partitions of } [a, b] \quad \ni \quad U(f; P) - L(f; P) < \varepsilon$$

Proof. "$\impliedby$" Fix $\varepsilon > 0$. Then there exists P partition of $[a, b]$ s.t. $U(f; P) - L(f; P) < \varepsilon$

$$\begin{aligned} &\implies U(f) \leq U(f; P) < L(f; P) + \varepsilon \leq L(f) + \varepsilon \\ &\implies \left.\begin{matrix} U(f) < L(f) + \varepsilon \\ \varepsilon > 0 \text{ was arbitrary} \end{matrix}\right\} \implies \left.\begin{matrix} U(f) \leq L(f) \\ L(f) \leq U(f) \end{matrix}\right\} \implies U(f) = L(f) \\ &\implies f \text{ is Darboux integrable} \end{aligned}$$

"$\implies$" Fix $\varepsilon > 0$, f is Darboux integrable implies

$$U(f) = L(f)$$

Then

$$\begin{aligned} U(f) = \inf_P U(f; P) &\implies \exists P_1 \text{ partition of } [a, b] \text{ s.t. } U(f; P_1) < U(f) + \frac{\varepsilon}{2} \\ L(f) = \sup_P L(f; P) &\implies \exists P_2 \text{ partition of } [a, b] \text{ s.t. } L(f; P_2) > L(f) - \frac{\varepsilon}{2} \end{aligned}$$

Consider the partition $P_1 \cup P_2$. Then

$$L(f; P_2) \leq L(f; P_1 \cup P_2) \leq U(f; P_1 \cup P_2) \leq U(f; P_1)$$

So

$$U(f; P_1 \cup P_2) - L(f; P_1 \cup P_2) < U(f) + \frac{\varepsilon}{2} - \left(L(f) - \frac{\varepsilon}{2}\right) = \varepsilon$$ □

Definition 47.4 (Mesh) — Let $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ be a partition of $[a, b]$. The mesh of P is given by

$$\text{mesh}(P) = \max_{1 \leq k \leq n} (t_k - t_{k-1})$$

Theorem 47.5

Let $f : [a, b] \to \mathbb{R}$ be bounded. Then f is Darboux integrable if and only if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \text{ s.t. if } P \text{ is a partition of } [a,b] \text{ with } \mathrm{mesh}(P) < \delta$$

then

$$U(f;P) - L(f;P) < \varepsilon$$

Proof. " $\Longleftarrow$ " By the previous theorem, it suffices to show that $\forall \delta > 0\ \exists P$ partition of $[a, b]$ with $\mathrm{mesh}(P) < \delta$. For $\delta > 0$, let $P = \{a = t_0 < \ldots < t_n = b\}$ where

$$t_k = a + k \cdot \frac{\delta}{2} \quad \text{for} \quad 0 \le k \le \lfloor \frac{2(b-a)}{\delta} \rfloor = n - 1$$

and $t_n = b$. Clearly,

$$\mathrm{mesh}(P) = \frac{\delta}{2} < \delta$$

" $\Longrightarrow$ " Fix $\varepsilon > 0$. By the previous theorem, as f is Darboux integrable, there exists a partition $P_0 = \{a = s_0 < \ldots < s_m = b\}$ of $[a, b]$ s.t.

$$U(f;P_0) - L(f;P_0) < \frac{\varepsilon}{2}$$

Let $0 < \delta < \mathrm{mesh}(P_0)$ to be chosen later and let $P = \{a = t_0 < \ldots < t_n = b\}$ be a partition of $[a, b]$ with $\mathrm{mesh}(P) < \delta$

$$\begin{aligned} U(f;P) - L(f;P) &\le U(f;P) - U(f;P_0) + U(f;P_0) - L(f;P_0) + L(f;P_0) - L(f;P) \\ &\le \frac{\varepsilon}{2} + U(f;P) - U(f;P_0) + L(f;P_0) - L(f;P) \end{aligned}$$

Consider the partition $P \cup P_0$. Then

$$U(f;P) - U(f;P_0) \le U(f;P) - U(f;P \cup P_0)$$

As $\mathrm{mesh}(P) < \delta < \mathrm{mesh}(P_0)$, there must be at most one point from P_0 in each $[t_{k-1}, t_k]$. Only subintervals $[t_{k-1}, t_k]$ with an $s_j \in P_0 \cap [t_{k-1}, t_k]$ contribute to $U(f;P) - U(f;P_0 \cup P)$. There are only m many such intervals. The contribution of one such interval to $U(f;P) - U(f;P_0 \cup P)$ is

$$M(f;[t_{k-1},t_k])(t_k - t_{k-1}) - M(f;[t_{k-1},s_j])(s_j - t_{k-1}) - M(f;[s_j,t_k])(t_k - s_j)$$

As f is bounded, $\exists M > 0$ s.t. $|f(x)| \le M\ \forall x \in [a, b]$. Note

$$\begin{aligned} &M(f;[t_{k-1},t_k]) \le M \\ &M(f;[t_{k-1},s_j]) \ge -M; \qquad M(f;[s_j,t_k]) \ge -M \end{aligned}$$

So

$$M(f;[t_{k-1},t_k])(t_k - t_{k-1}) - M(f;[t_{k-1},s_j])(s_j - t_{k-1}) - M(f;[s_j,t_k])(t_k - s_j)$$

which is smaller than or equal to

$$M(t_k - t_{k-1}) - (-M)[(s_j - t_{k-1}) + (t_k - s_j)] = 2M(t_k - t_{k-1}) < 2M \cdot \mathrm{mesh}(P)$$

Thus

$$U(f;P) - U(f;P_0) < m \cdot 2M \cdot \mathrm{mesh}(P)$$

Similarly,

$$L(f;P_0) - L(f;P) < m \cdot 2M \cdot \text{mesh}(P)$$

which requires

$$4Mm \cdot \text{mesh}(P) < \frac{\varepsilon}{2} \iff \text{mesh}(P) < \frac{\varepsilon}{8Mm}$$

Thus, $\delta < \min\left\{\frac{\varepsilon}{8Mm}, \text{mesh}(P_0)\right\}$. □

§48 Lec 20: May 12, 2021

§48.1 Riemann Integral

Definition 48.1 (Riemann Sum) — Let $f : [a, b] \to \mathbb{R}$ be a function and let $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ be a partition of $[a, b]$. A Riemann sum of f associated to P is a sum of the form

$$S = \sum_{k=1}^{n} f(x_k)(t_k - t_{k-1}) \qquad \text{where } x_k \in [t_{k-1}, t_k] \quad \forall 1 \le k \le n$$

<u>Note</u>: If S is a Riemann sum associated with a partition P of $[a, b]$ then

$$L(f; P) \le S \le U(f; P)$$

Definition 48.2 (Riemann Integrable) — We say that f is Riemann integrable if $\exists r \in \mathbb{R}$ s.t. $\forall \varepsilon > 0\ \exists \delta > 0$ s.t.

$$|S - r| < \varepsilon$$

for any Riemann sum S of f associated with a partition P with mesh$(P) < \delta$. Then r is called the Riemann integral of f and we write

$$r = \mathcal{R}\int_a^b f(x)\,dx$$

Lemma 48.3

If $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then f is bounded.

Proof. Let $r = \mathcal{R}\int_a^b f(x)\,dx$. Taking $\varepsilon = 1$ we find $\delta > 0$ s.t. $|S - r| < 1$ for any Riemann sum S of f associated to a partition P with mesh$(P) < \delta$.

Let $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ with mesh$(P) < \delta$. Fix $1 \le k \le n$. Fix $x_l \in [t_{l-1}, t_l]$ for $1 \le l \le n$, $l \ne k$. For $x \in [t_{k-1}, t_k]$ we have

$$\left| \sum_{l \ne k} f(x_l)(t_l - t_{l-1}) + f(x)(t_k - t_{k-1}) - r \right| < 1$$

$$\left. \begin{aligned} &\tfrac{r-1-\sum_{l\ne k} f(x_l)(t_l-t_{l-1})}{t_k-t_{k-1}} < f(x) < \tfrac{1+r-\sum_{l\ne k} f(x_l)(t_l-t_{l-1})}{t_k-t_{k-1}} \\ &x \in [t_{k-1}, t_k] \text{ is arbitrary} \end{aligned} \right\} \implies$$

$$\implies \left. \begin{aligned} &f \text{ is bounded on } [t_{k-1}, t_k] \\ &1 \le k \le n \text{ is arbitrary} \end{aligned} \right\} \implies f \text{ is bounded on } [a, b]$$

□

Theorem 48.4

Let $f:[a,b]\to\mathbb{R}$. The following are equivalent

1. f is Riemann integrable.

2. f is bounded and Darboux integrable.

If either conditions holds, then the integrals agree.

Proof. 2) $\implies$ 1) Fix $\varepsilon > 0$.

f is Darboux integrable $\implies \exists\delta > 0$ s.t. $U(f;P) - L(f;P) < \varepsilon$ for any partition P with $\operatorname{mesh}(P) < \delta$. Let P be a partition of $[a,b]$ with $\operatorname{mesh}(P < \delta)$. If S is a Riemann sum of f associated to P, then

$$\left.\begin{aligned} S \le U(f;P) < L(f;P) + \varepsilon \le L(f) + \varepsilon = \textstyle\int_a^b f(x)\,dx + \varepsilon \\ S \ge L(f;P) > U(f;P) - \varepsilon \ge U(f) - \varepsilon = \textstyle\int_a^b f(x)\,dx - \varepsilon \end{aligned}\right\} \implies \left| s - \int_a^b f(x)dx \right| < \varepsilon$$

By definition, f is Riemann integrable and $\mathcal{R}\int_a^b f(x)dx = \int_a^b f(x)dx$.

1) $\implies$ 2) By the previous lemma, f is bounded. Fix $\varepsilon > 0$. Let $r = \mathcal{R}\int_a^b f(x)dx$. Then $\exists\delta > 0$ s.t.

$$|S - r| < \frac{\varepsilon}{2}$$

for any Riemann sum of f associated with a partition of P with $\operatorname{mesh}(P) < \delta$. Fix $P = \{a = t_0 < t_1 < \dots . < t_n = b\}$ be a partition with $(\operatorname{mesh}(P) < \delta$. There exist $x_k, y_k \in [t_{k-1}, t_k]$ s.t.

$$\begin{aligned} f(x_k) &> M\left(f;[t_{k-1},t_k]\right) - \frac{\varepsilon}{2(b-a)} \\ f(y_k) &< m\left(f;[t_{k-1},t_k]\right) + \frac{\varepsilon}{2(b-a)} \end{aligned}$$

Then

$$\begin{aligned} S_1 = \sum_{k=1}^n f(x_k)\left(t_k - t_{k-1}\right) &> U(f;P) - \frac{\varepsilon}{2(b-a)} \sum_{k=1}^n (t_k - t_{k-1}) \\ &= U(f;P) - \frac{\varepsilon}{2} \\ S_2 = \sum_{k=1}^n f(y_k)\left(t_k - t_{k-1}\right) &< L(f;P) + \frac{\varepsilon}{2(b-a)} \sum_{k=1}^n (t_k - t_{k-1}) \\ &= L(f;P) + \frac{\varepsilon}{2} \end{aligned}$$

However, $|S_1 - r| < \frac{\varepsilon}{2}$ and $|S_2 - r| < \frac{\varepsilon}{2}$. So

$$\left.\begin{aligned} U(f;P) - \tfrac{\varepsilon}{2} < S_1 < r + \tfrac{\varepsilon}{2} &\implies U(f) \le U(f;P) < r + \varepsilon \\ r - \tfrac{\varepsilon}{2} < S_2 < L(f;P) + \tfrac{\varepsilon}{2} &\implies r - \varepsilon < L(f;P) \le L(f) \end{aligned}\right\} \implies$$

$$\implies \left.\begin{aligned} &r - \varepsilon < L(f) \le U(f) < r + \varepsilon \\ &\varepsilon > 0 \text{ arbitrary} \end{aligned}\right\} \implies f \text{ is Darboux integrable and } \int_a^b f(x)\,dx = r$$

□

Theorem 48.5

Let $f : [a, b] \to \mathbb{R}$ be monotonic. Then f is integrable.

Proof. Assume f is increasing. Then

$$f(a) \le f(x) \le f(b) \qquad \forall x \in [a, b]$$

So f is bounded.
Let $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ with $\text{mesh}(P) < \delta$ for δ to be chosen later. Then

$$\begin{aligned} U(f;P) - L(f;P) &= \sum_{k=1}^{n} \left[M\left(f;[t_{k-1},t_k]\right) - m\left(f;[t_{k-1},t_k]\right)\right](t_k - t_{k-1}) \\ &= \sum_{k=1}^{n} \left[f(t_k) - f(t_{k-1})\right](t_k - t_{k-1}) \\ &\le \text{mesh}(P) \sum_{k=1}^{n} \left[f(t_k) - f(t_{k-1})\right] \\ &< \delta \cdot [f(b) - f(a)] \end{aligned}$$

Taking $\delta < \frac{\varepsilon}{f(b)-f(a)+1}$ we see that f is Darboux integrable. ☐

Theorem 48.6

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is integrable.

Proof. We have

$$\left.\begin{array}{l} f : [a,b] \to \mathbb{R} \text{ continuous} \\ [a,b] \text{ compact} \end{array}\right\} \implies f \text{ is bounded}$$

Fix $\varepsilon > 0$. As f is continuous on $[a, b]$ compact, f is uniformly continuous. So $\exists \delta > 0$ s.t.

$$|f(x) - f(y)| < \frac{\varepsilon}{b-a} \qquad \forall x, y \in [a, b] \text{ with } |x - y| < \delta$$

Let $P = \{a = t_0 < \ldots < t_n = b\}$ with $\text{mesh}(P) < \delta$.

$$U(f;P) - L(f;P) = \sum_{k=1}^{n} \left[M\left(f;[t_{k-1},t_k]\right) - m\left(f;[t_{k-1},t_k]\right)\right](t_k - t_{k-1})$$

f continuous on $[t_{k-1}, t_k]$ compact implies $\exists x_k, y_k \in [t_{k-1}, t_k]$ s.t.

$$\begin{aligned} f(x_k) &= M\left(f;[t_{k-1},t_k]\right) \\ f(y_k) &= m\left(f;[t_{k-1},t_k]\right) \end{aligned}$$

So

$$\begin{aligned} U(f;P) - L(f;P) &= \sum_{k=1}^{n} \left[f(x_k) - f(y_k)\right](t_k - t_{k-1}) \\ &< \sum_{k=1}^{n} \frac{\varepsilon}{b-a}(t_k - t_{k-1}) = \varepsilon \end{aligned}$$

Then f is Darboux integrable. ☐

Theorem 48.7

Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable.

1. For any $\alpha \in \mathbb{R}$, αf is Riemann integrable and

$$\int_a^b (\alpha f)(x)\, dx = \alpha \int_a^b f(x)\, dx$$

2. $f + g$ is Riemann integrable and

$$\int_a^b (f+g)(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$$

Proof. 1. If $\alpha = 0$ this is clear. Assume $\alpha > 0$. For any $S \subseteq [a, b]$

$$M(\alpha f; S) = \alpha M(f; S)$$
$$m(\alpha f; S) = \alpha m(f; S)$$

For by partition P of $[a, b]$,

$$\begin{aligned} U(\alpha f; P) = \alpha U(f; P) \implies U(\alpha f) &= \sup_P U(\alpha f; P) \\ &= \sup_P [\alpha \cdot U(f; P)] \\ &= \alpha \sup_P U(f; P) = \alpha U(f) \end{aligned}$$

Similarly,

$$L(\alpha f) = \alpha L(f)$$
$$L(f) = U(f)$$

$\implies$ αf is Darboux integrable and $\int_a^b (\alpha f)(x) dx = \alpha \int_a^b f(x) dx$. □

§49 Lec 21: May 14, 2021

§49.1 Riemann Integral (Cont'd)

Recall from last lecture, we have the following theorem,

Theorem 49.1

Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable.

1. For any $\alpha \in \mathbb{R}$, αf is Riemann integrable and
$$\int_a^b (\alpha f)(x)\, dx = \alpha \int_a^b f(x)\, dx$$
2. $f + g$ is Riemann integrable and
$$\int_a^b (f+g)(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$$

Proof. 1. Last time we proved the result for $\alpha \geq 0$. Assume $\alpha < 0$. For $S \subseteq [a, b]$, we have
$$M(\alpha f; S) = \alpha m(f; S) \quad \text{and} \quad m(\alpha f; S) = \alpha M(f; S)$$
If P is a partition of $[a, b]$,
$$U(\alpha f; P) = \alpha L(f; P) \quad \text{and} \quad L(\alpha f; P) = \alpha U(f; P)$$
Thus,
$$\left.\begin{array}{l} U(\alpha f) = \inf_P U(\alpha f; P) = \inf_P \alpha L(f; P) = \alpha \sup_P L(f; P) = \alpha L(f) \\ L(\alpha f) = \ldots = \alpha U(f) \\ f \text{ is Riemann integrable} \implies f \text{ bounded and } L(f) = U(f) = \int_a^b f(x)\, dx \end{array}\right\} \implies$$
$$\implies \alpha f \text{ is bounded and } L(\alpha f) = U(\alpha f) = \alpha \int_a^b f(x)\, dx$$
$$\implies \alpha f \text{ is Riemann integrable and } \int_a^b (\alpha f)(x)\, dx = \alpha \int_a^b f(x)\, dx$$

2. As f, g are Riemann integrable, $f + g$ is bounded and f, g are Darboux integrable. Fix $\varepsilon > 0$. Then, f is Darboux integrable implies $\exists P_1$ partition of $[a, b]$ s.t.
$$U(f; P_1) - L(f; P_1) < \frac{\varepsilon}{2}$$
g is Darboux integrable implies $\exists P_2$ partition of $[a, b]$ s.t.
$$U(g; P_2) - L(g; P_2) < \frac{\varepsilon}{2}$$
Let $P = P_1 \cup P_2$. Then, we have
$$U(f; P) - L(f; P) < \frac{\varepsilon}{2} \quad \text{and} \quad U(g; P) - L(g; P) < \frac{\varepsilon}{2}$$

For $S \subseteq [a, b]$,

$$M(f+g; S) \leq M(f; S) + M(g; S)$$
$$m(f+g; S) \geq m(f; S) + m(g; S)$$

So

$$\left.\begin{aligned} U(f+g; P) \leq U(f; P) + U(g; P) \\ L(f+g; P) \geq L(f; P) + L(g; P) \end{aligned}\right\} \implies$$

$$\implies U(f+g; P) - L(f+g; P) \leq U(f; P) - L(f; P) + U(g; P) - L(g; P) < \varepsilon$$

$$\implies \left.\begin{aligned} &f+g \text{ is Darboux integrable} \\ &f+g \text{ is bounded} \end{aligned}\right\} \implies f+g \text{ is Riemann integrable}$$

Moreover,

$$\begin{aligned} U(f+g) \leq U(f+g; P) &\leq U(f; P) + U(g; P) \\ &< L(f; P) + L(g; P) + \varepsilon \\ &\leq L(f) + L(g) + \varepsilon = \int_a^b f(x)dx + \int_a^b g(x)dx + \varepsilon \end{aligned}$$

Similarly,

$$\begin{aligned} L(f+g) \geq L(f+g; P) &\geq L(f; P) + L(g; P) \\ &> U(f; P) + U(g; P) - \varepsilon \\ &\geq U(f) + U(g) - \varepsilon = \int_a^b f(x)dx + \int_a^b g(x)dx - \varepsilon \end{aligned}$$

Let $\varepsilon \to 0$, we get

$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \qquad \square$$

Theorem 49.2

Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable. Assume $f(x) \leq g(x)$ $\forall x \in [a, b]$. Then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

Proof. By the previous theorem, $h : [a, b] \to \mathbb{R}$, $h = g - f$ is Riemann integrable. Moreover, since $h \geq 0$, we have

$$\int_a^b h(x)\,dx = L(h) = \sup_P L(h; P) \geq 0$$

which implies

$$0 \leq \int_a^b h(x)\,dx = \int_a^b (g-f)(x)\,dx = \int_a^b g(x)\,dx - \int_a^b f(x)\,dx \qquad \square$$

Theorem 49.3

Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Then $|f|$ is Riemann integrable and

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

Proof. Let f is Riemann integrable. Then, f is bounded and Darboux integrable. So $|f|$ is bounded. For $S \subseteq [a, b]$ we have

$$\begin{aligned}
M(|f|; S) - m(|f|; S) &= \sup_{x\in S} |f(x)| - \inf_{y\in S} |f(y)| \\
&= \sup_{x\in S} |f(x)| + \sup_{y\in S} -|f(y)| \\
&= \sup_{x,y\in S} \{|f(x)| - |f(y)|\} \\
&\leq \sup_{x,y\in S} |f(x) - f(y)| \\
&= \sup_{x,y\in S} \{f(x) - f(y)\} \\
&= \sup_{x\in S} f(x) - \inf_{y\in S} f(y) \\
&= M(f; S) - m(f; S)
\end{aligned}$$

So for any partition P of $[a, b]$ we have

$$U(|f|; P) - L(|f|; P) \leq U(f; P) - L(f; P)$$

f Darboux integrable $\implies \forall \varepsilon > 0\ \exists P$ partition of $[a, b]$ s.t.

$$\begin{aligned}
&U(f; P) - L(f; P) < \varepsilon \\
\implies &\forall \varepsilon > 0\, \exists P \text{ partition of } [a, b] \text{ s.t. } U(|f|; P) - L(|f|; P) < \varepsilon \\
\implies &\left.\begin{matrix} |f| \text{ is Darboux integrable} \\ |f| \text{ is bounded} \end{matrix}\right\} \implies |f| \text{ is Riemann integrable}
\end{aligned}$$

We have

$$-|f(x)| \leq f(x) \leq |f(x)| \qquad \forall x \in [a, b]$$

By the previous theorem,

$$-\int_a^b |f(x)|\,dx = \int_a^b -|f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx$$

which implies

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

□

Theorem 49.4

Let $f : [a, b] \to \mathbb{R}$ be a function and let $a < c < b$. Assume f is Riemann integrable on $[a, c]$ and on $[c, b]$. Then f is Riemann integrable on $[a, b]$ and

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Proof. f is Riemann integrable on $[a, c]$ and on $[c, b]$

$$\begin{aligned}&\implies f \text{ bounded on } [a, c] \text{ and on } [c, b]\\ &\implies f \text{ bounded on } [a, b]\end{aligned}$$

Fix $\varepsilon > 0$. As f is Riemann integrable on $[a, c]$, f is Darboux integrable on $[a, c]$

$$\implies \exists P_1 \text{ partition of } [a, c] \text{ s.t. } U_a^c(f; P_1) - L_a^c(f; P_1) < \frac{\varepsilon}{2}$$

Similarly, as f is Riemann integrable on $[c, b] \implies f$ Darboux integrable on $[c, b]$

$$\implies \exists P_2 \text{ partition of } [c, b] \text{ s.t. } U_c^b(f; P_2) - L_c^b(f; P_2) < \frac{\varepsilon}{2}$$

Let $P = P_1 \cup P_2$ partition on $[a, b]$ and

$$\begin{aligned}U(f; P) &= U_a^c(f; P_1) + U_c^b(f; P_2)\\ L(f; P) &= L_a^c(f; P_1) + L_c^b(f; P_2)\end{aligned}$$

So

$$U(f; P) - L(f; P) < \frac{\varepsilon}{2}$$

Therefore, as f is Darboux integrable and bounded on $[a, b]$, f is Riemann integrable on $[a, b]$. Moreover,

$$\begin{aligned}U(f) \le U(f; P) = U_a^c(f; P_1) + U_c^b(f; P_2) &< L_a^c(f; P_1) + L_c^b(f; P_2) + \varepsilon\\ &\le \int_a^c f(x)\,dx + \int_c^b f(x)\,dx + \varepsilon\end{aligned}$$

Similarly,

$$L(f) \ge \int_a^c f(x)\,dx + \int_c^b f(x)\,dx - \varepsilon$$

Since $\varepsilon > 0$ is arbitrary,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

□

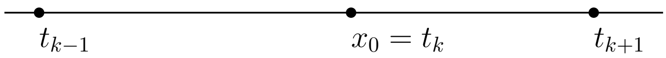
Lemma 49.5

Let $f, g : [a, b] \to \mathbb{R}$ be functions s.t. f is Riemann integrable and $g(x) = f(x)$ except at finitely many points in $[a, b]$. Then g is Riemann integrable and

$$\int_a^b g(x)\,dx = \int_a^b f(x)\,dx$$

Proof. Arguing by induction, we may assume that there exists exactly one point $x_0 \in [a, b]$ s.t. $f(x_0) \neq g(x_0)$. Let $B > 0$ s.t. $|f(x)| \le B$ and $|g(x)| \le B$ $\forall x \in [a, b]$. Let $P = \{a = t_0 < \ldots < t_n = b\}$. We consider

$$\begin{aligned}&U(f; P) - U(g; P)\\ &L(f; P) - L(g; P)\end{aligned}$$



The largest contribution occurs when $x_0 = t_k$ for some $1 \le k \le n-1$.

$$\begin{aligned}
|M(f;[t_{k-1},t_k]) - M(g;[t_{k-1},t_k])| &\le [B-(-B)](t_k - t_{k-1}) \\
&\le 2B \text{ mesh}(P) \\
\implies |U(f;P) - U(g;P)| &\le 4B \text{ mesh}(P)
\end{aligned}$$

Similarly,

$$\begin{aligned}
|m(f;[t_{k-1},t_k]) - m(g;[t_{k-1},t_k])| &\le 2B \text{ mesh}(P) \\
\implies |L(f;P) - L(g;P)| &\le 4B \text{ mesh}(P)
\end{aligned}$$

Thus,

$$\begin{aligned}
U(g;P) - L(g;P) &\le U(f;P) - L(f;P) + |U(f;P) - U(g;P)| \\
&\quad + |L(f;P) - L(g;P)| \\
&\le U(f;P) - L(f;P) + 8B \text{ mesh}(P)
\end{aligned}$$

f Darboux integrable $\implies \forall \varepsilon > 0\ \exists \delta > 0$ s.t.

$$U(f;P) - L(f;P) < \frac{\varepsilon}{2} \qquad \forall P \text{ partition with mesh}(P) < \delta$$

Choose δ even smaller if necessary so that

$$8B\delta < \frac{\varepsilon}{2} \iff \delta < \frac{\varepsilon}{16B}$$

Then $U(g;P) - L(g;P) < \varepsilon$ for all P partition with mesh$(P) < \delta$.

$$\left.\begin{array}{l} g \text{ is Darboux integrable} \\ g \text{ bounded} \end{array}\right\} \implies g \text{ is Riemann integrable}$$

Exercise 49.1. Show $\int_a^b g(x)\,dx = \int_a^b f(x)\,dx$. □

§50 Lec 22: May 17, 2021

§50.1 Riemann Integral (Cont'd)

Definition 50.1 (Piecewise Monotone) — We say that a function $f : [a, b] \to \mathbb{R}$ is piecewise monotone if there exists a partition $P = \{a = t_0 < \ldots < t_n = b\}$ s.t. f is monotone on (t_{k-1}, t_k) for each $1 \le k \le n$.

Definition 50.2 (Piecewise Continuous) — We say that $f : [a, b] \to \mathbb{R}$ is piecewise continuous if there exists a partition $P = \{a = t_0 < \ldots < t_n = b\}$ s.t. f is uniformly continuous on (t_{k-1}, t_k) for each $1 \le k \le n$.

Theorem 50.3

Let $f : [a, b] \to \mathbb{R}$ be a function that satisfies

1. f is bounded and piecewise monotone.

   or

2. f is piecewise continuous.

Then f is Riemann integrable.

Proof. Let $P = \{a = t_0 < \ldots < t_n = b\}$ be a partition of $[a, b]$ s.t. 1) f is monotone **or** 2) f is uniformly continuous on (t_{k-1}, t_k) $\forall 1 \le k \le n$.

If f is monotone on (t_{k-1}, t_k), then f can be extended to a monotone function on f_k on $[t_{k-1}, t_k]$. For example, if f is increasing on (t_{k-1}, t_k) we define

$$f_k(t) = \begin{cases} \inf_{t \in (t_{k-1}, t_k)} f(t), & t = t_{k-1} \\ f(t), & t \in (t_{k-1}, t_k) \\ \sup_{t \in (t_{k-1}, t_k)} f(t), & t = t_k \end{cases}$$

As f_k is monotone on $[t_{k-1}, t_k]$, f_k is Riemann integrable on $[t_{k-1}, t_k]$. As f differs from f_k at most two points, f is Riemann integrable on $[t_{k-1}, t_k]$ and

$$\int_{t_{k-1}}^{t_k} f(t)\, dt = \int_{t_{k-1}}^{t_k} f_k(t)\, dt$$

If f is uniformly continuous on (t_{k-1}, t_k), then f admits a continuous extension f_k to $[t_{k-1}, t_k]$. Then f_k is Riemann integrable on $[t_{k-1}, t_k]$ and so f is Riemann integrable on $[t_{k-1}, t_k]$ and

$$\int_{t_{k-1}}^{t_k} f(t)\, dt = \int_{t_{k-1}}^{t_k} f_k(t)\, dt$$

By the last theorem from last lecture, we conclude that f is Riemann integrable on $[a, b]$ and

$$\int_a^b f(t)\, dt = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f(t)\, dt \qquad \square$$

Theorem 50.4 (Intermediate Value Property for Integrals)
Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exists $c \in [a, b]$ s.t.

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx$$

Proof. f is continuous on $[a, b]$ compact which implies there exist $x_0, y_0 \in [a, b]$ s.t.

$$\begin{cases} f(x_0) = \inf_{x \in [a,b]} f(x) \\ f(y_0) = \sup_{x \in [a,b]} f(x) \end{cases}$$

So

$$(b-a)f(x_0) = \int_a^b f(x_0)\,dx \le \int_a^b f(x)\,dx \le \int_a^b f(y_0)\,dx = (b-a)f(y_0)$$
$$\left.\begin{array}{r} \implies f(x_0) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le f(y_0) \\ f \text{ is continuous} \implies f \text{ has the Darboux property} \end{array}\right\} \implies$$

$\implies \exists c$ between x_0 and y_0 s.t. $f(c) = \frac{1}{b-a}\int_a^b f(x)dx$. □

§50.2 Fundamental Theorem of Calculus

Definition 50.5 (Riemann Integrable – "Extension") — We say that a function $f : (a, b) \to \mathbb{R}$ is Riemann integrable on $[a, b]$ if every extension of f to $[a, b]$ is Riemann integrable. In this case, $\int_a^b f(t)dt$ does not depend on the values of the extension at a and at b.

Theorem 50.6 (Fundamental Theorem of Calculus Part II)
Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If f' is Riemann integrable on $[a, b]$ then

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Proof. Fix $\varepsilon > 0$. As f' is Riemann integrable on $[a, b]$, $\exists P = \{a = t_0 < \ldots < t_n = b\}$ s.t.

$$U(f'; P) - L(f'; P) < \varepsilon$$

where f is continuous on $[t_{k-1}, t_k]$ and differentiable on (t_{k-1}, t_k). So, by the Mean Value theorem, $\exists x_k \in (t_{k-1}, t_k)$ s.t.

$$f'(x_k) = \frac{f(t_k) - f(t_{k-1})}{t_k - t_{k-1}}$$

In particular,

$$\sum_{k=1}^n f'(x_k)(t_k - t_{k-1}) = \sum_{k=1}^n [f(t_k) - f(t_{k-1})] = f(b) - f(a)$$

is a Riemann sum of f' associated to the partition P. Moreover,

$$\left.\begin{aligned} L(f';P) \le f(b)-f(a) \le U(f';P) < L(f';P)+\varepsilon \\ L(f';P) \le \textstyle\int_a^b f'(x)\,dx \le U(f';P) < L(f';P)+\varepsilon \end{aligned}\right\} \implies$$

$$\left.\begin{aligned} &\implies \left|\textstyle\int_a^b f'(x)\,dx - [f(b)-f(a)]\right| < 2\varepsilon \\ &\varepsilon > 0 \text{ was arbitrary} \end{aligned}\right\} \implies \int_a^b f'(x)\,dx = f(b)-f(a)$$

□

Theorem 50.7 (Integration by Parts)
Let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If f' and g' are Riemann integrable on $[a,b]$, then

$$\int_a^b f(x)g'(x)\,dx + \int_a^b f'(x)g(x)\,dx = f(b)g(b) - f(a)g(a)$$

Proof. By Exc 1 from Hw 8, the product of two Riemann integrable functions is Riemann integrable. In particular, $f'g$ and fg' are Riemann integrable. Let $h : [a,b] \to \mathbb{R}$, $h(x) = f(x)g(x)$. We have h is continuous on $[a,b]$, differentiable on (a,b) and

$$h'(x) = f'(x)g(x) + f(x)g'(x)$$

h' is Riemann integrable on $[a,b]$. By Fundamental Theorem of Calculus Part II,

$$\int_a^b h'(x)\,dx = h(b) - h(a)$$

$$\implies \int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a)$$

□

Theorem 50.8 (Fundamental Theorem of Calculus Part I)
Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable. For $x \in [a,b]$, we define

$$F(x) = \int_a^x f(t)\,dt$$

Then F is continuous on $[a,b]$. Moreover, if f is continuous at a point $x_0 \in (a,b)$, then F is differentiable at x_0 and

$$F'(x_0) = f(x_0)$$

Proof. For $a \le x < y \le b$,

$$\begin{aligned} F(y) - F(x) &= \int_a^y f(t)\,dt - \int_a^x f(t)\,dt \\ &= \int_a^x f(t)\,dt + \int_x^y f(t)\,dt - \int_a^x f(t)\,dt \\ &= \int_x^y f(t)\,dt \end{aligned}$$

f is Riemann integrable $\implies$ f is bounded $\implies$ $\exists M > 0$ s.t.

$$|f(x)| \leq M \qquad \forall x \in [a,b]$$

So

$$|F(y) - F(x)| \leq \int_x^y |f(t)| \, dt \leq M \, |y - x|$$

This shows F is uniformly continuous on $[a,b]$. For each $\varepsilon > 0$ if $|y-x| < \frac{\varepsilon}{M}$ then

$$|F(y) - F(x)| < \varepsilon$$

Assume f is continuous at $x_0 \in (a,b)$. For $x \in [a,b] \setminus \{x_0\}$,

$$\begin{aligned}
\frac{F(x) - F(x_0)}{x - x_0} - f(x_0) &= \frac{1}{x - x_0} \int_{x_0}^x f(t) \, dt - f(x_0) \\
&= \frac{1}{x - x_0} \int_{x_0}^x f(t) \, dt - \frac{1}{x - x_0} \int_{x_0}^x f(x_0) \, dt \\
&= \frac{1}{x - x_0} \int_{x_0}^x \left[f(t) - f(x_0) \right] \, dt
\end{aligned}$$

Fix $\varepsilon > 0$. As f is continuous at x_0, $\exists \delta > 0$ s.t.

$$|f(x) - f(x_0)| < \varepsilon \qquad \forall \, |x - x_0| < \delta \quad x \in [a,b]$$

So for $x \in [a,b]$ with $0 < |x - x_0| < \delta$,

$$\begin{aligned}
\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| &\leq \frac{1}{|x - x_0|} \int_{x_0}^x |f(t) - f(x_0)| \, dt \\
&< \frac{1}{|x - x_0|} \int_{x_0}^x \varepsilon \, dt = \varepsilon
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, F is differentiable at x_0 and $F'(x_0) = f(x_0)$. $\square$

§51 Lec 23: May 19, 2021

§51.1 Change of Variables

Theorem 51.1 (Change of Variables)
Let J be an open interval in $\mathbb{R}$ and let $u : J \to \mathbb{R}$ be differentiable with u' continuous on J. Let I be an open interval in $\mathbb{R}$ s.t. $u(J) \subseteq I$ and let $f : I \to \mathbb{R}$ be continuous. Then $f \circ u : J \to \mathbb{R}$ is continuous and for any $a, b \in J$ with $a < b$ we have

$$\int_a^b f(u(x)) \cdot u'(x)\, dx = \int_{u(a)}^{u(b)} f(y)\, dy$$

Proof. As $f \circ u$ and u' are continuous on $[a, b]$, the function $x \mapsto (f \circ u)(x) \cdot u'(x)$ is continuous on $[a, b]$ and so it's Riemann integrable on $[a, b]$.

Fix $c \in I$ and consider $F(x) = \int_c^x f(t)dt$. By Fundamental Theorem of Calculus Part I, F is differentiable on I (because f is continuous on I) and $F'(x) = f(x)\ \forall x \in I$. Consider $x \mapsto (F \circ u)(x)$ is differentiable on J and

$$(F \circ u)'(x) = f(u(x)) \cdot u'(x) \qquad \forall x \in J$$

By the Fundamental Theorem of Calculus Part II,

$$\int_a^b (F \circ u)'(x)\, dx = (F \circ u)(b) - (F \circ u)(a)$$

which implies

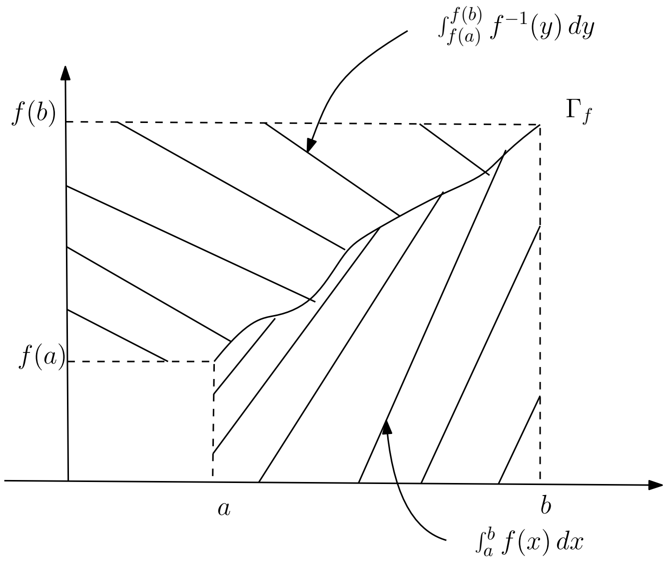
$$\implies \int_a^b f(u(x)) \cdot u'(x)\, dx = \int_c^{u(b)} f(y)\, dy - \int_c^{u(a)} f(y)\, dy = \int_{u(a)}^{u(b)} f(y)\, dy \qquad \square$$

Exercise 51.1. Let I be an open interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be injective and differentiable with f' continuous on I. Then $J = f(I)$ is an open interval and $f^{-1} : J \to I$ is differentiable.

Then for any $a, b \in I$ with $a < b$ we have

$$\int_a^b f(x)\, dx + \int_{f(a)}^{f(b)} f^{-1}(y)\, dy = bf(b) - af(a)$$

Proof. Consider:



$$\Gamma_f = \{(x, f(x)) : a \le x \le b\} = \left\{\left(f^{-1}(y), y\right) : y \text{ between } f(a) \text{ and } f(b)\right\}$$

We perform a change of variables:

$$\int_{f(a)}^{f(b)} f^{-1}(y)\,dy = \int_a^b f^{-1}(f(x))\,f'(x)\,dx$$

where $y = f(x)$ and $dy = f'dx$

$$\begin{aligned}\int_a^b f^{-1}(f(x))\,f'(x)\,dx &= \int_a^b x f'(x)\,dx \\ &= xf(x)\Big|_{x=a}^{x=b} - \int_a^b f(x)\,dx \\ &= bf(b) - af(a) - \int_a^b f(x)\,dx\end{aligned}$$

□

Theorem 51.2

Let $f_n : [a,b] \to \mathbb{R}$ be Riemann integrable s.t. $f_n \xrightarrow[n\to\infty]{u} f$ on $[a,b]$. Then f is Riemann integrable and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx = \int_a^b f(x)\,dx$$

Proof. For $n \ge 1$ let $d_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|$. As $f_n \xrightarrow[n\to\infty]{u} f$ on $[a,b]$ we have $d_n \xrightarrow[n\to\infty]{} 0$. In particular, $f_n(x) - d_n \le f(x) \le f_n(x) + d_n$ for all $x \in [a,b]$ (and thus f is bounded). For any partition P of $[a,b]$, we have

$$\begin{cases} U(f_n;P) - d_n(b-a) \le U(f;P) \le U(f_n;P) + d_n(b-a) \\ L(f_n;P) - d_n(b-a) \le L(f;P) \le L(f_n;P) + d_n(b-a) \end{cases}$$

So

$$U(f;P) - L(f;P) \le U(f_n;P) - L(f_n;P) + 2d_n(b-a)$$

Fix $\varepsilon > 0$. As $d_n \underset{n\to\infty}{\longrightarrow} 0$, $\exists n_\varepsilon \in \mathbb{N}$ s.t.

$$d_n < \frac{\varepsilon}{4(b-a)} \qquad \forall n \geq n_\varepsilon$$

Then for each $n \geq n_\varepsilon$ (fixed) there exists a partition $P = P(\varepsilon, n)$ of $[a, b]$ s.t.

$$U(f_n; P) - L(f_n; P) < \frac{\varepsilon}{2}$$

For $n \geq n_\varepsilon$ and $P = P(\varepsilon, n)$ as above we get

$$U(f; P) - L(f; P) < \varepsilon$$

As $\varepsilon > 0$ is arbitrary, this shows that f is Riemann integrable (since it's Darboux integrable and bounded). Moreover,

$$\begin{aligned}\int_a^b f(x)\,dx \leq U(f;P) &\leq U(f_n;P) + d_n(b-a)\\ &< L(f_n;P) + \frac{\varepsilon}{2} + \frac{\varepsilon}{4}\\ &\leq \int_a^b f_n(x)\,dx + \frac{3\varepsilon}{4}\end{aligned}$$

Similarly,

$$\begin{aligned}\int_a^b f(x)\,dx \geq L(f;P) &\geq L(f_n;P) - d_n(b-a)\\ &> U(f_n;P) - \frac{\varepsilon}{2} - \frac{\varepsilon}{4}\\ &\geq \int_a^b f_n(x)\,dx - \frac{3\varepsilon}{4}\end{aligned}$$

Thus,

$$\implies \left|\int_a^b f(x)\,dx - \int_a^b f_n(x)\,dx\right| < \frac{3\varepsilon}{4} \qquad \forall n \geq n_\varepsilon$$

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

□

§51.2 Lebesgue Criterion

Definition 51.3 (Zero Outer Measure) — A set $A \subseteq \mathbb{R}$ is said to have zero outer measure if for every $\varepsilon > 0$ there exists a countable collection of open intervals $\{(a_n, b_n)\}_{n\geq 1}$ s.t.

$$\begin{cases} A \subseteq \bigcup_{n\geq 1}(a_n, b_n)\\ \sum_{n\geq 1}(b_n - a_n) < \varepsilon\end{cases}$$

Remark 51.4.

1. If $A \subseteq \mathbb{R}$ has zero outer measure and $B \subseteq A$, then B has zero outer measure.
2. If $\{A_n\}_{n\geq 1}$ is a sequence of zero outer measure sets, then $\bigcup_{n\geq 1} A_n$ has zero outer measure.
3. If A is a set that is at most countable, then A has zero outer measure.

Proof. 2. Fix $\varepsilon > 0$. For each $n \geq 1$, let $\left\{\left(a_m^{(n)}, b_m^{(n)}\right)\right\}_{m\geq 1}$ be open intervals s.t.

$$\begin{cases} A_n \subseteq \bigcup_{m\geq 1}\left(a_m^{(n)}, b_m^{(n)}\right) \\ \sum_{n\geq 1}\left(b_m^{(n)} - a_m^{(n)}\right) < \frac{\varepsilon}{2^n} \end{cases}$$

Then $\left\{\left(a_m^{(n)}, b_m^{(n)}\right)\right\}_{m,n\geq 1}$ is a countable collection of open intervals s.t.

$$\begin{cases} \bigcup_{n\geq 1} A_n \subseteq \bigcup_{n,m\geq 1}\left(a_m^{(n)}, b_m^{(n)}\right) \\ \sum_{n\geq 1}\sum_{m\geq 1}\left(b_m^{(n)} - a_m^{(n)}\right) < \sum_{n\geq 1}\frac{\varepsilon}{2^n} = \varepsilon \end{cases}$$

□

Theorem 51.5 (Lebesgue Criterion)
Let $f : [a, b] \to \mathbb{R}$ be bounded. Then f is Riemann integrable if and only if the set

$$\mathscr{D}_f = \{x \in [a, b] : f \text{ is not continuous at } x\}$$

has zero outer measure.

Proof. We have

$$\mathscr{D}_f = \{x \in [a, b] : \omega(f, x) = 0\}$$

where

$$\begin{aligned} \omega(f, x) &= \inf_{\delta>0} \omega\left(f, B_\delta(x)\right) \\ &= \inf_{\delta>0}\left[\sup_{y\in B_\delta(x)} f(y) - \inf_{y\in B_\delta(x)} f(y)\right] \\ &= \inf_{\delta>0}\left[M\left(f; B_\delta(x)\right) - m\left(f; B_\delta(x)\right)\right] \end{aligned}$$

Then

$$\begin{aligned} \mathscr{D}_f &= \{x \in [a, b] : \omega(f, x) > 0\} \\ &= \bigcup_{n\geq 1}\underbrace{\left\{x \in [a, b] : \omega(f, x) \geq \frac{1}{n}\right\}}_{:=F_n} \end{aligned}$$

Key Observation: If $P = \{a = t_0 < \ldots < t_n = b\}$ then

$$\begin{aligned} U(f; P) - L(f; P) &= \sum_{k=1}^{n}\left[M\left(f; [t_{k-1}, t_k]\right) - m\left(f; [t_{k-1}, t_k]\right)\right](t_k - t_{k-1}) \\ &= \sum_{k=1}^{n}\omega\left(f; [t_{k-1}, t_k]\right)(t_k - t_{k-1}) \end{aligned}$$

We will continue with this proof in the next lecture. □

§52 Lec 24: May 21, 2021

§52.1 Lebesgue Criterion (Cont'd)

Proof. (Cont'd) " $\implies$ " Assume that f is Riemann integrable. We denote

$$\begin{aligned}\mathscr{D}_f &= \{x \in [a,b] : \omega(f,x) > 0\} \\ &= \bigcup_{n\ge 1} \left\{ x \in [a,b] : \omega(f,x) \ge \frac{1}{n} \right\}\end{aligned}$$

For $n \ge 1$, let $F_n = \left\{x \in [a,b] : \omega(f,x) \ge \frac{1}{n}\right\}$. To show that $\mathscr{D}_f$ has zero outer measure, it suffices to prove that F_n has zero outer measure for all $n \ge 1$.

Fix $N \ge 1$ and $\varepsilon > 0$. As f is Riemann integrable, there exists a partition $P = \{a = t_0 < \ldots < t_n = b\}$ s.t.

$$U(f;P) - L(f;P) < \frac{\varepsilon}{N}$$

Let $I = \{1 \le k \le n : F_N \cap (t_{k-1}, t_k) \ne \emptyset\}$. Then

$$F_N \subseteq \bigcup_{k\in I} (t_{k-1}, t_k) \cup P$$

As P is finite, it has zero outer measure. Thus, it suffices to show that

$$\sum_{k\in I} (t_k - t_{k-1}) < \varepsilon$$

Then,

$$\begin{aligned}\frac{\varepsilon}{N} > U(f;P) - L(f;P) &= \sum_{k=1}^{n} \left[M\left(f;[t_{k-1},t_k]\right) - m\left(f;[t_{k-1},t_k]\right)\right](t_k - t_{k-1}) \\ &\ge \sum_{k\in I} \omega\left(f;[t_{k-1},t_k]\right)(t_k - t_{k-1}) \\ &\ge \frac{1}{N}\sum_{k\in I}(t_k - t_{k-1})\end{aligned}$$

which implies

$$\sum_{k\in I}(t_k - t_{k-1}) < \varepsilon$$

" $\impliedby$ " Assume that $\mathscr{D}_f$ has zero outer measure.

$$f \text{ bounded } \implies \exists M > 0 \text{ s.t. } |f(x)| \le M \qquad \forall x \in [a,b]$$

Fix $\varepsilon > 0$ and let $\alpha > 0$ to be chosen later. Consider

$$\left.\begin{array}{l} F_\alpha = \{x \in [a,b] : \omega(f,x) \ge \alpha\} \subseteq \mathscr{D}_f \\ \mathscr{D}_f \text{ has zero outer measure}\end{array}\right\} \implies F_\alpha \text{ has zero outer measure}$$

$$\implies \exists \{(a_n,b_n)\}_{n\ge 1} \text{ s.t. } \begin{cases} F_\alpha \subseteq \bigcup_{n\ge 1}(a_n,b_n) \\ \sum_{n\ge 1}(b_n - a_n) < \varepsilon\end{cases}$$

Let $A = [a,b] \setminus F_\alpha$. For any $x \in A$, $\omega(f,x) < \alpha \implies \exists (c_x, d_x)$ neighborhood of x s.t.

$$\omega(f;[c_x,d_x]) < \alpha$$

So

$$\left.\begin{array}{l}[a,b] = F_\alpha \cup A \subseteq \bigcup_{n\geq 1}(a_n,b_n) \cup \bigcup_{x\in A}(c_x,d_x) \\ [a,b] \text{ is compact}\end{array}\right\}$$

which implies there exists $n_0 \in \mathbb{N}$ and $J \subseteq A$ finite s.t.

$$[a,b] \subseteq \bigcup_{k=1}^{n_0}(a_k,b_k) \cup \bigcup_{x\in J}(c_x,d_x)$$

Let P be a partition of $[a,b]$ formed by the points

$$\left(\{a,b\} \cup \bigcup_{k=1}^{n_0}\{a_x,b_x\} \cup \bigcup_{x\in J}\{c_x,d_x\}\right) \cap [a,b]$$

Say $P = \{a = t_0 < \ldots < t_n = b\}$. For any $1 \leq l \leq n$, we have

$$[t_{l-1},t_l] \subseteq [a_k,b_k] \text{ for some } 1 \leq k \leq n_0$$

or

$$[t_{l-1},t_l] \subseteq [c_x,d_x] \text{ for some } x \in J$$

Let

$$\begin{aligned} I_1 &= \{1 \leq l \leq n : \ [t_{l-1},t_l] \subseteq [a_k,b_k] \text{ for some } 1 \leq k \leq n_0\} \\ I_2 &= \{1,\ldots,n\} \setminus I_1 \end{aligned}$$

Note that

$$\sum_{l\in I_1}(t_l - t_{l-1}) \leq \sum_{k=1}^{n_0}(b_k - a_k) < \varepsilon$$
$$l \in I_2, \ \omega(f;[t_{l-1},t_l]) \leq \omega\left(f;[c_x,d_x]\right) < \alpha$$

Then,

$$\begin{aligned} U(f;P) - L(f;P) &= \sum_{l=1}^{n}\left[M\left(f;[t_{l-1},t_l]\right) - m\left(f;[t_{l-1},t_l]\right)\right](t_l - t_{l-1}) \\ &= \sum_{l\in I_1}\left[M\left(f;[t_{l-1},t_l]\right) - m\left(f;[t_{l-1},t_l]\right)\right](t_l - t_{l-1}) \\ &+ \sum_{l\in I_2}\omega\left(f;[t_{l-1},t_l]\right)(t_l - t_{l-1}) \end{aligned}$$

Notice that

$$\sum_{l\in I_1}\left[M\left(f;[t_{l-1},t_l]\right) - m\left(f;[t_{l-1},t_l]\right)\right](t_l - t_{l-1}) \leq 2M\sum_{l\in I_1}(t_l - t_{l-1}) < 2M\varepsilon$$

So

$$\begin{aligned} \sum_{l\in I_2}\omega\left(f;[t_{l-1},t_l]\right)(t_l - t_{l-1}) &< \alpha\sum_{l\in I_2}(t_l - t_{l-1}) \\ &\leq \alpha\sum_{l=1}^{n}(t_l - t_{l-1}) \\ &= \alpha(b-a) \end{aligned}$$

Choose $\alpha < \frac{\varepsilon}{b-a}$ to get

$$U(f;P) - L(f;P) < 2M\varepsilon + \varepsilon$$

As ε is arbitrary, this shows that f is Darboux integrable, and thus Riemann integrable.

□

§52.2 Improper Riemann Integrals

Definition 52.1 (Locally Riemann Integrable) — Let $-\infty < a < b \leq \infty$. We say that $f : [a, b) \to \mathbb{R}$ is locally Riemann integrable if f is integrable on $[a, c]$ for any $c \in (a, b)$.

Definition 52.2 (Improper Riemann Integral) — Let $-\infty < a < b \leq \infty$ and $f : [a, b) \to \mathbb{R}$ is locally Riemann integrable. In addition,

$$\lim_{c \to b} \int_a^c f(x)\, dx \text{ exists in } \mathbb{R}$$

We denote it $\int_a^b f(x)dx$ and we call it the improper Riemann integral of f. In this case we say that the improper Riemann integral of f converges. If

$$\lim_{c \to b} \int_a^c f(x)\, dx = \pm\infty$$

then we write $\int_a^b f(x)dx = \pm\infty$ and we say that the improper Riemann integral of f diverges to $\pm\infty$.

Remark 52.3. One can make a similar definition if $-\infty \leq a < b < \infty$ and $f : (a, b] \to \mathbb{R}$ or if $-\infty \leq a < b \leq \infty$ and $f : (a, b) \to \mathbb{R}$.

Theorem 52.4

Let $-\infty < a < b < \infty$ and let $f : [a, b) \to \mathbb{R}$ be locally Riemann integrable and bounded. Then the improper Riemann integral $\int_a^b f(x)dx$ converges. Moreover, any extension $\tilde{f} : [a, b] \to \mathbb{R}$ of f to $[a, b]$ is Riemann integrable and

$$\int_a^b \tilde{f}(x)\, dx = \int_a^b f(x)\, dx$$

Proof. Let $\tilde{f} : [a, b] \to \mathbb{R}$ be an extension of f to $[a, b]$. As f is bounded, $\exists M > 0$ s.t.

$$\left|\tilde{f}(x)\right| \leq M \qquad \forall x \in [a, b]$$

For $c \in (a, b)$,

$$U_a^b(\tilde{f}) = U_a^c(\tilde{f}) + U_c^b(\tilde{f}) = \int_a^c f(x)\, dx + U_c^b(\tilde{f}) \tag{*}$$

$$L_a^b(\tilde{f}) = L_a^c(\tilde{f}) + L_c^b(\tilde{f}) = \int_a^c f(x)\, dx + L_c^b(\tilde{f})$$

$$\left.\begin{aligned} &\implies U_a^b(\tilde{f}) - L_a^b(\tilde{f}) = U_c^b(\tilde{f}) - L_c^b(\tilde{f}) \\ &U_c^b(\tilde{f}) \leq M(b-c) \\ &\left|L_c^b(\tilde{f})\right| \leq M(b-c) \end{aligned}\right\} \implies U_a^b(\tilde{f}) - L_a^b(\tilde{f}) \leq \underbrace{2M(b-c)}_{\xrightarrow{c \to b} 0}$$

This shows that $\tilde{f}$ is Riemann integrable. Moreover, by (*),

$$\int_a^b \tilde{f}(x)\,dx = \lim_{c\to b}\int_a^c f(x)\,dx$$

Thus, the improper Riemann integral of f converges and

$$\int_a^b f(x)\,dx = \int_a^b \tilde{f}(x)\,dx \qquad \square$$

§53 Lec 25: May 24, 2021

§53.1 Improper Riemann Integrals (Cont'd)

Proposition 53.1

Let $-\infty < a < b \leq \infty$ and let $f, g : [a, b) \to \mathbb{R}$ be locally Riemann integrable s.t. the improper Riemann integrals of f and g converge. Then

1. For any $\alpha \in \mathbb{R}$, the improper Riemann integral of αf converges and

$$\int_a^b (\alpha f)(x)\, dx = \alpha \int_a^b f(x)\, dx$$

2. The improper Riemann integral of $f + g$ converges and

$$\int_a^b (f+g)(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$$

Proof. 1. Consider:

$$\begin{aligned} \mathbb{R} \ni \alpha \int_a^b f(x)\, dx = \alpha \lim_{c \to b} \int_a^c f(x)\, dx &= \lim_{c \to b} \alpha \int_a^c f(x)\, dx \\ (f \text{ is locally Riemann integrable}) &= \lim_{c \to b} \int_a^c (\alpha f)(x)\, dx \end{aligned}$$

So the improper Riemann integral of αf converges and

$$\int_a^b (\alpha f)(x)\, dx = \lim_{c \to b} \int_a^c (\alpha f)(x)\, dx = \alpha \int_a^b f(x)\, dx$$

2. Consider:

$$\begin{aligned} \mathbb{R} \ni \int_a^b f(x)\, dx + \int_a^b g(x)\, dx &= \lim_{c \to b} \int_a^c f(x)\, dx + \lim_{c \to b} \int_a^c g(x)\, dx \\ &= \lim_{c \to b} \left[\int_a^c f(x)\, dx + \int_a^c g(x)\, dx \right] \\ &= \lim_{c \to b} \int_a^c [f(x) + g(x)]\, dx \end{aligned}$$

So the improper Riemann integral of $f + g$ converges and

$$\int_a^b (f+g)(x)\, dx = \lim_{c \to b} \int_a^c (f+g)(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx \qquad \square$$

Remark 53.2. If $f, g : [a, b] \to \mathbb{R}$ are Riemann integrable functions, then

- $|f|$ is Riemann integrable.
- $f \cdot g$ is Riemann integrable.

However, if $f, g : [a, b)$ are locally integrable functions s.t. the improper Riemann integrals of f and g converge, then

- the improper Riemann integral of $|f|$ need not converge.
- the improper Riemann integral of $f \cdot g$ need not converge.

Example 53.3

Let $f, g : (0,1] \to \mathbb{R}$, $f(x) = g(x) = \frac{1}{\sqrt{x}}$. The improper Riemann integral of f converges

$$\int_c^1 f(x)\,dx = \int_c^1 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big|_{x=c}^{x=1} = 2 - 2\sqrt{c} \underset{c\to 0}{\longrightarrow} 2$$

The improper Riemann integral of $f \cdot g$ does not converge

$$\int_c^1 f(x)g(x)\,dx = \int_c^1 \frac{1}{x}\,dx = \ln x\Big|_{x=c}^{x=1} = -\ln c \underset{c\to 0}{\longrightarrow} \infty$$

More generally, we can take $f, g : (0,1] \to \mathbb{R}$

$$f(x) = \frac{1}{x^\alpha}, \quad g(x) = \frac{1}{x^\beta} \quad \text{with} \quad 0 < \alpha, \beta < 1 \quad \text{and} \quad \alpha + \beta \geq 1$$

Lemma 53.4 (Cauchy Criterion)

Let $-\infty < a < b \leq \infty$. Let $f : [a,b) \to \mathbb{R}$ be locally integrable. Then the improper Riemann integral of f converges if and only if

$$\forall \varepsilon > 0 \quad \exists c_\varepsilon \in (a,b) \text{ s.t. } \left|\int_{c_1}^{c_2} f(x)\,dx\right| < \varepsilon \quad \forall c_\varepsilon < c_1 < c_2 < b$$

Proof. " $\Longrightarrow$ " Assume that the improper Riemann integral of f converges. Let

$$\alpha = \int_a^b f(x)\,dx \in \mathbb{R}$$

We have

$$\alpha = \lim_{c\to b} \int_a^c f(x)\,dx$$

Then $\forall \varepsilon > 0\ \exists c_\varepsilon \in (a,b)$ s.t.

$$\left|\alpha - \int_a^c f(x)\,dx\right| < \frac{\varepsilon}{2} \qquad \forall c_\varepsilon < c < b$$

For $c_\varepsilon < c_1 < c_2 < b$ we have

$$\begin{aligned}\left|\int_{c_1}^{c_2} f(x)\,dx\right| &= \left|\int_a^{c_2} f(x)\,dx - \int_a^{c_1} f(x)\,dx\right| \\ &\leq \left|\int_a^{c_2} f(x)\,dx - \alpha\right| + \left|\alpha - \int_a^{c_1} f(x)\,dx\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

" $\Longleftarrow$ " Fix $\varepsilon > 0$ and let $c_\varepsilon \in (a,b)$ s.t.

$$\left|\int_{c_1}^{c_2} f(x)\,dx\right| < \varepsilon \qquad \forall c_\varepsilon < c_1 < c_2 < b$$

Let $\{c_n\}_{n\geq 1} \subseteq (a,b)$ s.t. $c_n \underset{n\to\infty}{\longrightarrow} b$. Then $\exists n_\varepsilon \in \mathbb{N}$ s.t. $c_\varepsilon < c_n < b$ for all $n \geq n_\varepsilon$. In particular,

$$\left|\int_a^{c_m} f(x)\,dx - \int_a^{c_n} f(x)\,dx\right| = \left|\int_{c_n}^{c_m} f(x)\,dx\right| < \varepsilon \qquad n,m \ge n_\varepsilon$$
$$\implies \left\{\int_a^{c_n} f(x)\,dx\right\}_{n\ge1} \subseteq \mathbb{R} \text{ is Cauchy and so convergent}$$

Let $\alpha = \lim_{n\to\infty} \int_a^{c_n} f(x)\,dx$. To prove that the Riemann integral of f converges, we need to show that α does not depend on $\{c_n\}_{n\ge1}$. Let $\{d_n\}_{n\ge1} \subseteq (a,b)$ s.t. $\lim_{n\to\infty} d_n = b$. Consider

$$x_n = \begin{cases} c_k & \text{if } n = 2k \\ d_k & \text{if } n = 2k-1 \end{cases} \qquad \text{for } k \ge 1$$

Then $x_n \underset{n\to\infty}{\longrightarrow} b$. From the same argument used for the sequence $\{c_n\}_{n\ge1}$, we conclude that $\left\{\int_a^{x_n} f(x)\,dx\right\}_{n\ge1}$ is Cauchy and so convergent. So

$$\lim_{n\to\infty} \int_a^{x_{2n}} f(x)\,dx = \lim_{n\to\infty} \int_a^{x_{2n-1}} f(x)\,dx$$
$$\alpha = \lim_{n\to\infty} \int_a^{c_n} f(x)\,dx = \lim_{n\to\infty} \int_a^{d_n} f(x)\,dx$$

□

Theorem 53.5 (Abel Criterion)
Let $-\infty < a < b \le \infty$ and let $f,g : [a,b) \to \mathbb{R}$ be locally integrable. Assume that g is decreasing and $\lim_{x\to b} g(x) = 0$. Assume also that there exists $M > 0$ s.t.

$$\left|\int_a^c f(x)\,dx\right| \le M \qquad \forall a < c < b$$

Then the improper Riemann integral of $f \cdot g$ converges.

Remark 53.6. Compare this with the series version

$$\left.\begin{array}{l} \{a_n\}_{n\ge1} \text{ is decreasing with } \lim_{n\to\infty} a_n = 0 \\ \exists M > 0 \text{ s.t. } \left|\sum_{k=1}^n b_k\right| \le M \quad \forall n \ge 1 \end{array}\right\} \implies \sum_{n\ge1} a_n b_n \text{ converges}$$

Proof. We'll use the Cauchy Criterion. Fix $\varepsilon > 0$.

$$\lim_{x\to b} g(x) = 0 \implies \exists c_\varepsilon \in (a,b) \text{ s.t. } |g(x)| < \varepsilon \quad \forall c_\varepsilon < x < b$$

Fix $c_\varepsilon < c_1 < c_2 < b$ and consider $\int_{c_1}^{c_2} f(x)g(x)dx$. Using exercise #6 in HW8, we can find $x_0 \in [c_1, c_2]$ s.t.

$$\begin{aligned}\int_{c_1}^{c_2} f(x)g(x)\,dx &= g(c_1)\int_{c_1}^{x_0} f(x)\,dx + g(c_2)\int_{x_0}^{c_2} f(x)\,dx \\ &= g(c_1)\left[\int_a^{x_0} f(x)\,dx - \int_a^{c_1} f(x)\,dx\right] \\ &\quad + g(c_2)\left[\int_a^{c_2} f(x)\,dx - \int_a^{x_0} f(x)\,dx\right]\end{aligned}$$

which implies

$$\begin{aligned}\left|\int_{c_1}^{c_2} f(x)g(x)\,dx\right| &\le g(c_1)\left[\left|\int_a^{x_0} f(x)\,dx\right| + \left|\int_a^{c_1} f(x)\,dx\right|\right] \\ &\quad + g(c_2)\left[\left|\int_a^{c_2} f(x)\,dx\right| + \left|\int_a^{x_0} f(x)\,dx\right|\right] \\ &< 4M\varepsilon\end{aligned}$$

As $c_\varepsilon < c_1, c_2 < b$ are arbitrary and $\varepsilon > 0$ is arbitrary, we conclude that the improper Riemann integral of fg converges. □

§54 Lec 26: May 26, 2021

§54.1 Improper Riemann Integrals (Cont'd)

Exercise 54.1. Show that the improper Riemann integral

$$\int_0^\infty \frac{\sin x}{x}\, dx \quad \text{converges}$$

but the improper Riemann integral

$$\int_0^\infty \left|\frac{\sin x}{x}\right| dx \quad \text{does not converge}$$

Proof. To show that $\int_0^\infty \frac{\sin x}{x} dx$ converges, we have to prove that

$$\lim_{M\to\infty} \int_0^M \frac{\sin x}{x}\, dx \quad \text{exists in } \mathbb{R}$$

Note that

$$x \mapsto \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

is continuous on on $[0,\infty)$ and so it is Riemann integrable on $[0, M]$ for each $M > 0$. For $M > 1$, we write

$$\int_0^M \frac{\sin x}{x}\, dx = \underbrace{\int_0^1 \frac{\sin x}{x}\, dx}_{\in\mathbb{R}} + \int_1^M \frac{\sin x}{x}\, dx$$

Note that $f, g : [1,\infty) \to \mathbb{R}$, $f(x) = \sin x$ and $g(x) = \frac{1}{x}$ are continuous and so Riemann integrable on $[1, M]$ $\forall M > 1$. Also,

- g is decreasing and $\lim_{x\to\infty} g(x) = 0$
- In addition,

$$\left|\int_1^M \sin x\, dx\right| = |\cos 1 - \cos M| \le 2 \qquad \forall M > 1$$

So by the Abel Criterion, the improper Riemann integral $\int_1^\infty \frac{\sin(x)}{x} dx$ converges. Moreover,

$$\begin{aligned}\int_0^\infty \frac{\sin x}{x}\, dx = \lim_{M\to\infty}\int_0^M \frac{\sin x}{x}\, dx &= \int_0^1 \frac{\sin x}{x}\, dx + \lim_{M\to\infty}\int_1^M \frac{\sin x}{x}\, dx \\ &= \int_0^1 \frac{\sin x}{x}\, dx + \int_1^\infty \frac{\sin x}{x}\, dx\end{aligned}$$

Let's show that the improper Riemann integral $\int_0^\infty \frac{|\sin x|}{x} dx$ diverges to ∞. We'll use that

$$|\sin x| \ge \frac{1}{2} \quad \text{on} \quad \left[k\pi + \frac{\pi}{6}, k\pi + \frac{5\pi}{6}\right]$$

for all $k \geq 0$. So

$$\begin{aligned}
\int_0^\infty \frac{|\sin x|}{x}\,dx &\geq \sum_{k\geq 0}\int_{k\pi+\frac{\pi}{6}}^{k\pi+\frac{5\pi}{6}} \frac{|\sin x|}{x}\,dx \\
&\geq \sum_{k\geq 0}\frac{1}{2}\cdot\frac{1}{k\pi+\frac{5\pi}{6}}\cdot\left[\left(k\pi+\frac{5\pi}{6}\right)-\left(k\pi+\frac{\pi}{6}\right)\right] \\
&\geq \sum_{k\geq 0}\frac{1}{2}\cdot\frac{1}{(k+1)\pi}\cdot\frac{2\pi}{3} = \frac{1}{3}\sum_{k\geq 0}\frac{1}{k+1} = \infty
\end{aligned}$$

□

Proposition 54.1

Let $-\infty < a < b \leq \infty$ and let $f : [a, b) \to \mathbb{R}$ be locally Riemann integrable s.t. the improper Riemann integral of $|f|$ converges. Then the improper Riemann integral of f converges and

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

Proof. As the improper Riemann integral of $|f|$ converges, by the Cauchy Criterion we have

$$\forall \varepsilon > 0 \quad \exists c_\varepsilon \in (a, b) \text{ s.t. } \int_{c_1}^{c_2} |f(x)|\,dx < \varepsilon \quad \forall c_\varepsilon < c_1 < c_2 < b$$

As f is locally integrable, f is integrable on $[c_1, c_2]$ and

$$\left|\int_{c_1}^{c_2} f(x)\,dx\right| \leq \int_{c_1}^{c_2} |f(x)|\,dx < \varepsilon \qquad \forall c_\varepsilon < c_1 < c_2 < b$$

By the Cauchy Criterion, the improper Riemann integral of f converges. Moreover,

$$\begin{aligned}
\left|\int_a^b f(x)\,dx\right| = \left|\lim_{c\to b}\int_a^c f(x)\,dx\right| &= \lim_{c\to b}\left|\int_a^c f(x)\,dx\right| \\
(f \text{ is locally integrable}) &\leq \lim_{c\to b}\int_a^c |f(x)|\,dx \\
&= \int_a^b |f(x)|\,dx
\end{aligned}$$

□

Definition 54.2 (Absolute Convergence – Integral) — Let $-\infty < a < b \leq \infty$ and $f : [a, b) \to \mathbb{R}$ be locally integrable. We say that the improper Riemann integral of f converges absolutely if the improper Riemann integral of $|f|$ converges.

Remark 54.3.

1. If the improper Riemann integral of f converges absolutely, then it converges.
2. The improper Riemann integral of f converges absolutely if and only if

$$\lim_{c\to b}\int_a^c |f(x)|\,dx \in \mathbb{R} \iff \exists M>0 \text{ s.t. } \int_a^c |f(x)|\,dx \le M \quad \forall c\in[a,b)$$

3. If $f,g:[a,b)\to\mathbb{R}$ are locally integrable s.t. $|f(x)|\le|g(x)|\ \forall x\in[a,b)$ and the improper Riemann integral of g converges absolutely, then the improper Riemann integral of f converges absolutely.
4. If $f,g:[a,b)\to\mathbb{R}$ are locally integrable and their improper Riemann integrals converge absolutely, then the improper Riemann integral of $f+g$ converges absolutely.
5. If $f,g:[a,b)\to\mathbb{R}$ are locally integrable s.t. f is bounded and the improper Riemann integral of g converges absolutely, then the improper Riemann integral of $f\cdot g$ converges absolutely.

§54.2 Continuous 1-Periodic Functions

Definition 54.4 (Convolution) — Let $f,g:\mathbb{R}\to\mathbb{C}$ be continuous functions with period 1, that is,

$$f(x+1)=f(x) \quad\text{and}\quad g(x+1)=g(x) \qquad x\in\mathbb{R}$$

Their convolution $f*g:\mathbb{R}\to\mathbb{C}$ is defined via

$$(f*g)(x)=\int_0^1 f(y)g(x-y)\,dy$$

<u>**Claim 1**</u>:

$$(f*g)(x)=\int_a^{a+1} f(y)g(x-y)\,dy \qquad \forall a\in\mathbb{R},\ \forall x\in\mathbb{R}$$

This is obviously true if $a=k\in\mathbb{Z}$. For $y=k+z$,

$$\begin{aligned}\int_k^{k+1} f(y)g(x-y)\,dy &= \int_0^1 f(k+z)g(x-z-k)\,dz\\ (f\&g\text{ periodic}) &= \int_0^1 f(z)g(x-z)\,dz=(f*g)(x)\end{aligned}$$

Next, decomposing $a=\underbrace{[a]}_{\in\mathbb{Z}}+\underbrace{\{a\}}_{\in[0,1)}$ we see that it suffices to prove the claim for $a\in(0,1)$.

$$\begin{aligned}\int_a^{a+1} f(y)g(x-y)\,dy &= \int_a^1 f(y)g(x-y)\,dy+\int_1^{1+a} f(y)g(x-y)\,dy\\ &= \int_a^1 f(y)g(x-y)\,dy+\int_0^a f(z+1)g(x-z-1)\,dz\\ &= \int_a^1 f(y)g(x-y)\,dy+\int_0^a f(z)g(x-z)\,dz\\ &= \int_0^1 f(y)g(x-y)\,dy=(f*g)(x)\end{aligned}$$

Claim 2: $f * g$ is 1-periodic.

$$(f * g)(x+1) = \int_0^1 f(y)g(x+1-y)\,dy = \int_0^1 f(y)g(x-y)\,dy = (f*g)(x)$$

Claim 3: $f * g$ is continuous

$$\begin{aligned} |(f*g)(x_1) - (f*g)(x_2)| &= \left| \int_0^1 f(y)\left[g(x_1-y) - g(x_2-y)\right]\,dy \right| \\ &\le \int_0^1 |f(y)|\,|g(x_1-y) - g(x_2-y)|\,dy \end{aligned}$$

g continuous on $[0,2]$ compact $\implies$ g is uniformly continuous on $[0,2]$, and since g is 1-periodic, we conclude that g is uniformly continuous on $\mathbb{R}$. So $\forall \varepsilon > 0\ \exists \delta > 0$ s.t.

$$|g(x) - g(y)| < \varepsilon \qquad \forall\,|x-y| < \delta$$

f is continuous on $[0,1]$ compact $\implies$ $M > 0$ s.t.

$$|f(x)| \le M \qquad \forall x \in [0,1]$$

So

$$|(f*g)(x_1) - (f*g)(x_2)| \le \int_0^1 M\cdot\varepsilon\,dy = M\cdot\varepsilon \quad \forall\,|x_1 - x_2| < \delta$$

Claim 4: $f * g = g * f$. For $z = x - y$,

$$\begin{aligned} (g*f)(x) = \int_0^1 g(y)f(x-y)\,dy &= -\int_x^{x-1} g(x-z)f(z)\,dz \\ &= \int_{x-1}^{x} f(y)g(x-y)\,dy \\ &= \int_0^1 f(y)g(x-y)\,dy \\ &= (f*g)(x) \end{aligned}$$

Claim 5: For all $\alpha \in \mathbb{C}$,

$$(\alpha f) * g = f * (\alpha g) = \alpha(f*g)$$

Claim 6: If f, g, h are continuous, 1-periodic functions,

$$\begin{cases} f*(g+h) = f*g + f*h \\ (f*g)*h = f*(g*h) \end{cases}$$

Left as exercise!

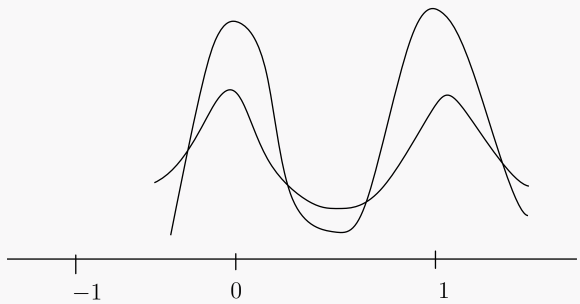
§55 Lec 27: May 28, 2021

§55.1 Continuous 1-Periodic Functions (Cont'd)

Definition 55.1 (Approximation to the Identity) — A sequence of continuous, 1-periodic functions $K_n : \mathbb{R} \to \mathbb{C}$ is called an approximation to the identity if it satisfies the following:

1. $\int_0^1 K_n(x)\,dx = 1 \ \forall n \geq 1$
2. $\exists M > 0$ s.t. $\int_0^1 |K_n(x)|\,dx \leq M \ \forall n \geq 1$
3. $\forall \delta > 0$, $\int_\delta^{1-\delta} |K_n(x)|\,dx \underset{n\to\infty}{\longrightarrow} 0$.

Remark 55.2. While 1) says that K_n assigns mass 1 to each period, 3) says that this mass is concentrating at the integers as $n \to \infty$.



Theorem 55.3

Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous, 1-periodic function and let $\{K_n\}_{n\geq 1}$ be an approximation to the identity. Then

$$K_n * f \underset{n\to\infty}{\overset{u}{\longrightarrow}} f \text{ on } \mathbb{R}$$

Proof. Fix $x \in \mathbb{R}$.

$$\begin{aligned}(K_n * f)(x) - f(x) &= \int_0^1 K_n(y) f(x-y)\,dy - f(x)\int_0^1 K_n(y)\,dy \\ &= \int_0^1 K_n(y)\,[f(x-y) - f(x)]\,dy \\ \implies |(K_n * f)(x) - f(x)| &\leq \int_0^1 |K_n(y)|\,|f(x-y) - f(x)|\,dy\end{aligned}$$

f is continuous and 1-periodic $\implies$ f is uniformly continuous.

Let $\varepsilon > 0$. Then $\exists \delta > 0$ s.t. $|f(x) - f(y)| < \varepsilon$ for all $|x - y| < \delta$

$$\int_0^\delta |K_n(y)| \underbrace{|f(x-y) - f(x)|}_{<\varepsilon} \, dy < \varepsilon \int_0^\delta |K_n(y)| \, dy$$
$$\leq \varepsilon \int_0^1 |K_n(y)| \, dy \leq \varepsilon M$$
$$\int_{1-\delta}^1 |K_n(y)| \, |f(x-y) - f(x)| \, dy \overset{y=1+z}{=} \int_{-\delta}^0 |K_n(1+z)| \, |f(x-z-1) - f(x)| \, dz$$
$$= \int_{-\delta}^0 |K_n(z)| \underbrace{|f(x-z) - f(x)|}_{<\varepsilon} \, dz$$
$$< \varepsilon \int_{-1}^0 |K_n(z)| \, dz \leq \varepsilon M$$
$$\int_\delta^{1-\delta} |K_n(y)| \, |f(x-y) - f(x)| \, dy \leq \int_\delta^{1-\delta} |K_n(y)| \, [|f(x-y)| + |f(x)|] \, dy$$
$$\leq 2 \sup_{x \in [0,1]} |f(x)| \int_\delta^{1-\delta} |K_n(y)| \, dy$$

As $\int_\delta^{1-\delta} |K_n(y)| \, dy \underset{n\to\infty}{\longrightarrow} 0$, $\exists n_\varepsilon \in \mathbb{N}$ s.t.

$$\int_\delta^{1-\delta} |K_n(y)| \, dy < \frac{\varepsilon}{2\|f\|_\infty + 1}$$

So collecting our estimates, we get

$$|(K_n * f)(x) - f(x)| \leq 2\varepsilon M + \varepsilon \qquad \forall x \in \mathbb{R}, \ \forall n \geq n_\varepsilon$$

As $\varepsilon > 0$ is arbitrary, we get $K_n * f \xrightarrow[n\to\infty]{u} f$. □

§55.2 Fourier Series

Definition 55.4 (Orthonormal Family) — For $n \in \mathbb{Z}$, let $e_n(x) = e^{2\pi i n x} = \cos(2\pi n x) + i \sin(2\pi n x)$. Note $e_n : \mathbb{R} \to \mathbb{C}$ is continuous, 1-periodic.

$$\int_0^1 e_n(x) \, dx = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

So

$$\int_0^1 e_n(x) \overline{e_m(x)} \, dx = \int_0^1 e_{n-m}(x) \, dx = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$$

$\implies \{e_n\}_{n \geq 1}$ form an orthonormal family.

Definition 55.5 (Trigonometric Polynomial) — A trigonometric polynomial takes the form

$$\sum_{|n| \leq N} c_n e_n(x)$$

where $c_n \in \mathbb{C}$ for all $|n| \leq N$.

Definition 55.6 (Fourier Series) — Given a continuous, 1-periodic function $f : \mathbb{R} \to \mathbb{C}$, we define its n^{th} Fourier coefficient via

$$\hat{f}(n) = \int_0^1 f(x)\overline{e_n(x)}\,dx = \int_0^1 f(x)e^{-2\pi inx}\,dx$$

The Fourier series of f is given by $\sum_{n\in\mathbb{Z}} \hat{f}(n)e_n(x)$.

Question 55.1. Can we recover f from its Fourier series?

If $f \in C^2$, then

$$\sum_{n\in\mathbb{Z}} \hat{f}(n)e_n(x) \xrightarrow[n\to\infty]{u} f(x)$$

In 1966, Carleson proved that the Fourier series of an integrable function converges pointwise to f outside a set of measure zero.
For $N \geq 0$, let

$$\begin{aligned} S_N(f)(x) = \sum_{|n|\leq N} \hat{f}(n)e_n(x) &= \sum_{|n|\leq N} \int_0^1 f(y)\overline{e_n(y)}\,dy \cdot e_n(x) \\ &= \sum_{|n|\leq N} \int_0^1 f(y)e_n(x-y)\,dy \\ &= \int_0^1 f(y)\left(\sum_{|n|\leq N} e_n\right)(x-y)\,dy \\ &= \left[f * \left(\sum_{|n|\leq N} e_n\right)\right](x) \end{aligned}$$

For $N \geq 0$, let $D_N = \sum_{|n|\leq N} e_n$ denote the Dirichlet Kernel. Note that

$$\int_0^1 D_N(x)\,dx = \sum_{|n|\leq N} \int_0^1 e_n(x)\,dx = 1 \qquad \forall N \geq 0$$

$\{D_N\}_{N\geq 0}$ do not form an approximation to the identity since

$$\int_0^1 |D_N(x)|\,dx \xrightarrow[N\to\infty]{} \infty$$

We have

$$\begin{aligned} D_N &= \sum_{|n|\leq N} e_n \\ (e_1 - 1)D_N &= \sum_{n=-N+1}^{N+1} e_n - \sum_{n=-N}^{N} e_n = e_{N+1} - e_{-N} \\ \implies D_N &= \frac{e_{N+1} - e_{-N}}{e_1 - 1} \end{aligned} \tag{1}$$

In addition,

$$\begin{aligned} D_N(x) = \frac{e^{2\pi i(N+1)x} - e^{-2\pi iNx}}{e^{2\pi ix} - 1} &= \frac{e^{\pi ix}\left(e^{2\pi i\left(N+\frac{1}{2}\right)x} - e^{-2\pi i\left(N+\frac{1}{2}\right)x}\right)}{e^{\pi ix}\left(e^{\pi ix} - e^{-\pi ix}\right)} \\ &= \frac{\sin\left(2\pi\left(N+\frac{1}{2}\right)x\right)}{\sin(\pi x)} \end{aligned}$$

Also,

$$\begin{aligned}\int_0^1 |D_N(x)|\,dx &\geq \int_0^1 \frac{\left|\sin\left(2\pi\left(N+\frac{1}{2}\right)x\right)\right|}{\pi x}\,dx \\ &\underset{y=2\pi\left(N+\frac{1}{2}\right)x}{=} \int_0^{2\pi\left(N+\frac{1}{2}\right)} \frac{|\sin(y)|}{\pi\cdot\frac{y}{2\pi\left(N+\frac{1}{2}\right)}}\cdot\frac{dy}{2\pi\left(N+\frac{1}{2}\right)} \\ &= \frac{1}{\pi}\int_0^{2\pi\left(N+\frac{1}{2}\right)} \frac{|\sin(y)|}{y}\,dy \underset{N\to\infty}{\longrightarrow} \infty\end{aligned}$$

The average of the Dirichlet kernels do form an approximation to the identity. For $N \geq 1$, let $F_N = \frac{D_0+\ldots+D_{N_1}}{N}$ denote the Fejer Kernels. Note that

$$\int_0^1 F_N(x)\,dx = \frac{1}{N}\sum_{k=0}^{N-1}\int_0^1 D_k(x)\,dx = 1 \qquad N \geq 1$$

We will show that $F_N \geq 0$ and so

- $\int_0^1 |F_N(x)|\,dx = \int_0^1 F_N(x)dx = 1\ \forall N \geq 1$
- $\forall \delta > 0,\ \int_\delta^{1-\delta} |F_N(x)|\,dx \underset{N\to\infty}{\longrightarrow} 0$

Consequently, we obtain the following

Theorem 55.7

If $f : \mathbb{R} \to \mathbb{C}$ is a continuous, 1-periodic function, then

$$F_N * f \underset{N\to\infty}{\overset{u}{\longrightarrow}} f \text{ on } \mathbb{R}$$

if and only if

$$\sigma(f) = \frac{1}{N}\sum_{k=0}^{N-1} S_N(f) \underset{N\to\infty}{\overset{u}{\longrightarrow}} f \text{ on } \mathbb{R}$$

Corollary 55.8

If $f : \mathbb{R} \to \mathbb{C}$ is a continuous, 1-periodic function, with $\hat{f}(n) = 0\ \forall n \in \mathbb{Z}$, then $f \equiv 0$.

Corollary 55.9

Every continuous, 1-periodic function can be approximated uniformly by trigonometric polynomials.

§56 Lec 28: Jun 2, 2021

§56.1 Fourier Series (Cont'd)

Recall that for $n \in \mathbb{Z}$ we define the character $e_n : \mathbb{R} \to \mathbb{C}$

$$e_n(x) = e^{2\pi inx}$$

For a continuous, 1-periodic function $f : \mathbb{R} \to \mathbb{C}$, we define its n^{th} Fourier coefficient via

$$\hat{f}(n) = \int_0^1 f(x)\overline{e_n(x)}\,dx = \int_0^1 f(x)e^{-2\pi inx}\,dx \qquad \forall n \in \mathbb{Z}$$

and the partial Fourier series

$$[S_N(f)](x) = \sum_{|n|\le N} \hat{f}(n)e_n(x) \qquad \forall N \ge 0$$

We observed $S_N(f) = f * D_N$ where D_N denotes the Dirichlet kernel

$$D_N = \sum_{|n|\le N} e_n \qquad \forall N \ge 0$$

Using

$$D_N = \frac{e_{N+1} - e_{-N}}{e_1 - 1} \tag{1}$$

We obtained the explicit formula

$$D_N(x) = \frac{\sin\left(2\pi\left(N + \frac{1}{2}\right)x\right)}{\sin(\pi x)}$$

and computed

$$\int_0^1 |D_N(x)|\,dx \xrightarrow[N\to\infty]{} \infty$$

In particular, $\{D_N\}_{N\ge1}$ do not form an approximation to the identity. Instead, we define the Fejer Kernel

$$F_N = \frac{D_0 + \ldots + D_{N-1}}{N} \qquad \forall N \ge 1$$

So

$$\sigma(f) = f * F_N = \frac{1}{N}\sum_{n=0}^{N-1} f * D_n = \frac{1}{N}\sum_{n=0}^{N-1} S_n(f)$$

Claim 56.1. $\{F_N\}_{N\ge1}$ form an approximation to the identity and thus $\sigma(f) \xrightarrow[n\to\infty]{u} f$ for any continuous, 1-periodic $f : \mathbb{R} \to \mathbb{C}$.

Proof. First, we have

$$\int_0^1 e_n(x)\,dx = \int_0^1 \cos(2\pi nx)\,dx + i\int_0^1 \sin(2\pi ni)\,dx = \begin{cases} 1, & n = 0 \\ 0, & n \ne 0 \end{cases}$$

we get

$$\int_0^1 D_N(x)\,dx = \sum_{|n|\le N}\int_0^1 e_n(x)\,dx = 1 \qquad \forall N \ge 0$$

and so

$$\int_0^1 F_N(x)\,dx = \frac{1}{N}\sum_{n=0}^{N-1}\int_0^1 D_n(x)\,dx = 1 \qquad \forall N \geq 1$$

Net, we compute an explicit formula for F_N

$$\begin{aligned} NF_N &= D_0 + \ldots + D_{N-1} \\ &\overset{(1)}{=} \frac{e_1 - e_0}{e_1 - 1} + \frac{e_2 - e_{-1}}{e_1 - 1} + \ldots + \frac{e_N - e_{-N+1}}{e_1 - 1} \\ &= \frac{(e_1 + e_2 + \ldots + e_N) - (e_0 + e_{-1} + \ldots + e_{-N+1})}{e_1 - 1} \\ &= \frac{(e_1 - 1)(e_1 + e_2 + \ldots + e_N) - (e_1 - 1)(e_0 + e_{-1} + \ldots + e_{-N+1})}{(e_1 - 1)^2} \end{aligned}$$

Notice that

$$\begin{aligned} (e_1 - 1)(e_1 + \ldots + e_N) &= e_2 + \ldots + e_{N+1} - e_1 - \ldots - e_N = e_{N+1} - e_1 \\ (e_1 - 1)(e_0 + \ldots + e_{-N+1}) &= e_1 + \ldots + e_{-N+2} - e_0 - \ldots - e_{-N+1} = e_1 - e_{-N+1} \end{aligned}$$

So

$$\begin{aligned} NF_N(x) &= \frac{e_{N+1}(x) + e_{-N+1}(x) - 2e_1(x)}{\left(e^{2\pi i x} - 1\right)^2} \\ &= \frac{e_1(x)\left(e^{2\pi i N x} + e^{-2\pi i N x} - 2\right)}{e_1(x)\left(e^{\pi i x} - e^{-\pi i x}\right)^2} \\ &= \frac{2\left(\cos(2\pi N x) - 1\right)}{\left[2i\sin(\pi x)\right]^2} \\ &= \left[\frac{\sin(\pi N x)}{\sin(\pi x)}\right]^2 \end{aligned}$$

which implies

$$F_N(x) = \frac{1}{N}\left[\frac{\sin(\pi N x)}{\sin(\pi x)}\right]^2 \geq 0 \qquad \forall N \geq 1$$

Thus,

$$\int_0^1 |F_N(x)|\,dx = \int_0^1 F_N(x)\,dx = 1 \qquad \forall N \geq 1$$

Lastly, we have to verify that $\forall 0 < \delta < 1$

$$\int_\delta^{1-\delta} |F_N(x)|\,dx \underset{N\to\infty}{\longrightarrow} 0$$

Fix $\delta > 0$. Then

$$\delta \leq x \leq 1 - \delta \implies \pi\delta \leq \pi x \leq \pi - \pi\delta$$

$\implies \exists c_\delta > 0$ s.t.

$$|\sin(\pi x)|^2 \geq c_\delta \qquad \forall x \in [\delta, 1-\delta]$$

So

$$\begin{aligned} \int_\delta^{1-\delta} |F_N(x)|\,dx &= \frac{1}{N}\int_\delta^{1-\delta}\left|\frac{\sin(\pi N x)}{\sin(\pi x)}\right|^2 dx \\ &\leq \frac{1}{N}\int_\delta^{1-\delta}\frac{1}{c_\delta}\,dx \\ &= \frac{1}{N}\frac{1-2\delta}{c_\delta} \underset{N\to\infty}{\longrightarrow} 0 \end{aligned}$$

This proves that $\{F_N\}_{N\geq 1}$ form an approximation to the identity. □

§56.2 Topology Addendum

Lemma 56.1

Let (X,d) be a metric space. A set $A \subseteq X$ is dense in X if and only if $A \cap W \neq \emptyset$ for every non-empty open set $W \subseteq X$.

Proof. " $\implies$ " Let $A \subseteq X$ be such that $\overline{A} = X$. Assume, towards a contradiction that $\exists \emptyset \neq W = \mathring{W} \subseteq X$ s.t.

$$\begin{aligned} A \cap W = \emptyset &\implies W \subseteq {}^cA \\ &\implies W = \mathring{W} \subseteq \widehat{{}^cA}^{\circ} = {}^c(\overline{A}) = {}^cX = \emptyset \end{aligned}$$

which is a contradiction as $W \neq \emptyset$.
" $\impliedby$ " Assume, towards a contradiction, that

$$\overline{A} \neq X \implies \left. \begin{matrix} {}^c(\overline{A}) \neq \emptyset \\ {}^c(\overline{A}) = \widehat{{}^cA}^{\circ} \end{matrix} \right\} \implies \widehat{{}^cA}^{\circ} \neq \emptyset$$

which implies

$$\exists x \in {}^cA \text{ and } \exists r > 0 \text{ s.t. } B_r(x) \subseteq {}^cA$$

So $\underbrace{B_r(x)}_{\neq \emptyset \text{ open}} \cap A \neq \emptyset$ – contradiction! □

Theorem 56.2

Let (X,d) be a complete metric space. Then X has the property of Baire, that is, for every sequence $\{A_n\}_{n \geq 1}$ of open dense sets we have

$$\overline{\bigcap_{n \geq 1} A_n} = X$$

Proof. Using the lemma, it suffices to show

$$\bigcap_{n \geq 1} A_n \cap W \neq \emptyset \qquad \forall \emptyset \neq W = \mathring{W} \subseteq X$$

Fix $\emptyset \neq W = \mathring{W} \subseteq X$.

$$\overline{A_1} = x \implies A_1 \cap W \neq \emptyset \implies \exists x_1 \in \underbrace{A_1 \cap W}_{\text{open}} \implies \exists 0 < r_1 < 1 \text{ s.t.}$$

$$K_{r_1}(x_1) = \{y \in X : d(y, x_1) \leq r_1\} \subseteq A_1 \cap W$$

$$\overline{A_2} = X \implies A_2 \cap B_{r_1}(x_1) \neq \emptyset \implies \exists x_2 \in \underbrace{A_2 \cap B_{r_1}(x_1)}_{\text{open}} \implies \exists 0 < r_2 < \frac{1}{2} \text{ s.t.}$$

$$K_{r_2}(x_2) \subseteq A_1 \cap B_{r_1}(x_1)$$

Proceeding inductively, we find a sequence $\{x_n\}_{n \geq 1} \subseteq X$ and $\{r_n\}_{n \geq 1}$ s.t.

$$\begin{cases} 0 < r_n < \frac{1}{n} & \forall n \geq 1 \\ K_{r_{n+1}}(x_{n+1}) \subseteq A_{n+1} \cap B_{r_n}(x_n) \subseteq K_{r_n}(x_n) & \forall n \geq 1 \end{cases}$$

Note that $\{K_{r_n}(x_n)\}_{n\geq 1}$ is a sequence of nested closed sets whose diameters decrease to zero. As (X,d) is complete, we find

$$\bigcap_{n\geq 1} K_{r_n}(x_n) = \{x\}$$

for some $x \in X$. In addition,

$$\{x\} = \bigcap_{n\geq 1} K_{r_n}(x_n) \subseteq A_1 \cap W \cap \bigcap_{n\geq 2} A_n \cap B_{r_{n-1}}(x_{n-1}) \subseteq \left(\bigcap_{n\geq 1} A_n\right) \cap W$$

which implies $\left(\bigcap_{n\geq 1} A_n\right) \cap W \neq \emptyset$. ☐

Lemma 56.3

Let (X,d) be a metric space. Then the following are equivalent:

1. For every $\{A_n\}_{n\geq 1}$ of open dense sets we have $\overline{\bigcap_{n\geq 1} A_n} = X$.
2. For every $\{F_n\}_{n\geq 1}$ of closed sets with empty interiors, we have

$$\overbrace{\bigcup_{n\geq 1} F_n}^{\circ} = \emptyset$$

Proof. Left as exercise. ☐

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§57.1 Topology Addendum (Cont'd)

Lemma 57.1

Let (X,d) be a metric space that has the Baire property. If $\emptyset \neq W = \mathring{W} \subseteq X$, then W has the Baire property.

Proof. Fix $\emptyset \neq W = \mathring{W} \subseteq X$. Let $\{D_n\}_{n\geq 1}$ be open dense sets in W.
D_n open in $W \implies \exists G_n$ open in X s.t. $D_n = G_n \cap W$ open in X as G_n and W are open.
D_n dense in $W \implies \overline{D_n} \cap W = W \implies W \subseteq \overline{D_n} \implies \overline{W} \subseteq \overline{D_n}$.
Define $A_n = D_n \cup {}^c(\overline{W})$ open in X.

$$\overline{A_n} = \overline{D_n \cup {}^c(\overline{W})} = \overline{D_n} \cup \overline{{}^c(\overline{W})} = \overline{D_n} \cup {}^c\left(\mathring{\overline{W}}\right) \supseteq \overline{W} \cup {}^c(\overline{W}) = X$$

Thus $\{A_n\}_n$ are dense open sets in X and as X has the Baire property,

$$\overline{\bigcap_{n\geq 1} A_n} = X$$

Then,

$$X = \overline{\bigcap_{n\geq 1} A_n} = \overline{\bigcap_{n\geq 1} [D_n \cup {}^c(\overline{W})]} = \overline{\left(\bigcap_{n\geq 1} D_n\right) \cup {}^c(\overline{W})} = \overline{\bigcap_{n\geq 1} D_n} \cup {}^c\left(\mathring{\overline{W}}\right)$$

which implies

$$\begin{aligned} W &= \left[\overline{\bigcap_{n\geq 1} D_n} \cup {}^c\left(\mathring{\overline{W}}\right)\right] \cap W \\ &= \left[\overline{\textstyle\bigcap_{n\geq 1} D_n} \cap W\right] \cup \left[{}^c\left(\mathring{\overline{W}}\right) \cap W\right] \\ \mathring{\overline{W}} \supseteq \mathring{W} = W &\implies {}^c\left(\mathring{\overline{W}}\right) \subseteq {}^cW \implies {}^c\left(\mathring{\overline{W}}\right) \cap W = \emptyset \end{aligned}$$

$\implies \overline{\bigcap_{n\geq 1} D_n} \cap W = W$ i.e. $\bigcap_{n\geq 1} D_n$ is dense in W. □

Theorem 57.2

Let (X,d) be a metric space with the Baire property. Let $f_n : X \to \mathbb{R}$ be continuous function that converges pointwise to a function $f : X \to \mathbb{R}$. Then the set

$$C = \{x \in X : \ f \text{ is continuous at } x\} \text{ is dense in } X$$

Proof. We can observe that it suffices to prove the theorem under the additional hypothesis

$$|f_n(x)| \leq 1 \qquad \forall x \in X \quad \forall n \geq 1$$

Indeed, if $\{f_n\}_{n\geq 1}$ is as in the theorem, then we consider

$\phi : \mathbb{R} \to (-1,1)$, $\quad \phi(x) = \dfrac{x}{1+|x|}$ continuous, bijective, with the inverse $\phi^{-1}(y) = \dfrac{y}{1-|y|}$

So $\phi \circ f_n : X \to (-1,1)$ is continuous and $|\phi \circ f_n(x)| \leq 1$ for all $n \geq 1$ and $x \in X$. Also, $f_n \underset{n\to\infty}{\longrightarrow} f$ pointwise $\implies$ $\phi \circ f_n \underset{n\to\infty}{\longrightarrow} \phi \circ f$ pointwise. If the theorem holds with the additional uniform boundedness hypothesis, we get

$$\left.\begin{matrix} \{x \in X :\ \phi \circ f \text{ is continuous at } x\} \\ \{x \in X :\ f \text{ is continuous at } x\} \end{matrix}\right\} \text{ is dense in } X$$

So without the loss of generality, we assume

$$|f_n(x)| \leq 1 \qquad \forall n \geq 1 \quad \forall x \in X \tag{1}$$

Then,

$$\begin{aligned} C &= \{x \in X :\ f \text{ is continuous at } x\} \\ &= \{x \in X :\ \omega(f,x) = 0\} \\ &= \bigcap_{n\geq 1} \underbrace{\left\{x \in X :\ \omega(f,x) < \frac{1}{n}\right\}}_{=:G_n \text{ open in } X} = \bigcap_{n\geq 1} G_n \end{aligned}$$

As X has the Baire property, to prove $\overline{C} = X$ it suffices to show $\overline{G_n} = X$ $\forall n \geq 1$. Fix $N \geq 1$. We will show that $G_N = \left\{x \in X : \omega(f,x) < \frac{1}{N}\right\}$ is dense in X. By a lemma from last lecture, it suffices to show

$$G_N \cap W \neq \emptyset \qquad \forall \emptyset \neq W = \mathring{W} \subseteq X$$

Fix $\emptyset \neq W = \mathring{W} \subseteq X$. For $n \geq 1$ and $x \in X$, we define

$$u_n(x) = \inf_{m\geq n} f_m(x) \quad \text{and} \quad v_n(x) = \sup_{m\geq n} f_m(x)$$

Then $\{u_n(x)\}_{n\geq 1}$ is increasing and $\{v_n(x)\}_{n\geq 1}$ is decreasing. As $\lim_{n\to\infty} f_n(x) = f(x)$, we have

$$\lim_{n\to\infty} u_n(x) = f(x) = \lim_{n\to\infty} v_n(x) \tag{2}$$

For $n \geq 1$, let

$$\begin{aligned} F_n &= \left\{x \in X :\ v_n(x) - u_n(x) \leq \frac{1}{4N}\right\} \\ &= \left\{x \in X :\ \sup_{m\geq n} f_m(x) - \inf_{l\geq n} f_l(x) < \frac{1}{4N}\right\} \\ &= \left\{x \in X :\ \sup_{m,l\geq n} [f_m(x) - f_l(x)] \leq \frac{1}{4N}\right\} \\ &= \bigcap_{m,l\geq n} \left\{x \in X :\ f_m(x) - f_l(x) \leq \frac{1}{4N}\right\} \\ &\overset{(1)}{=} \bigcap_{m,l\geq n} (f_m - f_l)^{-1}\left(\left[-2, \frac{1}{4N}\right]\right) \end{aligned}$$

$f_m - f_l$ is continuous $\forall m, l \geq n$ and $\left[-2, \frac{1}{4N}\right]$ is closed, so

$$(f_m - f_l)^{-1}\left(\left[-2, \frac{1}{4N}\right]\right) \text{ is closed} \qquad \forall m, l \geq n$$

So F_n is closed in X for all $n \geq 1$. Also,

$$X = \bigcup_{n\geq 1} F_n \qquad \text{by (2)}$$

So

$$\left.\begin{array}{l} W = \left(\bigcup_{n\geq 1} F_n\right) \cap W = \bigcup_{n\geq 1}(F_n \cap W) \\ W = \mathring{W} \neq \emptyset \\ W \text{ has the Baire property} \end{array}\right\} \implies \exists n_1 \in \mathbb{N} \text{ s.t. } \widehat{F_{n_1} \cap W}^{\circ} \neq \emptyset$$

Let $x_0 \in \widehat{F_{n_1} \cap W}^{\circ}$ and let $\delta > 0$ s.t. $B_\delta(x_0) \subseteq F_{n_1} \cap W$. As f_{n_1} is continuous at x_0, shrinking δ if necessary, we may assume

$$\omega(f_{n_1}, B_\delta(x_0)) < \frac{1}{4N}$$

We compute

$$\begin{aligned} \omega(f, x_0) \leq \omega(f, B_\delta(x_0)) &= \sup_{x\in B_\delta(x_0)} f(x) - \inf_{y\in B_\delta(x_0)} f(y) \\ &= \sup_{x,y\in B_\delta(x_0)} [f(x) - f(y)] \\ &\leq \sup_{x,y\in B_\delta(x_0)} [v_{n_1}(x) - u_{n_1}(y)] \\ &= \sup_{x,y\in B_\delta(x_0)} [v_{n_1}(x) - u_{n_1}(x) + v_{n_1}(y) - u_{n_1}(y) + u_{n_1}(x) - v_{n_1}(y)] \\ (B_\delta(x_0) \subseteq F_{n_1}) &\leq \frac{1}{4N} + \frac{1}{4N} + \sup_{x,y\in B_\delta(x_0)} [u_{n_1}(x) - v_{n_1}(y)] \\ &\leq \frac{1}{2N} + \sup_{x,y\in B_\delta(x_0)} [f_{n_1}(x) - f_{n_1}(y)] \\ &= \frac{1}{2N} + \omega(f_{n_1}; B_\delta(x_0)) \\ &\leq \frac{1}{2N} + \frac{1}{4N} < \frac{1}{N} \end{aligned}$$

This proves $x_0 \in G_n \cap W \implies G_N \cap W \neq \emptyset$. As $\emptyset \neq W = \mathring{W} \subseteq X$ was arbitrary, we conclude G_N is dense in X. □