# Math 131BH – Honors Real Analysis II

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This is math 131BH – Honors Real Analysis II, and it is instructed by Professor Visan. It's the second class in the undergrad real analysis sequence at UCLA. We meet weekly on MWF from 10:00 – 10:50 am for online lectures. Similar to 131AH, there are two textbooks associated to the course, *Principles of Mathematical Analysis* by *Rudin* and *Metric Spaces* by *Copson*. You can find the previous analysis lecture notes along with the other course notes through my github. Please email me if you notice any significant mathematical errors/typos that needs to be addressed. Thank you, and I hope you find this helpful for your study!

## Contents

## List of Theorems

## List of Definitions

# §1 Lec 1: Mar 29, 2021

## §1.1 Compactness

**Definition 1.1** (Open Cover) — Let $(X, d)$ be a metric space and let $A \subseteq X$. An open cover of $A$ is a family $\{G_i\}_{i \in I}$ of open sets in $X$ such that

$$A \subseteq \bigcup_{i \in I} G_i$$

The open cover is called finite if the cardinality of $I$ is finite. If it's not finite, the open cover is called infinite.

**Definition 1.2** (Compactness & Precompactness) — Let $(X, d)$ be a metric space and let $K \subseteq X$.

1. We say that $K$ is a compact set if every open cover $\{G_i\}_{i \in I}$ of $K$ admits a finite subcover, that is,

$$\exists n \geq 1 \text{ and } \exists i_1, \ldots, i_n \in I \text{ s.t. } K \subseteq \bigcup_{j=1} G_{i_j}$$

2. We say that a set $A \subseteq X$ is precompact if $\overline{A}$ is compact.

**Lemma 1.3**
Let $(X, d)$ be a metric space and let $\emptyset \neq Y \subseteq X$. We equip $Y$ with the induced metric $d_1 : Y \times Y \to \mathbb{R}$, $d_1(y_1, y_2) = d(y_1, y_2)$. Let $K \subseteq Y \subseteq X$. The followings are equivalent:

1. $K$ is compact in $(X, d)$.
2. $K$ is compact in $(Y, d_1)$.

*Proof.* 1) $\implies$ 2) Assume $K$ is compact in $(X, d)$. Let $\{V_i\}_{i \in I}$ be a family of open sets in $(Y, d_1)$ s.t.

$$K \subseteq \bigcup_{i \in I} V_i$$

For $i \in I$ fixed, $V_i$ is open in $(Y, d_1) \implies \exists G_i \subseteq X$ open in $(X, d)$ s.t.

$$V_i = G_i \cap Y$$

Then

$$\left.\begin{array}{l} K \subseteq \bigcup_{i\in I} V_i \subseteq \bigcup_{i\in I} G_i \\ K \text{ compact in } (X,d) \end{array}\right\} \implies \exists n \geq 1 \text{ and } \exists i_1,\dots,i_n \in I \text{ s.t.}$$

$$\left.\begin{array}{l} K \subseteq \bigcup_{j=1}^n G_{i_j} \\ K \subseteq Y \end{array}\right\} \implies K \subseteq \left(\bigcup_{j=1}^n G_{i_j}\right) \cap Y = \bigcup_{j=1}^n \left(G_{i_j} \cap Y\right) = \bigcup_{j=1}^n V_{i_j}$$

So $K$ is compact in $(Y, d_1)$.
2) $\implies$ 1) Assume $K$ is compact in $(Y, d_1)$. Let $\{G_i\}_{i\in I}$ be a family of open sets in $(X,d)$ s.t.

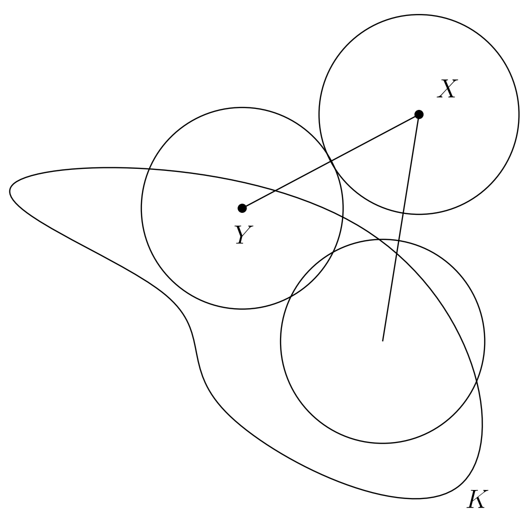
$$\left.\begin{array}{l} K \subseteq \bigcup_{i\in I} G_i \\ K \subseteq Y \end{array}\right\} \implies \left.\begin{array}{l} K \subseteq \left(\bigcup_{i\in I} G_i\right) \cap Y = \bigcup_{i\in I} \underbrace{(G_i \cap Y)}_{\text{open in } Y} \\ K \text{ is compact in } (Y, d_1) \end{array}\right\} \implies$$

$$\implies \exists n \geq 1 \text{ and } \exists i_1,\dots,i_n \in I \text{ s.t. } K \subseteq \bigcup_{j=1}^n \left(G_{i_j} \cap Y\right) \subseteq \bigcup_{j=1}^n G_{i_j}.$$ □

**Proposition 1.4**

Let $(X,d)$ be a metric space and let $K \subseteq X$ be compact. Then $K$ is closed and bounded.

*Proof.* Let's prove $K$ is closed. We'll show ${}^cK$ is open.
**Case 1:** ${}^cK = \emptyset$. This is open.
**Case 2:** ${}^cK \neq \emptyset$. Let $x \in {}^cK$
For $y \in K$ let $r_y = \frac{d(x,y)}{2}$. Note $r_y > 0$ (since $x \in {}^cK$ and $y \in K$).




Note

$$\left.\begin{array}{l} K \subseteq \bigcup_{y\in K} \underbrace{B_{r_y}(y)}_{\text{open}} \\ K \text{ is compact} \end{array}\right\} \implies \exists n \geq 1 \text{ and } \exists y_1,\dots,y_n \in K \text{ s.t. } K \subseteq \bigcup_{j=1}^n B_{r_j}(y_j)$$

where we use the shorthand $r_j = r_{y_j}$.
Let $r = \min_{1\le j\le n} r_j > 0$.
By construction, $B_r(x) \cap B_{r_j}(y_j) = \emptyset \quad \forall 1 \le j \le n$.

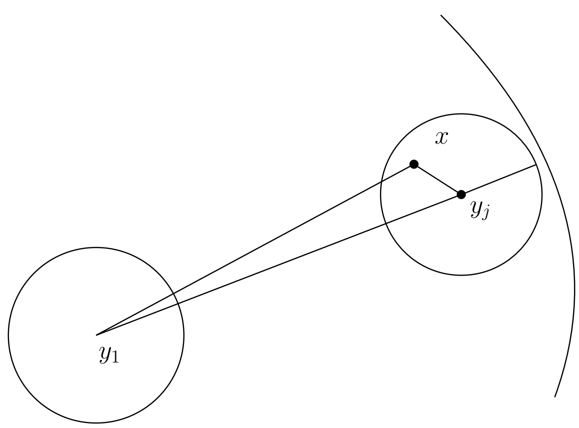
$$\implies B_r(x) \subseteq {}^cB_{r_j}(y_j) \quad \forall 1 \le j \le n$$
$$\implies B_r(x) \subseteq \bigcap_{j=1}^{n} {}^cB_{r_j}(y_j) = {}^c\left(\bigcup_{j=1}^{n} B_{r_j}(y_j)\right) \subseteq {}^cK$$
$$\implies \left.\begin{array}{l} x \in \mathring{\widehat{{}^cK}} \\ x \in {}^cK \text{ was arbitrary} \end{array}\right\} \implies {}^cK = \mathring{\widehat{{}^cK}}$$

Let's show $K$ is bounded. Note

$$\left.\begin{array}{l} K \subseteq \bigcup_{y\in K} \underbrace{B_1(y)}_{\text{open}} \\ K \text{ compact} \end{array}\right\} \implies \exists n \ge 1 \text{ and } \exists y_1, \dots, y_n \in K \text{ s.t. } K \subseteq \bigcup_{j=1}^{n} B_1(y_j)$$

For $2 \le j \le n$, let $r_j = d(y_1, y_j) + 1$.

**Claim 1.1.** $B_1(y_j) \subseteq B_{r_j}(y_1)$




Indeed, if $x \in B_1(y_j) \implies d(x, y_j) < 1$. By the triangle inequality

$$d(y_1, x) \le d(y_j, x) + d(y_1, y_j) < 1 + d(y_1, y_j) = r_j \implies x \in B_{r_j}(y_1)$$

So with $r = \max_{2\le j\le n} r_j$,

$$K \subseteq \bigcup_{j=1}^{n} B_1(y_j) \subseteq B_r(y_1)$$

□

**Proposition 1.5**

Let $(X, d)$ be a metric space and let $F \subseteq K \subseteq X$ such that $F$ is closed in $X$ and $K$ is compact. Then $F$ is compact.

*Proof.* Let $\{G_i\}_{i\in I}$ be a family of open sets in $X$ s.t.

$$F \subseteq \bigcup_{i\in I} G_i$$

Then

$$\left.\begin{array}{l} K \subseteq F \cup {}^cF \subseteq \bigcup_{i\in I} G_i \cup \underbrace{{}^cF}_{\text{open in } X} \\ K \text{ compact} \end{array}\right\} \implies$$

$\implies \exists n \geq 1$ and $\exists i_1, \dots, i_n \in I$ s.t.

$$\left.\begin{array}{l} K \subseteq \bigcup_{j=1}^n G_{i_j} \cup {}^cF \\ F \subseteq K \end{array}\right\} \implies F = \left(\bigcup_{j=1}^n G_{i_j} \cup {}^cF\right) \cap F \subseteq \bigcup_{j=1}^n G_{i_j}$$

So $F$ is compact. □

**Corollary 1.6**
Let $(X, d)$ be a metric space and let $F \subseteq X$ be closed and let $K \subseteq X$ be compact. Then $K \cap F$ is compact.

*Proof.* $K$ is compact. So

$$\left.\begin{array}{r} K \text{ closed} \\ F \text{ closed} \end{array}\right\} \implies \left.\begin{array}{l} K \cap F \text{ is closed} \\ K \cap F \subseteq K \text{ compact} \end{array}\right\} \implies K \cap F \text{ is compact}$$

□

## §1.2 Sequential Compactness

**Definition 1.7** (Sequential Compactness) — Let $(X, d)$ be a metric space. A set $K \subseteq X$ is called sequentially compact if every sequence $\{x_n\}_{n\geq 1} \subseteq K$ admits a subsequence that converges in $K$.

# §2 Lec 2: Mar 31, 2021

## §2.1 Sequential Compactness (Cont'd)

**Theorem 2.1** (Bolzano – Weierstrass)
Let $(X, d)$ be a metric space and let $K \subseteq X$ be infinite. The following are equivalent:

1. $K$ is sequentially compact.
2. For every infinite $A \subseteq K$ we have $A' \cap K \neq \emptyset$.

*Proof.* 1) $\implies$ 2) Let $A \subseteq K$ be infinite. As every infinite set has a countable subset we can find a sequence $\{a_n\}_{n\geq 1} \subseteq A$ such that $a_n \neq a_m \, \forall n \neq m$. As $K$ is sequentially compact, $\exists \{a_{k_n}\}_{n\geq 1}$ subsequence of $\{a_n\}_{n\geq 1}$ s.t.

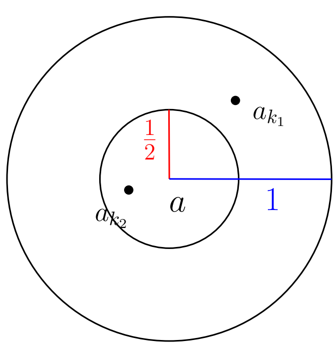
$$a_{k_n} \xrightarrow[n\to\infty]{d} a \in K$$

**Claim 2.1.** $a \in A' \iff \forall r > 0 \; B_r(a) \cap A \setminus \{a\} \neq \emptyset$.

Indeed, fix $r > 0$.

$$a_{k_n} \xrightarrow[n\to\infty]{d} a \implies \exists n_r \in \mathbb{N} \text{ s.t. } d(a, a_{k_n}) < r \quad \forall n \geq n_r$$

As $a_n \neq a_m \, \forall n \neq m$, $\exists n_0 \geq n_r$ s.t. $a_{k_{n_0}} \neq a$. Then $a_{k_{n_0}} \in B_r(a) \cap A \setminus \{a\}$. We get $a \in A' \cap K$.
2) $\implies$ 1) Let $\{a_n\}_{n\geq 1} \subseteq K$. We distinguish two cases:
**Case 1:** The sequence $\{a_n\}_{n\geq 1}$ contains a constant subsequence. That subsequence converges to an element in $K$.
**Case 2:** $\{a_n\}_{n\geq 1}$ does not contain a constant subsequence. Then $A = \{a_n : n \geq 1\}$ is infinite and $A \subseteq K$. So $A' \cap K \neq \emptyset$. Let $a \in A' \cap K$. Then $\exists \{a_{k_n}\}_{n\geq 1}$ subsequence of $\{a_n\}_{n\geq 1}$ s.t. $a_{k_n} \xrightarrow[n\to\infty]{d} a$.




$$B_1(a) \cap A \setminus \{a\} \neq \emptyset$$

$$B_{\min\left\{\frac{1}{2}, d(a, a_{k_1})\right\}}(a) \cap A \setminus \{a\} \neq \emptyset$$

$$k_2 > k_1$$

$\square$

**Theorem 2.2**

Let $(X, d)$ be a metric space and let $K \subseteq X$ be compact. Then $K$ is sequentially compact.

*Proof.* If $K$ is finite, then any sequence $\{x_n\}_{n\geq 1} \subseteq K$ will have a constant subsequence.

Assume now $K$ is infinite. We will use the Bolzano – Weierstrass theorem. It suffices to prove that for any infinite $A \subseteq K$ we have $A' \cap K \neq \emptyset$.

$$\left.\begin{array}{l}\text{Note } A \subseteq K \text{ then } A' \subseteq K' \\ K \text{ compact} \implies K \text{ closed} \implies K' \subseteq K\end{array}\right\} \implies A' \subseteq K \implies A' \cap K = A'$$

We argue by contradiction. Assume $A' = \emptyset$. Then for $x \in K$ we have $x \notin A' \implies \exists r_x > 0$ s.t. $B_{r_x}(x) \cap A \setminus \{x\} = \emptyset$. So

$$\left.\begin{array}{l}K \subseteq \bigcup_{x\in K} \underbrace{B_{r_x}(x)}_{\text{open}} \\ K \text{ compact}\end{array}\right\} \implies \exists n \geq 1 \text{ and } \exists x_1, \ldots, x_n \in K \text{ s.t.}$$

$$K \subseteq \bigcup_{j=1}^{n} B_{r_j}(x_j) \text{ where } r_j = r_{x_j}$$

In particular,

$$\left.\begin{array}{l}A = \left(\bigcup_{j=1}^{n} B_{r_j}(x_j)\right) \cap A = \bigcup_{j=1}^{n} \left[B_{r_j}(x_j) \cap A\right] \\ \text{By construction, } B_{r_j}(x_j) \cap A \subseteq \{x_j\}\end{array}\right\} \implies \underbrace{A}_{\text{infinite}} \subseteq \underbrace{\bigcup_{j=1}^{n} \{x_j\}}_{\text{finite}}$$

– Contradiction! So $A' \neq \emptyset$. □

**Proposition 2.3**

Let $(X, d)$ be a metric space and let $K \subseteq X$ be sequentially compact. Then $K$ is closed and bounded.

*Proof.* Let's show $K$ is closed $\iff K = \overline{K}$.

We know $K \subseteq \overline{K}$. We need to show $\overline{K} \subseteq K$. Let $x \in \overline{K} \implies \exists \{x_n\}_{n\geq 1} \subseteq K$ s.t. $x_n \xrightarrow[n\to\infty]{d} x$.

$K$ sequentially compact $\implies \exists \{x_{k_n}\}_{n\geq 1}$ subsequence of $\{x_n\}_{n\geq 1}$ s.t.

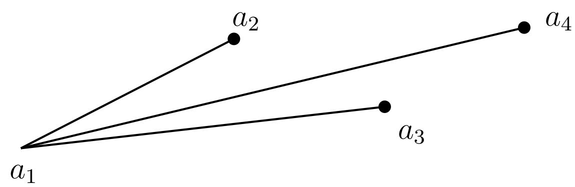
$$\left.\begin{array}{l}x_{k_n} \xrightarrow[n\to\infty]{d} y \in K \\ x_n \xrightarrow[n\to\infty]{d} x \implies x_{k_n} \xrightarrow[n\to\infty]{d} x \\ \text{Limits of convergent sequences are unique}\end{array}\right\} \implies x = y \in K$$

As $x \in \overline{K}$ was arbitrary, we get $\overline{K} \subseteq K$.

Let's show $K$ is bounded. We argue by contradiction. Assume $K$ is not bounded. Let $a_1 \in K$.

$$K \text{ not bounded} \implies K \nsubseteq B_1(a_1) \implies \exists a_2 \in K \text{ s.t. } d(a_1,a_2) \geq 1$$
$$K \text{ not bounded} \implies K \nsubseteq B_{1+d(a_1,a_2)}(a_1) \implies \exists a_3 \in K \text{ s.t. } d(a_1,a_3) \geq 1 + d(a_1,a_2)$$

Proceeding inductively, we find a sequence $\{a_n\}_{n\geq 1} \subseteq K$ s.t. $d(a_1,a_{n+1}) \geq 1 + d(a_1,a_n)$.




By construction,

$$|d(a_1,a_m) - d(a_1,a_n)| \geq |n-m| \quad \forall n,m \geq 1$$

By the triangle inequality,

$$d(a_n,a_m) \geq |d(a_1,a_n) - d(a_1,a_m)| \geq |n-m| \quad \forall n,m \geq 1$$

This sequence cannot have a convergent (Cauchy) subsequence, thus contradiction the hypothesis that $K$ is sequentially compact. So $K$ is bounded. □

**Definition 2.4** (Totally Bounded) — Let $(X,d)$ be a metric space. A set $A \subseteq X$ is totally bounded if for every $\varepsilon > 0$, $A$ can be covered by finitely many balls of radius $\varepsilon$.

**Remark 2.5.**

1. $A$ totally bounded $\implies$ $A$ bounded.

   Indeed, taking $\varepsilon = 1$, $\exists n \geq 1$ and $\exists x_1, \ldots, x_n \in X$ s.t.

   $$A \subseteq \bigcup_{j=1}^{n} B_1(x_j) \subseteq B_r(x_1)$$

   where $r = 1 + \max_{2 \leq j \leq n} d(x_1, x_j)$.

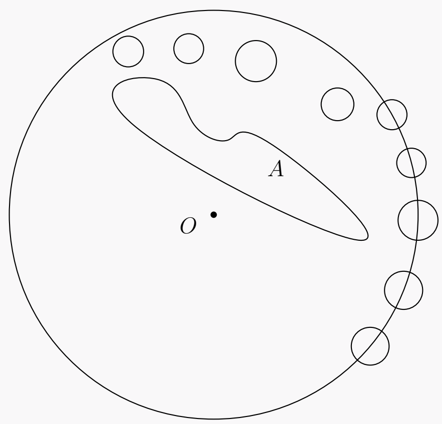
2. $A$ bounded $\not\Longrightarrow$ $A$ totally bounded.

   Consider $\mathbb{N}$ equipped with the discrete metric

   $$d(n,m) = \begin{cases} 0, \; n = m \\ 1, \; n \neq m \end{cases}$$

   Then $\mathbb{N} = B_2(1)$, but $\mathbb{N}$ cannot be covered by finitely many balls of radius $\frac{1}{2}$ since $B_{\frac{1}{2}}(n) = \{n\}$.

3. On $(\mathbb{R}^n, d_2)$, $A$ bounded $\implies$ $A$ totally bounded. Indeed, $A$ bounded $\implies$ $A \subseteq B_R(0)$ for some $R > 0$. $B_R(0)$ can be covered by $10^6 \left(\frac{R}{\varepsilon}\right)^n$ many balls of radius $\varepsilon$.



# §3 Lec 3: Apr 2, 2021

## §3.1 Heine – Borel Theorem

**Theorem 3.1**

Let $(X, d)$ be a metric space and let $K \subseteq X$. The following are equivalent:

1. $K$ is sequentially compact.
2. $K$ is complete and totally bounded.

*Proof.* 1) $\implies$ 2) Let's show $K$ is complete. Let $\{x_n\}_{n\geq 1}$ be a Cauchy sequence with $x_n \in K \quad \forall n \geq 1$.
$K$ sequentially compact $\implies \exists \{x_{k_n}\}_{n\geq 1}$ subsequence of $\{x_n\}_{n\geq 1}$ s.t.

$$\left.\begin{aligned} &x_{k_n} \xrightarrow[n\to\infty]{d} y \in K \\ &\{x_n\}_{n\geq 1} \text{ is Cauchy} \end{aligned}\right\} \implies x_n \xrightarrow[n\to\infty]{d} y \in K$$

As $\{x_n\}_{n\geq 1} \subseteq K$ was arbitrary, we get that $K$ is complete.
Let's show $K$ is totally bounded. Fix $\varepsilon > 0$ and $a_1 \in K$.

- If $K \subseteq B_\varepsilon(a_1)$, then $K$ is totally bounded.
- If $K \nsubseteq B_\varepsilon(a_1)$, then $\exists a_2 \in K$ s.t. $d(a_1, a_2) \geq \varepsilon$
- If $K \subseteq B_\varepsilon(a_1) \cup B_\varepsilon(a_2)$, then $K$ is totally bounded.
- If $K \nsubseteq B_\varepsilon(a_1) \cup B_\varepsilon(a_2)$, then $\exists a_3 \in K$ s.t. $d(a_1, a_3) \geq \varepsilon$ and $d(a_2, a_3) \geq \varepsilon$.

We distinguish two cases:
**Case 1:** The process terminates in finitely many steps $\implies K$ is totally bounded.
**Case 2:** The process does not terminate in finitely many steps. Then we find $\{a_n\}_{n\geq 1} \subseteq K$ s.t. $d(a_n, a_m) \geq \varepsilon \quad \forall n \neq m$. This sequence does not admit a convergent subsequence, contradicting the fact that $K$ is sequentially compact.
2) $\implies$ 1) Let $\{a_n\}_{n\geq 1} \subseteq K$. $K$ totally bounded $\implies \mathcal{J}_1$ finite and $\left\{x_j^{(1)}\right\}_{j\in\mathcal{J}_1} \subseteq X$ s.t.

$$\left.\begin{aligned} &K \subseteq \textstyle\bigcup_{j\in\mathcal{J}_1} B_1(x_j^{(1)}) \\ &\{a_n\}_{n\geq 1} \subseteq K \end{aligned}\right\} \implies \exists j_1 \in \mathcal{J}_1 \text{ s.t. } \left|\left\{n : a_n \in B_1(x_{j_1}^{(1)})\right\}\right| = \aleph_0$$

Let $\left\{a_n^{(1)}\right\}_{n\geq 1}$ be the corresponding subsequence.
$K$ totally bounded $\implies \exists \mathcal{J}_2$ finite and $\left\{x_j^{(2)}\right\}_{j\in\mathcal{J}_2} \subseteq X$ s.t.

$$\left.\begin{aligned} &K \subseteq \textstyle\bigcup_{j\in\mathcal{J}_2} B_{\frac{1}{2}}(x_j^{(2)}) \\ &\left\{a_n^{(1)}\right\}_{n\geq 1} \subseteq K \end{aligned}\right\} \implies \exists j_2 \in \mathcal{J}_2 \text{ s.t. } \left|\left\{n : a_n^{(1)} \in B_{\frac{1}{2}}(x_{j_2}^{(2)})\right\}\right| = \aleph_0$$

Let $\left\{a_n^{(2)}\right\}_{n\geq 1}$ denote the corresponding subsequence.
We proceed inductively. We find that $\forall k \geq 1$

- $\left\{a_n^{(k+1)}\right\}_{n\geq 1}$ subsequence of $\left\{a_n^{(k)}\right\}_{n\geq 1}$
- $\left\{a_n^{(k)}\right\}_{n\geq 1} \subseteq B_{\frac{1}{k}}\left(x_{j_k}^{(k)}\right)$ for some $x_{j_k}^{(k)} \in X$.

We consider the subsequence $\left\{a_n^{(n)}\right\}_{n\geq 1}$ of $\{a_n\}_{n\geq 1}$.

$$\begin{aligned}
\left\{a_n^{(1)}\right\}_{n\geq 1} &= \left(a_1^{(1)}, \quad a_2^{(1)}, \quad a_3^{(1)}, \quad \dots\right) \\
\left\{a_n^{(2)}\right\}_{n\geq 1} &= \left( \qquad a_1^{(2)}, \quad a_2^{(2)}, \quad a_3^{(2)}, \quad \dots\right) \\
\left\{a_n^{(3)}\right\}_{n\geq 1} &= \left( \qquad\qquad a_1^{(3)}, \quad a_2^{(3)}, \qquad a_3^{(3)}, \quad \dots\right)
\end{aligned}$$

For $n, m \geq k$ the $a_n^{(n)}, a_m^{(m)}$ belong to the subsequence $\left\{a_n^{(k)}\right\}_{n\geq 1}$. In particular,

$$d(a_n^{(n)}, a_m^{(m)}) \leq d(a_n^{(n)}, x_{j_k}^{(k)}) + d(a_m^{(m)}, x_{j_k}^{(k)}) < \frac{2}{k} \quad \forall n, m \geq k$$

This shows $\left\{a_n^{(n)}\right\}_{n\geq 1}$ is Cauchy and $K$ is complete, so $a_n^{(n)} \xrightarrow[n\to\infty]{d} a \in K$. As $\{a_n\}_{n\geq 1}$ was arbitrary, we get that $K$ is sequentially compact. □

**Lemma 3.2**

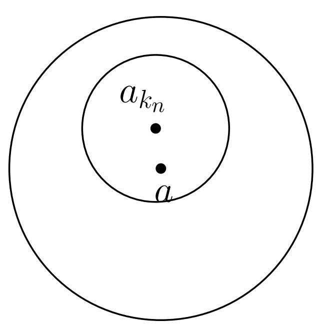
Let $(X, d)$ be a sequentially compact metric space. Let $\{G_i\}_{i\in I}$ be an open cover of $X$. Then there exists $\varepsilon > 0$ such that every ball of radius $\varepsilon$ is contained in at least one $G_i$.

*Proof.* We argue by contradiction. Then

$$\forall n \geq 1 \quad \exists a_n \in X \text{ s.t. } B_{\frac{1}{n}}(a_n) \text{ is not contained in any } G_i$$

$X$ is sequentially compact $\implies \exists \{a_{k_n}\}_{n\geq 1}$ subsequence of $\{a_n\}_{n\geq 1}$ s.t.

$$\begin{gathered}
a_{k_n} \xrightarrow[n\to\infty]{d} a \in X = \bigcup_{i\in I} G_i \implies \exists i_0 \in I \text{ s.t. } a \in G_{i_0} \\
G_{i_0} \text{ open } \implies \exists r > 0 \text{ s.t. } B_r(a) \subseteq G_{i_0} \\
a_{k_n} \xrightarrow[n\to\infty]{d} a \implies \exists n_1(r) \in \mathbb{N} \text{ s.t. } d(a_1, a_{k_n}) < \frac{r}{2} \forall n \geq n_1
\end{gathered}$$




Let $n_2(r)$ s.t. $n_2 > \frac{2}{r}$.

**Claim 3.1.** $\forall n \geq n_r = \max\{n_1, n_2\}$ we have $B_{\frac{1}{k_n}}(a_{k_n}) \subseteq B_r(a) \subseteq G_{i_0}$ thefore giving a contradiction!

Fix $x \in B_{\frac{1}{k_n}}(a_{k_n})$. Then

$$d(a,x) \leq d(x, a_{k_n}) + d(a_{k_n}, a) < \frac{1}{k_n} + \frac{r}{2} < \frac{r}{2} + \frac{r}{2} = r$$

□

**Theorem 3.3**

A sequentially compact metric space $(X, d)$ is compact.

*Proof.* Let $\{G_i\}_{i\in I}$ be an open cover of $X$. Let $\varepsilon$ be given by the previous lemma. $X$ sequentially compact $\implies$ $X$ totally bounded $\implies$ $\exists n \geq 1$ and

$$\left.\begin{array}{l} \exists x_1, \dots, x_n \in X \text{ s.t. } X = \bigcup_{j=1}^n B_\varepsilon(x_j) \\ \forall 1 \leq j \leq n \quad \exists i_j \in I \text{ s.t. } B_\varepsilon(x_j) \subseteq G_{i_j} \end{array}\right\} \implies X = \bigcup_{j=1}^n G_{i_j}$$

□

Collecting our results so far we obtain

**Theorem 3.4** (Heine – Borel)

Let $(X, d)$ be a metric space and let $K \subseteq X$. The following are equivalent:

1. $K$ is compact,
2. $K$ is sequentially compact,
3. $K$ is complete and totally bounded,
4. Every infinite subset of $K$ has an accumulation point in $K$.

**Remark 3.5.** In $\mathbb{R}^n$, $K$ is compact $\iff$ $K$ is closed and bounded.

**Definition 3.6** (Finite Intersection Property) — An infinite family $\{F_i\}_{i\in I}$ of closed sets is said to have the finite intersection property if $\forall \mathcal{J} \subseteq I$ finite we have

$$\bigcap_{j\in\mathcal{J}} F_j \neq \emptyset$$

**Theorem 3.7**

A metric space $(X, d)$ is compact if and only if every infinite family $\{F_i\}_{i\in I}$ of closed sets with the finite intersection property satisfies

$$\bigcap_{i\in I} F_i \neq \emptyset$$

*Proof.* " $\implies$ " We argue by contradiction. Assume $\exists \{F_i\}_{i\in I}$ closed sets with the finite intersection property s.t. $\bigcap_{i\in I} F_i = \emptyset$

$$\left.\begin{aligned} &X = {}^c\left(\textstyle\bigcap_{i\in I} F_i\right) = \textstyle\bigcup_{i\in I} \underbrace{{}^cF_i}_{\text{open}} \\ &X \text{ compact} \end{aligned}\right\} \implies \exists \mathcal{J} \subseteq I \text{ finite s.t. } X = \bigcup_{j\in\mathcal{J}} {}^cF_j$$

$$\implies \emptyset = {}^c\left(\bigcup_{j\in\mathcal{J}} {}^cF_j\right) = \bigcap_{j\in\mathcal{J}} F_j \text{ – Contradiction!}$$

" $\impliedby$ " We argue by contradiction. Assume $\exists \{G_i\}_{i\in I}$ open cover of $X$ that does not admit a finite subcover.

So $\forall \mathcal{J} \subseteq I$ finite $X \neq \bigcup_{j\in\mathcal{J}} G_j \implies \emptyset \neq \bigcap_{j\in\mathcal{J}} \underbrace{{}^cG_j}_{\text{closed}}$. So $\{{}^cG_i\}_{i\in I}$ is a family of closed sets with the finite intersection property. Then

$$\bigcap_{i\in I} {}^cG_i \neq \emptyset \implies \bigcup_{i\in I} G_i \neq X$$

Contradiction! □

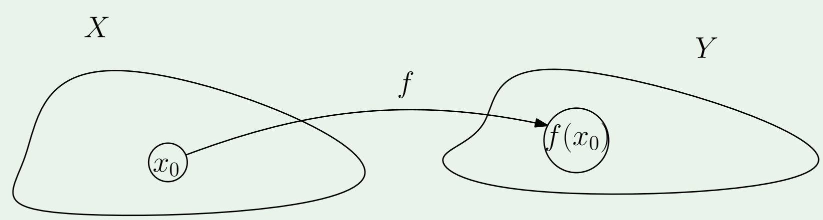
# §4 Lec 4: Apr 5, 2021

## §4.1 Continuity

**Definition 4.1** (Continuous Function) — Let $(X, d_X)$ and $(Y, d_Y)$ be two metric spaces. We say that a function $f : X \to Y$ is continuous at a point $x_0 \in X$ if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \text{ s.t. } d_X(x, x_0) < \delta \text{ then } d_Y(f(x), f(x_0)) < \varepsilon$$

We say $f$ is continuous (on $X$) if $f$ is continuous at every point in $X$.




**Remark 4.2.** $f : X \to Y$ is continuous at every isolated point in $X$. Indeed, if $x_0 \in X$ is isolated, then $\exists \delta > 0$ s.t. $B_\delta^X(x_0) = \{x_0\}$. Then $d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) = 0$

**Proposition 4.3**

Let $(X, d_X), (Y, d_Y)$ be two metric spaces and $f : X \to Y$ be a function. The following are equivalent:

1. $f$ is continuous at $x_0 \in X$.
2. For any $\{x_n\}_{n\geq 1} \subseteq X$ s.t. $x_n \xrightarrow[n\to\infty]{d_X} x_0$ we have $f(x_n) \xrightarrow[n\to\infty]{d_Y} f(x_0)$.

*Proof.* 1) $\implies$ 2) Let $\{x_n\}_{n\geq 1} \subseteq X$ s.t. $x_n \xrightarrow[n\to\infty]{d_X} x_0$.
Let $\varepsilon > 0$. $f$ continuous at $x_0 \implies \exists \delta > 0$ s.t.

$$\left.\begin{aligned} & d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \varepsilon \\ & x_n \xrightarrow[n\to\infty]{d_X} x_0 \implies \exists n_\delta \in \mathbb{N} \text{ s.t. } d_X(x_n, x_0) < \delta \, \forall n \geq n_\delta \end{aligned}\right\} \implies d_X(f(x_n), f(x_0)) < \varepsilon$$

for each $n \geq n_\delta$.
2) $\implies$ 1) We argue by contradiction. Assume

$$\exists \varepsilon_0 > 0 \text{ s.t. } \forall \delta > 0 \quad \exists x_\delta \in X \text{ s.t. } d_X(x_\delta, x_0) < \delta \text{ but } d_Y(f(x_\delta), f(x_0)) \geq \varepsilon_0$$

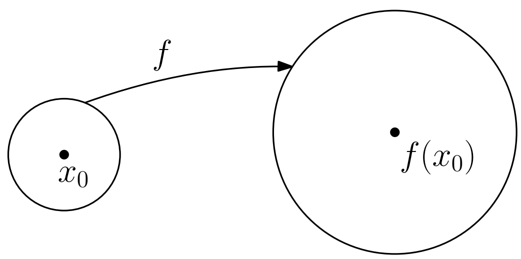
Letting $\delta = \frac{1}{n}$ we find $\{x_n\}_{n\geq 1} \subseteq X$ s.t. $d_X(x_n, x_0) < \frac{1}{n}$ but $d_Y(f(x_n), f(x_0)) \geq \varepsilon_0$ – Contradiction! ☐

**Theorem 4.4**

Let $(X, d_X), (Y, d_Y)$ be two metric spaces and let $f : X \to Y$ be a function. The following are equivalent:

1. $f$ is continuous.
2. for any $G$ open in $Y$, $f^{-1}(G) = \{x \in X : f(X) \in G\}$ is open in $X$.
3. for any $F$ closed in $Y$, $f^{-1}(F)$ is closed in $X$.
4. for any $B \subseteq Y$, $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.
5. for any $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.

*Proof.* We will show 1) $\implies$ 2) $\implies$ 3) $\implies$ 4) $\implies$ 5) $\implies$ 1).
1) $\implies$ 2) Let $G \subseteq Y$ be open.




Let $x_0 \in f^{-1}(G)$

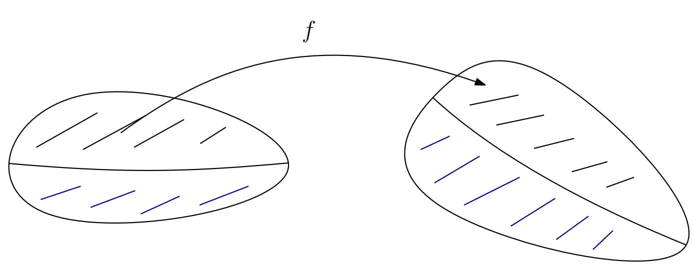
$$\implies \left.\begin{array}{l} f(x_0) \in G \\ G \text{ open in } Y \end{array}\right\} \implies \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon^Y(f(x_0)) \subseteq G$$

$f$ is continuous

$$\implies \exists \delta > 0 \text{ s.t. } f\left(B_\delta^X(x_0)\right) \subseteq B_\varepsilon^Y(f(x_0)) \subseteq G$$
$$\implies B_\delta^X(x_0) \subseteq f^{-1}(G) \implies x_0 \in \overset{\circ}{\widehat{f^{-1}(G)}}$$

So $f^{-1}(G)$ is open in $X$.
2) $\implies$ 3) Let $F \subseteq Y$ be closed $\implies$ ${}^cF = Y \setminus F$ is open in $Y$. By assumption,

$$\left.\begin{array}{l} f^{-1}\left({}^cF\right) \text{ is open in } X \\ f^{-1}\left({}^cF\right) = {}^c\left[f^{-1}(F)\right] = X \setminus f^{-1}(F) \end{array}\right\} \implies f^{-1}(F) \text{ is closed in } X$$



$$f^{-1}(Y \setminus F) = f^{-1}(Y) \setminus f^{-1}(F) = X \setminus f^{-1}(F)$$

3) $\implies$ 4) Let $B \subseteq Y \implies \overline{B}$ closed in $Y$. By assumption,

$$\left.\begin{matrix} f^{-1}(\overline{B}) \text{ closed in } X \\ f^{-1}(\overline{B}) \supseteq f^{-1}(B) \end{matrix}\right\} \implies \overline{f^{-1}(B)} \subseteq \overline{f^{-1}(\overline{B})} = f^{-1}(\overline{B})$$

4) $\implies$ 5) Let $A \subseteq X$. Use the hypothesis with $B = f(A)$. We have

$$\overline{A} \subseteq \overline{f^{-1}(f(A))} \subseteq f^{-1}\left(\overline{f(A)}\right) \implies f(\overline{A}) \subseteq \overline{f(A)}$$

5) $\implies$ 1) We argue by contradiction. Assume $\exists x_0 \in X$ s.t. $f$ is not continuous at $x_0$. Then $\exists \varepsilon_0 > 0$ and $\exists x_n \xrightarrow[n\to\infty]{d_X} x_0$ but $d_Y(f(x_n), f(x_0)) \geq \varepsilon_0$.
Let $A = \{x_n : n \geq 1\}$. Then $x_0 \in \overline{A}$ but $f(x_0) \notin \overline{\{f(x_n) : n \geq 1\}} = \overline{f(A)}$. On the other hand, we must have

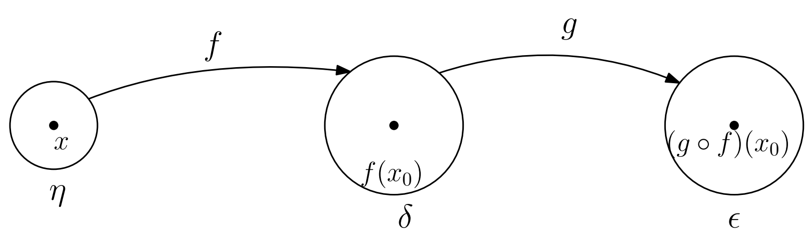
$$\left.\begin{matrix} f(\overline{A}) \subseteq \overline{f(A)} \\ x_0 \in \overline{A} \end{matrix}\right\} \implies f(x_0) \in \overline{f(A)}$$

Contradiction! □

**Proposition 4.5**
Let $(X, d_X), (Y, d_Y), (Z, d_Z)$ be metric spaces and assume $f : X \to Y$ is continuous at $x_0 \in X$ and $g : Y \to Z$ is continuous at $f(x_0) \in Y$. Then $g \circ f : X \to Z$ is continuous at $x_0$.

*Proof.* Fix $\varepsilon > 0$.

$$g \text{ continuous at } f(x_0) \implies \exists \delta > 0 \text{ s.t. } d_Y(y, f(x_0)) < \delta \implies d_Z(g(y), g(f(x_0))) < \varepsilon$$
$$f \text{ continuous at } x_0 \implies \exists \eta > 0 \text{ s.t. } d_X(x, x_0) < \eta \implies d_Y(f(x), f(x_0)) < \delta$$




So if $d_X(x, x_0) < \eta$ then $d_Z(g(f(x)), g(f(x_0))) < \varepsilon$. □

**Exercise 4.1.** Let $(X, d)$ be a metric space and let $f, g : X \to \mathbb{R}$ be continuous at $x_0 \in X$. Then $f \pm g, f \cdot g$ are continuous at $x_0$. If $g(x_0) \neq 0$ then $\frac{f}{g} : X \to \mathbb{R}$ is continuous at $x_0$.

**Exercise 4.2.** Let $(X, d)$ be a metric space and let $f_1, \ldots, f_n : X \to \mathbb{R}$. Then $f = (f_1, \ldots, f_n) : X \to \mathbb{R}^n$ is continuous at $x_0 \in X$ if and only if $f_1, \ldots, f_n$ are continuous at $x_0$.

Hint: $|f_i(x) - f_i(x_0)| \leq d_2(f(x), f(x_0)) = \sqrt{\sum_{j=1}^n |f_j(x) - f_j(x_0)|^2}$.

## §4.2 Continuity and Compactness

**Theorem 4.6**
Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f : X \to Y$ be continuous. If $K$ is compact in $X$, then $f(K)$ is compact in $Y$.

*Proof.* Method 1: Let $\{G_i\}_{i\in I}$ be a family of open sets in $Y$ s.t.

$$f(K) \subseteq \bigcup_{i\in I} G_i \implies K \subseteq f^{-1}\left(\bigcup_{i\in I} G_i\right) = \bigcup_{i\in I} \underbrace{f^{-1}(G_i)}_{\text{open in } X}$$

$K$ compact $\implies \exists n \geq 1$ and $\exists i, \ldots, i_n \in I$ s.t.

$$K \subseteq \bigcup_{j=1}^{n} f^{-1}\left(G_{i_j}\right) = f^{-1}\left(\bigcup_{j=1}^{n} G_{i_j}\right) \implies f(K) \subseteq \bigcup_{j=1}^{n} G_{i_j}$$

Method 2: Let's show $f(K)$ is sequentially compact. Let $\{y_n\}_{n\geq 1} \subseteq f(K)$.

$$y_n \in f(K) \implies \exists x_n = f^{-1}(y_n) \in K$$

As $K$ is sequentially compact, $\exists \{x_{k_n}\}_{n\geq 1}$ subsequence of $\{x_n\}_{n\geq 1}$ s.t.

$$\left.\begin{array}{l} x_{k_n} \xrightarrow[n\to\infty]{d_X} x_0 \in K \\ f \text{ is continuous} \end{array}\right\} \implies \underbrace{f\left(x_{k_n}\right)}_{=y_{k_n}} \xrightarrow[n\to\infty]{d_Y} f(x_0) \in f(K)$$

□

# §5 Lec 5: Apr 7, 2021

## §5.1 Continuity and Compactness (Cont'd)

**Corollary 5.1**

Let $(X, d_X)$ be a compact metric space and let $f : X \to \mathbb{R}^n$ be continuous. Then $f(X)$ is closed and bounded.

**Corollary 5.2**

Let $(X, d_X)$ be a compact metric space and let $f : X \to \mathbb{R}$ be continuous. Then there exists $x_1, x_2 \in X$ s.t.

$$f(x_1) = \inf \{f(x) : x \in X\} \text{ and } f(x_2) = \sup \{f(x) : x \in X\}$$

*Proof.* $f(x)$ is closed and bounded.

$$\begin{aligned} \text{Boundedness} &\implies \inf f(x) \text{ and } \sup f(x) \text{ are well defined} \\ \text{Closedness} &\implies \inf f(x), \sup f(x) \in \overline{f(x)} = f(x) \end{aligned}$$

□

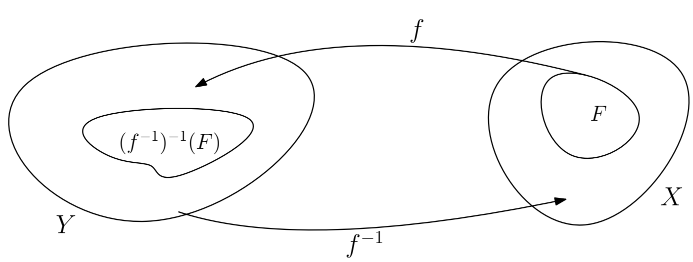
**Proposition 5.3**

Let $(X, d_X), (Y, d_Y)$ be metric spaces s.t. $X$ is compact. Let $f : X \to Y$ be bijective and continuous. Then $f^{-1} : Y \to X$ is continuous.

*Proof.* If suffices to show that for every closed set $F \subseteq X$, we have

$$\left(f^{-1}\right)^{-1}(F) = \left\{y \in Y : f^{-1}(y) \in F\right\}$$

is closed in $Y$.




But $\left(f^{-1}\right)^{-1}(F) = f(F)$.

$$\left.\begin{aligned} &F \text{ closed in } X \text{ compact} \implies F \text{compact} \\ &f : X \to Y \text{ is continuous} \end{aligned}\right\} \implies f(F) \text{ is compact and closed}$$

□

**Definition 5.4** (Uniform Continuity) — Let $(X, d_X), (Y, d_Y)$ be metric spaces. We say that a function $f : X \to Y$ is uniformly continuous if

$$\forall \varepsilon > 0 \quad \exists \delta = \delta(\varepsilon) \text{ s.t. } d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

Compare this with $g : X \to Y$ is continuous if

$$\forall x \in X \quad \forall \varepsilon > 0 \quad \exists \delta = \delta(\varepsilon, x) \text{ s.t. } d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

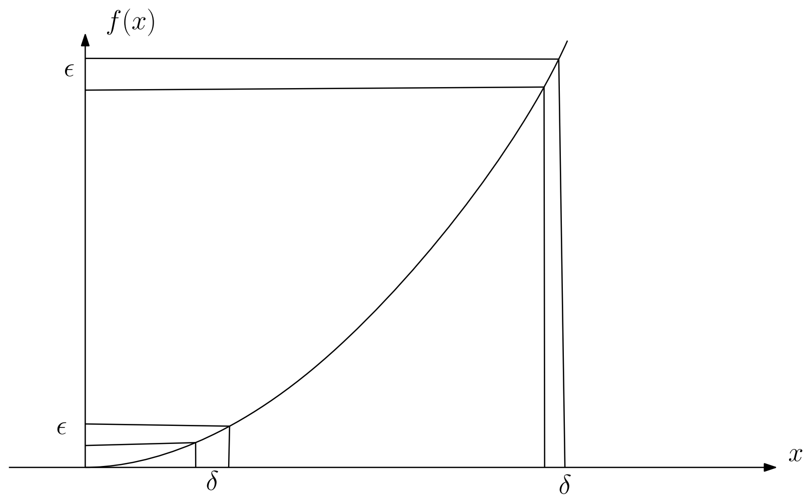
**Remark 5.5.**
1. Continuity is defined pointwise. Uniform continuity is a property of a function on a set.
2. Uniform continuity $\implies$ continuity.
3. There are continuous functions that are not uniformly continuous.

For example, consider

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$$

Let $x_n = n + \frac{1}{n}$, $y_n = n$

$$|x_n - y_n| = \frac{1}{n} \underset{n\to\infty}{\longrightarrow} 0$$
$$|f(x_n) - f(y_n)| = \left(n + \frac{1}{n}\right)^2 - n^2 = 2 + \frac{1}{n^2} > 2$$




**Theorem 5.6**

Let $(X, d_X), (Y, d_Y)$ be metric spaces with $X$ compact. Let $f : X \to Y$ continuous. Then $f$ is uniformly continuous.

*Proof.* We argue by contradiction. Assume $f$ is not uniformly continuous $\implies \exists \varepsilon_0 > 0$ s.t. $\forall \delta > 0\, \exists x_\delta, y_\delta \in X$ s.t. $d_X(x_\delta, y_\delta) < \delta$ but $d_Y(f(x_\delta), f(y_\delta)) \geq \varepsilon_0$.

Let $\delta = \frac{1}{n}$ to get $\{x_n\}_{n\geq 1}, \{y_n\}_{n\geq 1} \subseteq X$ s.t. $d_X(x_n, y_n) < \frac{1}{n}$ but $d_Y(f(x_n), f(y_n)) \geq \varepsilon_0$
$X$ compact $\implies \exists \{x_{k_n}\}_{n\geq 1}$ subsequence of $\{x_n\}_{n\geq 1}$ s.t.

$$x_{k_n} \xrightarrow[n\to\infty]{d_X} x_0 \in X$$

By the triangle inequality,

$$d(y_{k_n}, x_0) \leq \underbrace{d(x_{k_n}, y_{k_n})}_{<\frac{1}{k_n}\leq\frac{1}{n}\xrightarrow[n\to\infty]{}0} + \underbrace{d(x_{k_n}, x_0)}_{\xrightarrow[n\to\infty]{}0} \xrightarrow[n\to\infty]{} 0 \implies y_{k_n} \xrightarrow[n\to\infty]{d_X} x_0$$

$$f \text{ continuous} \implies \begin{cases} f(x_{k_n}) \xrightarrow[n\to\infty]{d_Y} f(x_0) \\ f(y_{k_n}) \xrightarrow[n\to\infty]{d_Y} f(x_0) \end{cases}$$

But

$$\varepsilon_0 \leq d_Y(f(x_{k_n}), f(y_{k_n})) \leq \underbrace{d_Y(f(x_{k_n}), f(x_0))}_{\to 0} + \underbrace{d_Y(f(x_0), f(y_{k_n}))}_{\to 0} \xrightarrow[n\to\infty]{} 0$$

Contradiction! □

## §5.2 Continuity and Connectedness

**Theorem 5.7**

Let $(X, d_X), (Y, d_Y)$ be metric spaces s.t. $X$ is connected. Let $f : X \to Y$ be continuous. Then $f(X)$ is connected.

*Proof.* Method 1: Abusing notation we write $f : X \to f(x)$. It suffices to show that if $\emptyset \neq B \subseteq f(x)$ is both open and closed in $f(x)$ then $B = f(x)$.

As $f$ is continuous, $f^{-1}(B) \neq \emptyset$ is both open and closed in $X$. But $X$ is connected which implies $f^{-1}(B) = X$ and $f(x) = B$.

Method 2: Assume that $f(x)$ is not connected. Then $\exists \emptyset \neq B_1 \subseteq Y$, $\exists \emptyset \neq B_2 \subseteq Y$ s.t. $f(x) \subseteq B_1 \cup B_2$ and

$$\overline{B_1} \cap B_2 = \emptyset = B_1 \cap \overline{B_2}$$

let

$$A_1 = f^{-1}(B_1) \neq \emptyset$$
$$A_2 = f^{-1}(B_2) \neq \emptyset$$

Have

$$f(X) \subseteq B_1 \cup B_2 \implies X \subseteq f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2) = A_1 \cup A_2$$
$$\overline{A_1} \cap A_2 = \overline{f^{-1}(B_1)} \cap f^{-1}(B_2) \subseteq f^{-1}(\overline{B_1}) \cap f^{-1}(B_2) = f^{-1}(\overline{B_1} \cap B_2)$$
$$= f^{-1}(\emptyset) = \emptyset$$

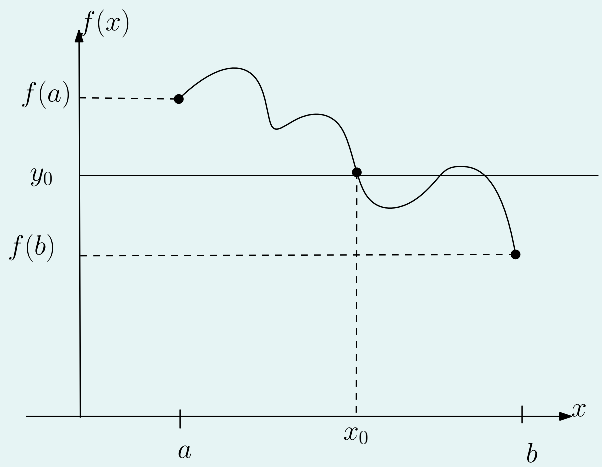
Similarly, $\overline{A_2} \cap A_1 = \emptyset$ . exercise

This contradicts that $X$ is connected. □

**Corollary 5.8** (Darboux's Property)

Let $(X, d_X)$ be a metric space and let $f : X \to \mathbb{R}$ be continuous. If $A \subseteq X$ is connected then $f(A)$ is an interval in $\mathbb{R}$.

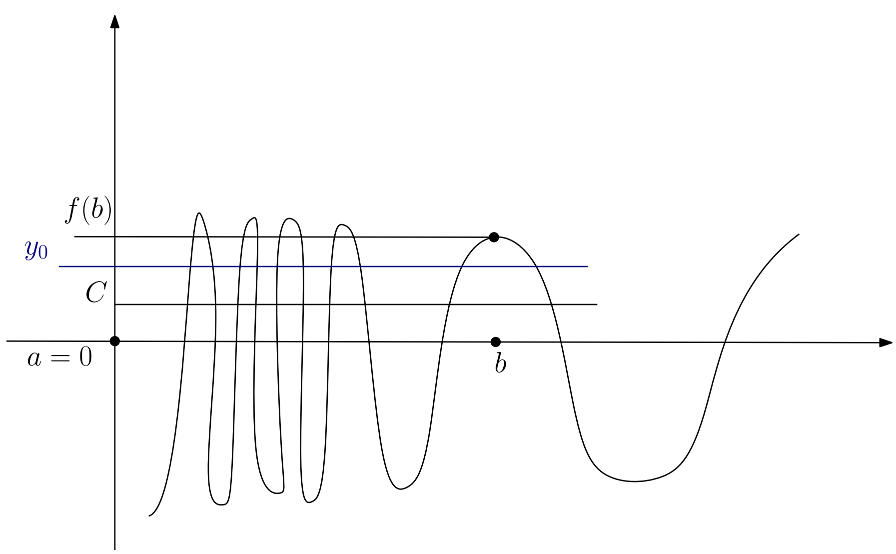
In particular, if $X = \mathbb{R}$, and $a, b \in \mathbb{R}$ s.t. $a < b$ and $y_0$ lies between $f(a)$ and $f(b)$, then $\exists x_0 \in (a, b)$ s.t. $f(x_0) = y_0$.




**Remark 5.9.** There are function that have the Darboux property, but are not continuous.

For example, consider

$$f : [0, \infty) \to \mathbb{R}, \quad f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ c, & x = 0 \end{cases} \quad \text{where } c \in [-1, 1]$$




Notice $f$ is continuous on $(0, \infty)$ implies $f$ has the Darboux property on $(0, \infty)$. $f$ has the Darboux property on $[0, \infty)$, but is not continuous at $x = 0$.

# §6 Lec 6: Apr 9, 2021

## §6.1 Continuity and Connectedness (Cont'd)

**Proposition 6.1**

Let $(X, d_X)$ and $(Y, d_Y)$ be two connected metric spaces. Then $(X \times Y, d)$ where

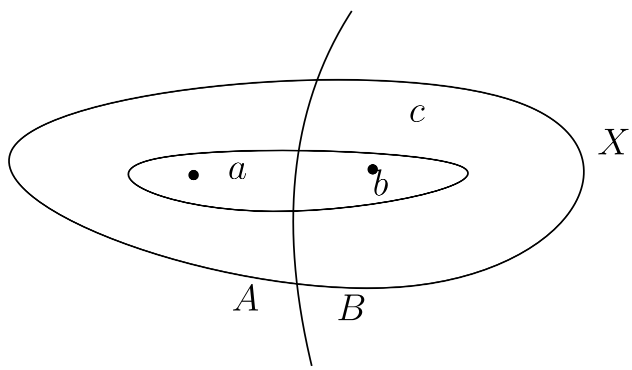
$$d\left((x_1, y_1), (x_2, y_2)\right) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}$$

is a connected metric space.

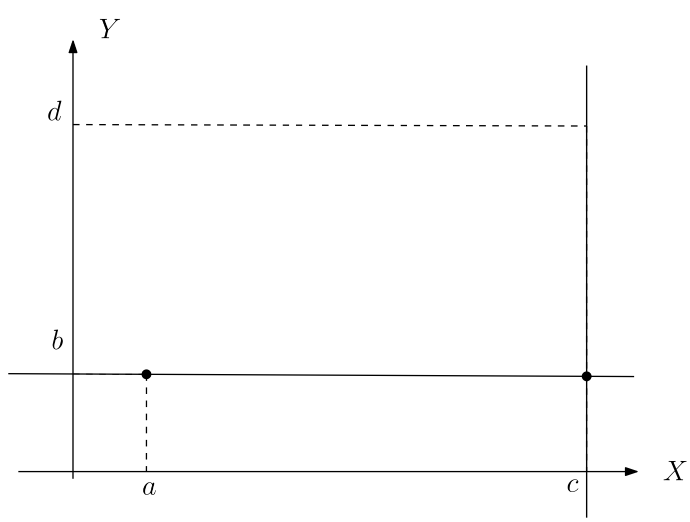
**Remark 6.2.** One could replace the distance $d$ by

$$d_1\left((x_1, y_1), (x_2, y_2)\right) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$
$$d_\infty\left((x_1, y_1), (x_2, y_2)\right) = \max\left\{d_X(x_1, x_2), d_Y(y_1, y_2)\right\}$$

*Proof.* We will use the fact that a metric space is connected if and only if any two points are contained in a connected subset of the metric space.




So to show $X \times Y$ is connected if suffices to show that if $(a, b), (c, d) \in X \times Y$, then there exists $C \subseteq X \times Y$ connected s.t. $(a, b), (c, d) \in C$.



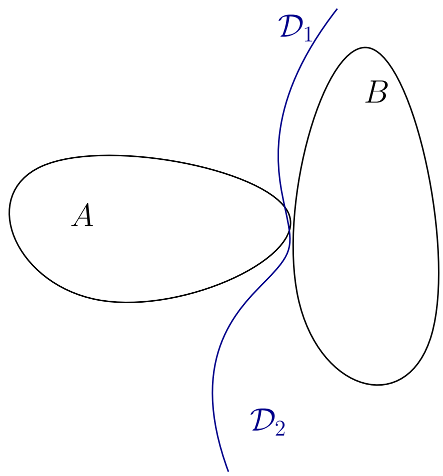
Let $f : X \to X \times Y$ where $f(x) = (x, b)$

**Claim 6.1.** $f$ is continuous.

Take $\delta = \varepsilon$ in the definition of continuity. As $X$ is connected, $f(X) = X \times \{b\}$ is connected.

Similarly, $g : Y \to X \times Y$, $g(y) = (c, y)$ is continuous and since $Y$ is connected, $g(Y) = \{c\} \times Y$ is connected.

Finally, $f(x) \cap g(y) \ni (c, b)$ and so $f(x)$, $g(y)$ are not separated. As the union of two connected not separated sets is connected we get $f(x) \cup g(y)$ is connected.




Note $(a, b), (c, d) \in f(x) \cup g(y)$. □

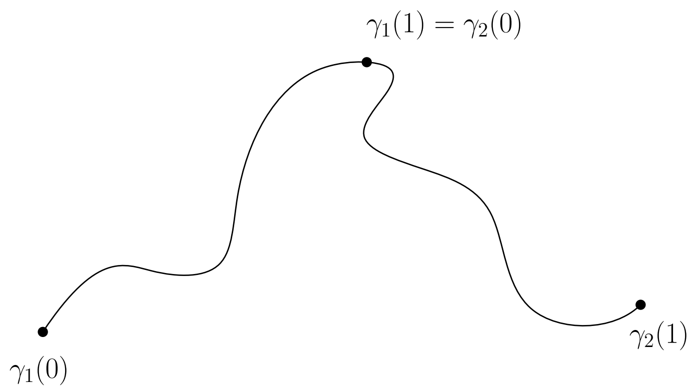
**Definition 6.3** (Path) — Let $(X, d)$ be a metric space. A path is a continuous function $\gamma : [0, 1] \to X$. $\gamma(0)$ is called the origin of the path and $\gamma(1)$ is called the end of the path.

As $[0, 1]$ is compact and connected and $\gamma$ is continuous, $\gamma([0, 1])$ is compact and connected.

Given $\gamma : [0, 1] \to X$ a path, we define

$$\gamma^- : [0, 1] \to X, \qquad \gamma^-(t) = \gamma(1 - t) \text{ is a path}$$

Given $\gamma_1, \gamma_2 : [0, 1] \to X$ paths s.t. $\gamma_1(1) = \gamma_2(0)$.



We define

$$\gamma_1 \vee \gamma_2 : [0,1] \to X$$

via

$$\gamma_1 \vee \gamma_2(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \gamma_2(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

**Proposition 6.4**

Let $(X,d)$ be a metric space and let $A \subseteq X$. Then 1) $\iff$ 2) $\implies$ 3) where

1. $\exists a \in A$ s.t. $\forall x \in A \, \exists \gamma_x : [0,1] \to A$ path s.t.

$$\gamma_x(0) = a \text{ and } \gamma_x(1) = x$$

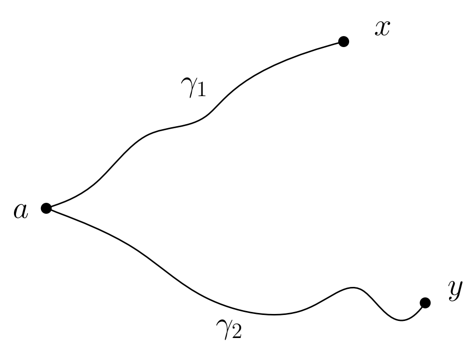
2. $\forall x, y \in A \, \exists \gamma_{x,y} : [0,1] \to A$ path s.t.

$$\gamma_{x,y}(0) = x \text{ and } \gamma_{x,y}(1) = y$$

3. $A$ is connected.

*Proof.* 1) $\implies$ 2) Let $x, y \in A$. By hypothesis, $\exists \gamma_x, \gamma_y : [0,1] \to A$ paths s.t.

$$\gamma_x(0) = \gamma_y(0) = a, \quad \gamma_x(1) = x, \quad \gamma_y(1) = y$$




Then $\gamma_x^- \vee \gamma_y : [0,1] \to A$ is the desired path.
2) $\implies$ 1)Choose $a \in A$ arbitrary.
1) $\implies$ 3) Given $x \in A$, let $A_x = \gamma_x([0,1])$ connected. Note

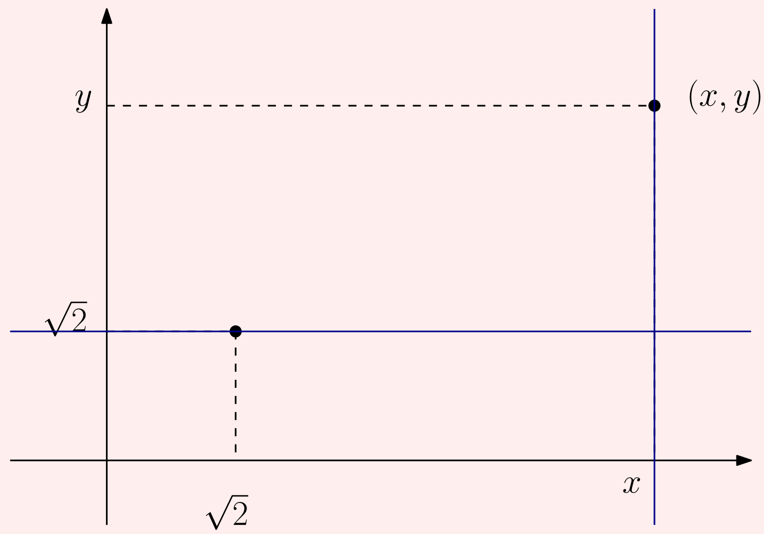
$$a \in \bigcap_{x \in A} A_x \implies \text{no two sets } A_x, A_y \text{ are separated}$$

Then $A = \bigcup_{x \in A} A_x$ is connected. □

**Definition 6.5** (Path Connected) — If either 1) or 2) holds in the Proposition 6.4, we say that $A$ is <u>path connected</u>. Note $A$ is path connected implies $A$ is connected.

**Example 6.6**

$\mathbb{R}^2 \setminus \mathbb{Q}^2$ is path connected.




We will show that any $(x,y) \in \mathbb{R}^2 \setminus \mathbb{Q}^2$ can be joined via path in $\mathbb{R}^2 \setminus \mathbb{Q}^2$ to $\left(\sqrt{2}, \sqrt{2}\right)$.

$$(x,y) \in \mathbb{R}^2 \setminus \mathbb{Q}^2 \implies x \notin \mathbb{Q} \text{ or } y \notin \mathbb{Q}$$

Say $x \notin \mathbb{Q}$. Then $\{x\} \times \mathbb{R} \subseteq \mathbb{R}^2 \setminus \mathbb{Q}^2$. Note also that $\mathbb{R} \times \left\{\sqrt{2}\right\} \subseteq \mathbb{R}^2 \setminus \mathbb{Q}^2$. Let $\gamma : [0,1] \to \mathbb{R}^2 \setminus \mathbb{Q}^2$, $\gamma = \gamma_1 \vee \gamma_2$ where

$$\gamma_1 : [0,1] \to \mathbb{R}^2 \setminus \mathbb{Q}^2,\ \gamma_1(t) = \left(\sqrt{2} + t(x - \sqrt{2}), \sqrt{2}\right) \text{ path}$$
$$\gamma_2 : [0,1] \to \mathbb{R}^2 \setminus \mathbb{Q}^2,\ \gamma_2(t) = \left(x, \sqrt{2} + t(y - \sqrt{2})\right) \text{ path}$$

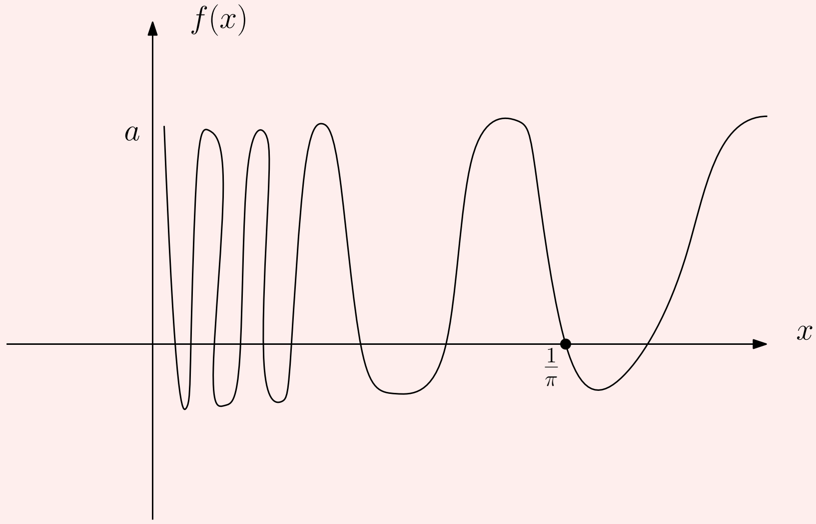
**Example 6.7**

A connected set which is not path connected. Let $f : [0, \infty) \to \mathbb{R}$ s.t.

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ a, & x = 0 \end{cases}$$

where $a \in [-1, 1]$ fixed.

Then $\Gamma_f = \{(x, f(x)) : x \in [0, \infty)\}$ is connected, but not path connected.




Let's show $\Gamma_f$ is connected. The function $g : [0, \infty) \to \mathbb{R}^2$, $g(x) = (x, f(x))$ is continuous on $(0, \infty) \implies g((0, \infty))$ is connected.

Also, $g(\{0\}) = \{(0, a)\}$ is connected. We will show that $(0, a) \in \overline{g((0, \infty))}$ and so $\{(0, a)\}, g((0, \infty))$ are not separated. Then

$$\Gamma_f = g([0, \infty)) = g(\{0\}) \cup g((0, \infty)) \text{ is connected}$$

To see $(0, a) \in \overline{g(0, \infty)}$ we need to find $x_n \to 0$ s.t.

$$\sin\left(\frac{1}{x_n}\right) = a$$

Take $x_n = \frac{1}{\arcsin a + 2n\pi}$ where $\arcsin a \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

**Example 6.8** (Cont'd from above)

Now let's show $\Gamma_f$ is not path connected. Assume towards a contradiction that there exists $\gamma : [0,1] \to \Gamma_f$ a path s.t.

$$\gamma(0) = (0, a), \qquad \gamma(1) = \left(\frac{1}{\Pi}, 0\right)$$

Note $\Pi_1 \circ \gamma : [0,1] \to \mathbb{R}$ is continuous

$$\left(\Pi_1 \circ \gamma\right)(0) = 0, \quad \left(\Pi_1 \circ \gamma\right)(1) = \frac{1}{\pi}$$

Let $b \in [-1,1] \setminus \{a\}$. By the Darboux property, $\exists t_n \in \left(0, \frac{1}{\pi}\right)$ s.t.

$$\left(\Pi_1 \circ \gamma\right)(t_n) = \frac{1}{\arcsin b + 2n\pi} \text{ where } \arcsin b \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

As $[0,1]$ is compact, $\exists t_{k_n} \underset{n\to\infty}{\longrightarrow} t_\infty \in [0,1]$.

$$\left.\begin{aligned} &\gamma \text{ continuous } \implies \gamma\left(t_{k_n}\right) \underset{n\to\infty}{\longrightarrow} \gamma(t_\infty) \\ &\gamma\left(t_{k_n}\right) = \left(\tfrac{1}{\arcsin b + 2k_n\pi}, b\right) \underset{n\to\infty}{\longrightarrow} (0, b) \end{aligned}\right\} \implies \gamma(t_\infty) = (0, b) \notin \Gamma_f$$

# §7 Lec 7: Apr 12, 2021

## §7.1 Continuity and Connectedness (Cont'd)

**Example 7.1**

Two connected sets $A, B \subseteq [-1,1] \times [-1,1]$ s.t. $(-1,-1),(1,1) \in A$, $(-1,1),(1,-1) \in B$, $A \cap B = \emptyset$. Let $f : [-1,1] \to [-1,1]$,

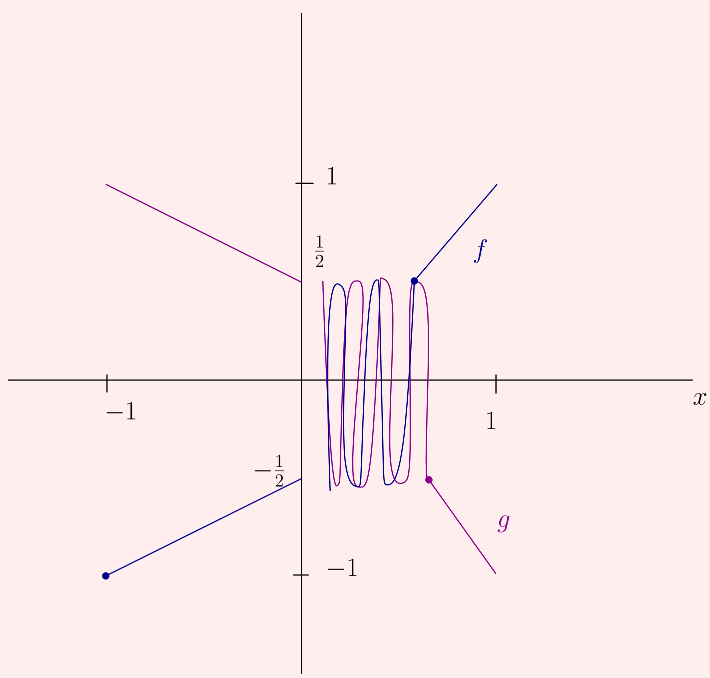
$$f(x) = \begin{cases} \frac{x-1}{2}, & -1 \le x \le 0 \\ x - \frac{1}{2}\sin\frac{\pi}{x}, & 0 < x \le \frac{1}{2} \\ x, & \frac{1}{2} \le x \le 1 \end{cases}$$

Let $g : [-1,1] \to [-1,1]$,

$$g(x) = \begin{cases} \frac{1-x}{2}, & -1 \le x \le 0 \\ -x - \frac{1}{2}\sin\frac{\pi}{x}, & 0 < x \le \frac{1}{2} \\ -x, & \frac{1}{2} \le x \le 1 \end{cases}$$

Let

$$A = \Gamma_f = \{(x_1 f(x)) : x \in [-1,1]\}$$
$$B = \Gamma_g = \{(x_1 g(x)) : x \in [-1,1]\}$$



**Example 7.2** (Cont'd from above)
Let's prove $A \cap B = \emptyset$. If

$$\begin{aligned} -1 \le x \le 0, &\quad f(x) = g(x) \iff \frac{x-1}{2} = \frac{1-x}{2} \iff x = 1 \\ 0 < x \le \frac{1}{2}, &\quad f(x) = g(x) \iff x = 0 \\ \frac{1}{2} \le x \le 1, &\quad f(x) = g(x) \iff x = 0 \end{aligned}$$

Also

$$\begin{aligned} f(-1) = -1 &\implies (-1,-1) \in A \\ f(1) = 1 &\implies (1,1) \in A \\ g(-1) = 1 &\implies (-1,1) \in B \\ g(1) = -1 &\implies (1,-1) \in B \end{aligned}$$

Let's show that $A$ is connected. A similar argument can be used to prove that $B$ is connected.

We write $A = A_1 \cup A_2$ where $A_1 = \{(x, f(x)) : -1 \le x \le 0\}$ and $A_2 = \{(x, f(x)) : 0 < x \le 1\}$. Note that $h : [-1,1] \to \mathbb{R}^2$ where $h(x) = (x, f(x))$ is continuous on $[-1,0]$ and $(0,1]$.

Since $[-1,0]$ and $(0,1]$ are connected sets, we get that $h([-1,0]) = A_1$ and $h((0,1]) = A_2$ are connected.

To show that $A = A_1 \cup A_2$ is connected, it suffices to show that $A_1$ and $A_2$ are not separated. We will show $\left(0, -\frac{1}{2}\right) \in A_1 \cap \overline{A_2}$. It's clear that $f(0) = -\frac{1}{2} \implies \left(0, -\frac{1}{2}\right) \in A_1$. To show that $\left(0, -\frac{1}{2}\right) \in \overline{A_2}$ we need to find a decreasing sequence $x_n \to 0$ s.t.

$$f(x_n) = x_n - \frac{1}{2}\sin\frac{\pi}{x_n} \xrightarrow[n\to\infty]{} -\frac{1}{2}$$

We take $x_n$ s.t. $\sin\frac{\pi}{x_n} = 1 \iff \frac{\pi}{x_n} = \frac{\pi}{2} + 2n\pi \iff x_n = \frac{2}{4n+1} \to 0$. Notice that

$$f(x_n) = \frac{2}{4n+1} - \frac{1}{2} \xrightarrow[n\to\infty]{} -\frac{1}{2}$$

## §7.2 Convergent Sequences of Functions

**Definition 7.3** (Pointwise Convergence) — Let $(X, d_X), (Y, d_Y)$ be two metric spaces and let $f_n : X \to Y$ be a sequence of functions. We say that $\{f_n\}_{n\ge1}$ converges pointwise if for all $x \in X$ the sequence $\{f_n(x)\}_{n\ge1}$ converges in $Y$. The limit $\lim_{n\to\infty} f_n(x) = f(x)$ defines a function $f : X \to Y$.

**Remark 7.4.** $\{f_n\}_{n\ge1}$ converges pointwise to $f$ if

$$\forall x \in X \quad \forall \varepsilon > 0 \quad \exists n(\varepsilon, x) \in \mathbb{N} \text{ s.t. } d_Y(f_n(x), f(x)) < \varepsilon \quad \forall n \ge n(\varepsilon, x)$$

Note that for $\varepsilon > 0$ fixed, $n(\varepsilon, \cdot) : X \to \mathbb{N}$ can be bounded or unbounded. If it is bounded, we get the following

**Definition 7.5** (Uniform Convergence) — Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f_n : X \to Y$ be a sequence of functions. We say that $\{f_n\}_{n\geq 1}$ converges uniformly to a function $f : X \to Y$ if

$$\forall \varepsilon > 0 \quad \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } d_Y\left(f(x), f_n(x)\right) < \varepsilon \quad \forall n \geq n_\varepsilon \, \forall x \in X$$

We denote $f_n \xrightarrow[n\to\infty]{u} f$.

**Remark 7.6.** Let $(X, d_X), (Y, d_Y)$ be metric spaces, $B(X, Y) = \{f : X \to Y;\ f \text{ is bounded}\}$, $d :\ B(X, Y) \times B(X, Y) \to \mathbb{R}$ via

$$d(f, g) = \sup_{x\in X} d_Y\left(f(x), g(x)\right)$$

**Exercise 7.1.** Show that $(B(X, Y), d)$ is a metric space.

Note that $f_n \xrightarrow[n\to\infty]{u} f \iff M_n = d\left(f_n, f\right) \xrightarrow[n\to\infty]{} 0$.

" $\Longleftarrow$ " $\forall \varepsilon > 0\ \exists n_\varepsilon \in \mathbb{N}$ s.t. $M_n < \varepsilon\ \forall n \geq n_\varepsilon$

$$\begin{aligned} &\implies d(f_n, f) = \sup_{x\in X} d_Y\left(f_n(x), f(x)\right) < \varepsilon \quad \forall n \geq n_\varepsilon \\ &\implies d_Y\left(f_n(x), f(x)\right) < \varepsilon \quad \forall n \geq n_\varepsilon \quad \forall x \in X \end{aligned}$$

" $\implies$ "

$$\begin{aligned} f_n \xrightarrow[n\to\infty]{u} f &\implies \forall \varepsilon > 0 \quad \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } d_Y\left(f_n(x), f(x)\right) < \frac{\varepsilon}{2} \quad \forall n \geq n_\varepsilon \, \forall x \in X \\ &\implies \underbrace{\sup_{x\in X} d_Y\left(f_n(x), f(x)\right)}_{d(f_n, f) = M_n} \leq \frac{\varepsilon}{2} < \varepsilon \quad \forall n \geq n_\varepsilon \end{aligned}$$

**Remark 7.7.**
1. Uniform convergence $\implies$ pointwise convergence
2. Pointwise convergence $\not\Longrightarrow$ uniform convergence

$f_n : [0, 1] \to \mathbb{R},\ f_n(x) = x^n$

$$\{f_n\}_{n\geq 1} \text{ converges pointwise} : \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

Let

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

Note $f_n \underset{n\to\infty}{\overset{u}{\not\to}} f$ since

$$d(f_n, f) = \sup_{x\in[0,1]} \left|f_n(x) - f(x)\right| = \sup_{x\in[0,1)} |x^n| = 1 \underset{n\to\infty}{\not\to} 0$$

**Theorem 7.8** (Weierstrass)
Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f_n : X \to Y$ be a sequence of functions that converges uniformly to a function $f : X \to Y$. If $\forall n \geq 1$, $f_n$ is continuous at $x_0 \in X$ then $f$ is continuous at $x_0$.

**Corollary 7.9**
A uniform limit of continuous functions is a continuous function.

*Proof.* (of theorem) Fix $\varepsilon > 0$.

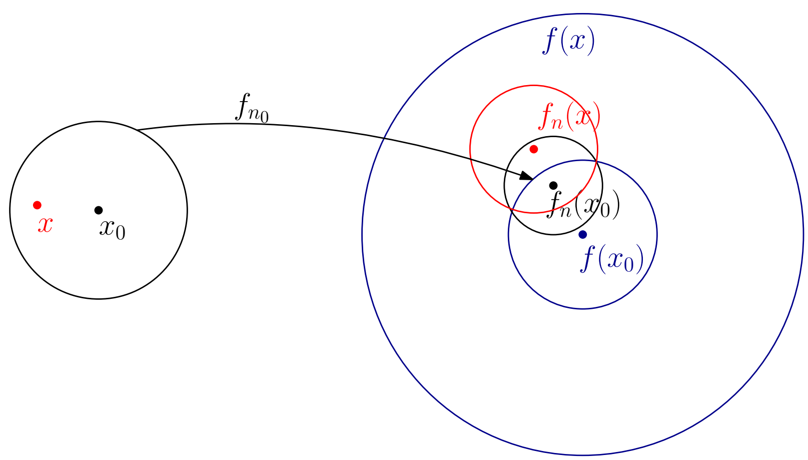
$$f_n \xrightarrow[n\to\infty]{u} f \implies \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } d_Y\left(f_n(x), f(x)\right) < \frac{\varepsilon}{3} \quad \forall n \geq n_\varepsilon \, \forall x \in X$$

Fix $n_0 \geq n_\varepsilon$. $f_{n_0}$ is continuous at $x_0$

$$\implies \exists \delta > 0 \text{ s.t. if } d_X(x_0, x) < \delta$$

then

$$d_Y\left(f_{n_0}(x_0), f_{n_0}(x)\right) < \frac{\varepsilon}{3}$$




Then for $x \in B_\delta(x_0)$ we have

$$\begin{aligned} d_Y\left(f(x), f(x_0)\right) &\leq d_Y\left(f(x), f_{n_0}(x)\right) + d\left(f_{n_0}(x), f_{n_0}(x_0)\right) + d\left(f_{n_0}(x_0), f(x_0)\right) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

By definition, $f$ is continuous at $x_0$. □

# §8 Lec 8: Apr 14, 2021

## §8.1 Convergent Sequences of Functions (Cont'd)

**Theorem 8.1** (Dini)

Let $(X, d)$ be a compact metric space and let $f_n : X \to \mathbb{R}$ be a sequence of continuous functions that converges pointwise to a continuous function $f : X \to \mathbb{R}$. Assume that $\{f_n\}_{n\geq 1}$ is monotone in the sense that either $\{f_n(x)\}_{n\geq 1}$ is increasing for all $x \in X$ or $\{f_n(x)\}_{n\geq 1}$ is decreasing for all $x \in X$. Then,

$$f_n \xrightarrow[n\to\infty]{u} f \text{ i.e. } d(f_n, f) = \sup_{x\in X} |f_n(x) - f(x)| \xrightarrow[n\to\infty]{} 0$$

*Proof.* Assume that $\{f_n\}_{n\geq 1}$ is increasing. Then $\{f - f_n\}_{n\geq 1}$ is decreasing and for all $x \in X$ we have

$$\lim_{n\to\infty} [f(x) - f_n(x)] = \inf_{n\to\infty} [f(x) - f_n(x)] = 0$$

Then $\forall \varepsilon > 0 \quad \exists n(\varepsilon, x) \in \mathbb{N}$ s.t. $\forall n \geq n(\varepsilon, x)$ we have

$$0 \leq f(x) - f_n(x) \leq f(x) - f_{n_{\varepsilon,x}}(x) < \varepsilon$$

As $f - f_{n_{\varepsilon,x}}$ is continuous at $x$, $\exists \delta(\varepsilon, x) > 0$ s.t.

$$d(x, y) < \delta_{\varepsilon,x} \implies \left|\left[f(x) - f_{n_{\varepsilon,x}}(x)\right] - \left[f(y) - f_{n_{\varepsilon,x}}(y)\right]\right| < \varepsilon$$

By the triangle inequality, we get

$$\begin{aligned} 0 \leq f(y) - f_{n_{\varepsilon,x}}(y) &\leq \left|\left[f(x) - f_{n_{\varepsilon,x}}(x)\right] - \left[f(y) - f_{n_{\varepsilon,x}}(y)\right]\right| + f(x) - f_{n_{\varepsilon,x}}(x) \\ &< \varepsilon + \varepsilon = 2\varepsilon \end{aligned}$$

whenever $y \in B_{\delta_{\varepsilon,x}}(x)$. In particular,

$$0 \leq f(y) - f_n(y) \leq f(y) - f_{n_{\varepsilon,x}}(y) < 2\varepsilon \quad \forall n \geq n_{\varepsilon,x}, \forall y \in B_{\delta_{\varepsilon,x}}(x) \tag{*}$$

Note

$$\left.\begin{matrix} X = \bigcup_{x\in X} B_{\delta_{\varepsilon,x}}(x) \\ X \text{ compact} \end{matrix}\right\} \implies \exists \mathcal{J} \subseteq \mathbb{N} \text{ finite and } \exists \{x_j\}_{j\in\mathcal{J}} \in X$$

s.t. $X = \bigcup_{j\in\mathcal{J}} B_{\delta_j}(x_j)$ and where $\delta_j = \delta(\varepsilon, x_j)$.
Let $n_\varepsilon = \max_{j\in\mathcal{J}} n(\varepsilon, x_j)$. Fix $n \geq n_\varepsilon$ and $x \in X$. As $x \in X = \bigcup_{j\in\mathcal{J}} B_{\delta_j}(x_j) \implies j \in \mathcal{J}$ s.t. $x \in B_{\delta_j}(x_j)$. By (*), we have

$$0 \leq f(x) - f_n(x) < 2\varepsilon$$

As $x \in X$ was arbitrary we get

$$d(f, f_n) \leq 2\varepsilon \qquad \forall n \geq n_\varepsilon$$

□

**Remark 8.2.** The compactness of $X$ is necessary in Dini's theorem.

**Example 8.3**

$f_n : (0,1) \to \mathbb{R}$, $f_n(x) = x^n$ continuous

$$f_{n+1}(x) \le f_n(x) \quad \forall n \ge 1 \quad \forall x \in (0,1)$$
$$f_n(x) \underset{n\to\infty}{\longrightarrow} 0 \quad \forall x \in (0,1)$$

Let $f : (0,1) \to \mathbb{R}$, $f(x) = 0 \quad \forall x \in (0,1)$. It's continuous. But

$$d(f_n, f) = \sup_{x\in(0,1)} |x^n| = 1 \underset{n\to\infty}{\not\longrightarrow} 0 \implies f_n \overset{u}{\underset{n\to\infty}{\not\longrightarrow}} f$$

Note that $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n$ continuous, $\{f_n\}_{n\ge1}$ is decreasing and converge pointwise to $f : [0,1] \to \mathbb{R}$,

$$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases} \qquad \text{which is not continuous}$$

This also shows that the continuity of the limit function is necessary in Dini's theorem.

**Remark 8.4.** Monotonicity is necessary in Dini's theorem.

**Example 8.5**

$f_n : [0,1] \to \mathbb{R}$ is continuous. $\{f_n\}_{n\ge1}$ converges pointwise to $f : [0,1] \to \mathbb{R}$, $f(x) = 0\,\forall x \in [0,1]$ figure here $f$ is continuous. But

$$d(f_n, f) = \sup_{x\in[0,1]} |f_n(x)| = 1 \underset{n\to\infty}{\not\longrightarrow} 0 \implies f_n \overset{u}{\underset{n\to\infty}{\not\longrightarrow}} f$$

Note that $\{f_n\}_{n\ge1}$ is not monotone!

## §8.2 Space of Functions

Fix $a, b \in \mathbb{R}$, $a < b$. We define

$$C([a,b]) = \{f : [a,b] \to \mathbb{R};\ f \text{ is continuous}\}$$

We equip $C([a,b])$ with the metric $d : C([a,b]) \times C([a,b]) \to \mathbb{R}$, given by

$$d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|$$

Then $(C([a,b]), d)$ is a metric space.
Completeness: Let $\{f_n\}_{n\ge1} \subseteq C([a,b])$ be Cauchy. So $\forall \varepsilon > 0\ \exists n_\varepsilon \in \mathbb{N}$ s.t. $d(f_n, f_m) < \varepsilon$ $\forall n, m \ge n_\varepsilon$

$$\implies |f_n(x) - f_m(x)| < \varepsilon \quad \forall n, m \ge n_\varepsilon \quad \forall x \in [a,b]$$

So $\{f_n(x)\}_{n\geq 1}$ is Cauchy $\forall x \in [a,b]$. As $\mathbb{R}$ is complete,

$$\forall x \in [a,b] \quad f_n(x) \underset{n\to\infty}{\longrightarrow} f(x) \in \mathbb{R}$$

This defines a function $f : [a,b] \to \mathbb{R}$. Recall that for all $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ s.t.

$$\begin{aligned} |f_n(x) - f(x)| \leq \varepsilon \quad \forall n \geq n_\varepsilon \quad \forall x \in [a,b] \\ \implies d(f_n, f) \leq \varepsilon \quad \forall n \geq n_\varepsilon \end{aligned}$$

So $f_n \xrightarrow[n\to\infty]{u} f$. By Weierstrass, $f \in C([a,b])$. Thus $(C([a,b]), d)$ is a complete metric space.

Compactness: Note that $(C([a,b]), d)$ is not bounded and so not compact.

**Example 8.6**

$f_n : [a,b] \to \mathbb{R}$, $f_n(x) = n$ for all $x \in [a,b]$.

Connectedness: $(C([a,b]), d)$ is path connected and so connected.

Let $f, g \in C([a,b])$. Define $\gamma : [0,1] \to C([a,b])$ via $\gamma(t) = f + t(g-f)$. Note $\forall t \in [0,1]$, $\gamma(t) \in C([a,b])$ and

$$\gamma(0) = f, \quad \gamma(1) = g$$

To see that $\gamma$ is a path we compute

$$\begin{aligned} d(\gamma(t), \gamma(s)) &= \sup_{x\in[a,b]} |\gamma(t;x) - \gamma(s;x)| \\ &= \sup_{x\in[a,b]} |t-s|\,|g(x) - f(x)| \\ &= |t-s| \underbrace{d(g,f)}_{\in\mathbb{R}} \underset{|t-s|\to 0}{\longrightarrow} 0 \end{aligned}$$

So $\gamma$ is a continuous function and so a path.

# §9 Lec 9: Apr 16, 2021

## §9.1 Arzela–Ascoli Theorem

For $a, b \in \mathbb{R}$ with $a < b$, we define

$$C([a,b]) = \{f : [a,b] \to \mathbb{R};\ f \text{ continuous}\}$$

We equip $C([a,b])$ with the uniform metric

$$d(f,g) = \sup_{x \in [a,b]} |f(x) - g(x)|$$

We showed that $(C([a,b]), d)$ is a complete, connected metric space, but it's not compact.

**Definition 9.1** (Equicontinuity) — We say that a set $\mathcal{F} \subseteq C([a,b])$ is <u>equicontinuous</u> if

$$\forall \varepsilon > 0 \quad \exists \delta(\varepsilon) > 0 \text{ s.t. } |f(x) - f(y)| < \varepsilon \quad \forall x, y \in [a,b] \text{ with } |x - y| < \delta(\varepsilon)$$

and for all $f \in \mathcal{F}$.

*<u>Note</u>*: For a fixed function $f \in \mathcal{F} \subseteq C([a,b])$, we have that $f$ is uniformly continuous (since $f$ is continuous on $[a,b]$ compact) which means for all $\varepsilon > 0$, there exists $\delta(\varepsilon, f) > 0$ s.t.

$$|f(x) - f(y)| < \varepsilon \quad \forall x, y \in [a,b] \text{ with } |x - y| < \delta(\varepsilon, f)$$

Note that for an equicontinuous family $\mathcal{F}$, $\delta_\varepsilon$ can be chosen uniformly for $f \in \mathcal{F}$.

**Definition 9.2** (Uniformly Bounded) — We say that a set $\mathcal{F} \subseteq C([a,b])$ is <u>uniformly bounded</u> if $\exists M > 0$ s.t. $|f(x)| \le M\ \forall x \in [a,b]\ \forall f \in \mathcal{F}$.

<u>Note</u>: For a fixed $f \in \mathcal{F} \subseteq C[a,b]$ we have that $f([a,b])$ is bounded (since $f$ continuous and $[a,b]$ compact which implies $f([a,b])$ is compact and so bounded). So $\exists M_f > 0$ s.t. $|f(x)| \le M_f\ \forall x \in [a,b]$. For a uniformly bounded family $\mathcal{F}$, we can choose the bound $M$ uniformly for $f \in \mathcal{F}$.

**Theorem 9.3** (Arzela-Ascoli)
Let $\mathcal{F} \subseteq C([a,b])$. The following are equivalent:

1. $\mathcal{F}$ is uniformly bounded and equicontinuous.
2. Every sequence in $\mathcal{F}$ admits a convergent subsequence.

<u>Caution</u>: We cannot guarantee that the limit of the convergent subsequence belongs to $\mathcal{F}$, unless $\mathcal{F}$ is closed in $C([a,b])$. If $\mathcal{F}$ is closed in $C([a,b])$, then the theorem becomes

$$\mathcal{F} \text{ is compact} \iff \mathcal{F} \text{ is uniformly bounded and equicontinuous}$$

*Proof.* 2) $\implies$ 1)

**Claim 9.1.** $\mathcal{F}$ is totally bounded.

Fix $\varepsilon > 0$. Let $f_1 \in \mathcal{F}$.

$$\begin{aligned}
&\text{If } \mathcal{F} \subseteq B_\varepsilon(f_1) \text{ then } \mathcal{F} \text{ is totally bounded} \\
&\text{If } \mathcal{F} \nsubseteq B_\varepsilon(f_1) \text{ then } \exists f_2 \in \mathcal{F} \text{ s.t. } d(f_1, f_2) \geq \varepsilon \\
&\text{If } \mathcal{F} \subseteq B_\varepsilon(f_1) \cup B_\varepsilon(f_2) \text{ then } \mathcal{F} \text{ is totally bounded} \\
&\text{If } \mathcal{F} \nsubseteq B_\varepsilon(f_1) \cup B_\varepsilon(f_2) \text{ then } \exists f_3 \in \mathcal{F} \text{ s.t. } \begin{cases} d(f_1, f_3) \geq \varepsilon \\ d(f_2, f_3) \geq \varepsilon \end{cases}
\end{aligned}$$

If the process terminates in finitely many steps, then $\mathcal{F}$ is totally bounded. Otherwise, we find $\{f_n\}_{n\geq 1} \subseteq \mathcal{F}$ s.t. $d(f_n, f_m) \geq \varepsilon \, \forall n \neq m$. This sequence does not admit a convergent subsequence, leading a contradiction.

Let's show that $\mathcal{F}$ is uniformly bounded. As $\mathcal{F}$ is totally bounded, $\exists n \geq 1$ and $\exists f_1, \ldots, f_n \in \mathcal{F}$ s.t.

$$\mathcal{F} \subseteq \bigcup_{j=1}^{n} B_1(f_j) \subseteq B_r(f_1)$$

where $r = 1 + \max_{2 \leq j \leq n} d(f_1, f_j)$. In particular, for all $f \in \mathcal{F}$,

$$d(f, f_1) < r$$

$f_1$ is continuous on compact $[a, b] \implies \exists M_{f_1} > 0$ s.t.

$$|f_1(x)| \leq M_{f_1} \quad \forall x \in [a, b]$$

So for $f \in \mathcal{F}$

$$|f(x)| \leq |f(x) - f_1(x)| + |f_1(x)| \leq d(f, f_1) + M_{f_1} < r + M_{f_1} \quad \forall x \in [a, b]$$

So $\mathcal{F}$ is uniformly bounded.
Let's show that $\mathcal{F}$ is equicontinuous. Let $\varepsilon > 0$. As $\mathcal{F}$ is totally bounded, $\exists n \geq 1$ and $\exists f_1, \ldots, f_n \in \mathcal{F}$ s.t.

$$\mathcal{F} \subseteq \bigcup_{j=1}^{n} B_{\frac{\varepsilon}{3}}(f_j)$$

For each $1 \leq j \leq n$, $f_j$ is uniformly continuous on $[a, b]$. So $\exists \delta_j(\varepsilon) > 0$ s.t.

$$|f_j(x) - f_j(y)| < \frac{\varepsilon}{3} \quad \forall x, y \in [a, b] \text{ with } |x - y| < \delta_j(\varepsilon)$$

Let $\delta_\varepsilon = \min_{1 \leq j \leq n} \delta_j(\varepsilon) > 0$.
Fix $f \in \mathcal{F} \implies \exists 1 \leq j \leq n$ s.t. $f \in B_{\frac{\varepsilon}{3}}(f_j)$. Then for $x, y \in [a, b]$ with $|x - y| < \delta_\varepsilon$ we have

$$\begin{aligned}
|f(x) - f(y)| &\leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)| \\
&\leq 2d(f, f_j) + |f_j(x) - f_j(y)| \\
&\leq \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

This shows $\mathcal{F}$ is equicontinuous.
1) $\implies$ 2) Let $\{f_n\}_{n\geq 1} \subseteq \mathcal{F}$. As $\mathcal{F}$ is uniformly bounded,

$$\exists M > 0 \text{ s.t. } |f(x)| \leq M \quad \forall x \in [a,b]\ \forall f \in \mathcal{F}$$

In particular, $|f_n(x)| \leq M\ \forall x \in [a,b]\ \forall n \geq 1$.
Let $\{r_n\}_{n\geq 1}$ denote an enumeration of the rationals in $[a,b]$. As $\{f_n(r_1)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded by $M$, $\exists \left\{f_n^{(1)}\right\}_{n\geq 1}$ subsequence of $\{f_n\}_{n\geq 1}$ s.t. $\left\{f_n^{(1)}(r_1)\right\}_{n\geq 1}$ converges. $\left\{f_n^{(1)}(r_2)\right\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded by $M \implies \exists \left\{f_n^{(2)}\right\}_{n\geq 1}$ subsequence of $\left\{f_n^{(1)}\right\}_{n\geq 1}$ s.t. $\left\{f_n^{(2)}(r_2)\right\}_{n\geq 1}$ converges.
Proceeding inductively we find $\forall k \geq 1$ $\left\{f_n^{(k+1)}\right\}_{n\geq 1}$ is a subsequence of $\left\{f_n^{(k)}\right\}_{n\geq 1}$ and $\left\{f_n^{(k)}(r_k)\right\}_{n\geq 1}$ converges.
We consider $\left\{f_n^{(n)}\right\}_{n\geq 1}$ subsequence of $\{f_n\}_{n\geq 1}$.
For $n, m \geq k$, $f_n^{(n)}$, $f_m^{(m)}$ are elements in $\left\{f_n^{(k)}\right\}_{n\geq 1}$. So $\left\{f_n^{(n)}\right\}_{n\geq 1}$ converges at $r_k$.
Caution: The convergence is not uniform in $k$.
Fix $\varepsilon > 0$. As $\mathcal{F}$ is equicontinuous, $\exists \delta > 0$ s.t.

$$|f(x) - f(y)| < \frac{\varepsilon}{3} \quad \forall x, y \in [a,b]\ |x-y| < \delta, \forall f \in \mathcal{F}$$

In particular,

$$|f_n(x) - f_n(y)| < \frac{\varepsilon}{3} \quad \forall x, y \in [a,b]\ |x-y| < \delta, \forall n \geq 1 \tag{*}$$

Let $r_1, \ldots, r_N \in \mathbb{Q} \cap [a,b]$ s.t. $a = r_0 < r_1 < \ldots < r_N < r_{N+1} = b$ and

$$|r_{j+1} - r_j| < \delta \qquad 0 \leq j \leq N$$

Note $N \sim \frac{|a-b|}{\delta}$. For each $1 \leq j \leq N$, $\exists n_j(\varepsilon) \in \mathbb{N}$ s.t.

$$\left|f_n^{(n)}(r_j) - f_m^{(m)}(r_j)\right| < \frac{\varepsilon}{3} \qquad \forall n, m \geq n_j(\varepsilon)$$

Let $n_\varepsilon = \max_{1\leq j\leq N} n_j(\varepsilon)$. Note

$$\left|f_n^{(n)}(r_j) - f_m^{(m)}(r_j)\right| < \frac{\varepsilon}{3} \quad \forall n, m \geq n_\varepsilon \quad \forall 1 \leq j \leq N \tag{**}$$

Let $x \in [a,b] \implies \exists 1 \leq j \leq N$ s.t. $|x - r_j| < \delta$. Then

$$\left|f_n^{(n)}(x) - f_m^{(m)}(x)\right| \leq \left|f_n^{(n)}(x) - f_n^{(n)}(r_j)\right| + \left|f_n^{(n)}(r_j) - f_m^{(m)}(r_j)\right| + \left|f_m^{(m)}(r_j) - f_m^{(m)}(x)\right|$$
$$\text{By (*) and (**)} < 2 \cdot \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \forall n, m \geq n_\varepsilon$$

So $\left\{f_n^{(n)}\right\}_{n\geq 1}$ is uniformly Cauchy and so uniformly convergent. □

**Remark 9.4.** One can replace $[a,b]$ by any other compact metric space $(X,d)$.

# §10 Lec 10: Apr 19, 2021

## §10.1 Arzela-Ascoli Theorem (Cont'd)

**Remark 10.1.** The compactness of the set on which the functions are defined is necessary in Arzela-Ascoli.

**Example 10.2**

$\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R}; |f(x) - f(y)| \le |x - y| \ \forall x, y \in \mathbb{R} \text{ and } \sup_{x\in\mathbb{R}} |f(x)| \le 1\}$. Note $\mathcal{F}$ is equicontinuous and uniformly bounded. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{1+x^2}$

**Claim 10.1.** $f \in \mathcal{F}$.

Indeed,

$$\sup_{x\in\mathbb{R}} |f(x)| = \sup_{x\in\mathbb{R}} \frac{1}{1+x^2} = 1$$

Moreover, for $x, y \in \mathbb{R}$

$$\begin{aligned}
|f(x) - f(y)| = \left|\frac{1}{1+x^2} - \frac{1}{1+y^2}\right| &= \frac{\left|x^2 - y^2\right|}{(1+x^2)(1+y^2)} \\
&= |x-y| \cdot \frac{|x+y|}{(1+x^2)(1+y^2)} \\
&\le |x-y| \, (\underbrace{\frac{|x|}{1+x^2}}_{\le \frac{1}{2}} + \underbrace{\frac{|y|}{1+y^2}}_{\le \frac{1}{2}}) \\
&\le |x-y|
\end{aligned}$$

So $f \in \mathcal{F}$.

For $n \ge 1$, let $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = f(x-n)$. Note $f_n \in \mathcal{F}$ since $\sup_{x\in\mathbb{R}} |f_n(x)| = \sup_{x\in\mathbb{R}} \frac{1}{1+(x-n)^2} = 1$.

$$\begin{aligned}
|f_n(x) - f_n(y)| = |f(x-n) - f(y-n)| &\le |(x-n) - (y-n)| \\
&= |x-y|
\end{aligned}$$

Note that $\{f_n\}_{n\ge1}$ converge pointwise to $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 0$ since $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{1+(x-n)^2} = 0$. However, $\{f_n\}_{n\ge1}$ does not admit a subsequence that converges uniformly since $\forall n \ge 1$

$$d(f_n, f) = \sup_{x\in\mathbb{R}} |f_n(x)| = 1 \overset{n\to\infty}{\not\to} 0$$

**Remark 10.3.** Uniform boundedness is necessary in Arzela-Ascoli.

**Example 10.4**

$\mathcal{F} = \{f : \underbrace{[0,1]}_{\text{compact}} \to \mathbb{R};\ f \text{ is continuous and } \underbrace{\sup_{x\in[0,1]} |f(x)| \le 1}_{\text{uniformly bounded}}\}$.

**Claim 10.2.** $\mathcal{F}$ is not equicontinuous.

For $n \ge 1$, let $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = \sin(nx)$. Note $f_n \in \mathcal{F}$. Let $x_n = \frac{3\pi}{2n}$, $y_n = \frac{\pi}{2n}$. Then $|x_n - y_n| = \frac{\pi}{n} \underset{n\to\infty}{\longrightarrow} 0$ but

$$|f_n(x_n) - f_n(y_n)| = 2$$

So $\{f_n\}_{n\ge1}$ is not equicontinuous $\implies \mathcal{F}$ is not equicontinuous.

**Claim 10.3.** $\{f_n\}_{n\ge1}$ does not admit a convergent subsequence.

Assume, towards a contradiction, that there exists a subsequence $\{f_{k_n}\}_{n\ge1}$ of $\{f_n\}_{n\ge1}$ that converges uniformly to $f : [0,1] \to \mathbb{R}$. By Weierstrass,

$$\left.\begin{array}{l} f \in C([0,1]) \\ \left.\begin{array}{l} f_{k_n}(0) = 0 \quad \forall n \ge 1 \\ f_{k_n}(0) \underset{n\to\infty}{\longrightarrow} f(0) \end{array}\right\} \implies f(0) = 0 \end{array}\right\} \implies \forall \varepsilon > 0\, \exists \delta > 0 \text{ s.t. } |f(x)| < \varepsilon\, \forall 0 < x < \delta$$

$f_{k_n} \xrightarrow[n\to\infty]{u} f \implies \exists n_\varepsilon \in \mathbb{N}$ s.t. $d(f_{k_n}, f) < \varepsilon\ \forall n \ge n_\varepsilon$. In particular, for $0 < x < \delta$ and $n \ge n_\varepsilon$ we have

$$|f_{k_n}(x)| \le |f_{k_n}(x) - f(x)| + |f(x)| < d(f_{k_n}, f) + \varepsilon < 2\varepsilon$$

Choosing $\varepsilon \le \frac{1}{2}$ and $N$ large so that $N \ge n_{\varepsilon=\frac{1}{2}}$ and $\frac{\pi}{2N} < \delta_{\varepsilon=\frac{1}{2}}$ we find

$$1 = \left|f_{k_N}\left(\frac{\pi}{2N}\right)\right| < 2\varepsilon \le 1 \qquad \text{Contradiction!}$$

## §10.2 The oscillation of a Real Function

**Definition 10.5** (Oscillation of a Function) — Let $(X, d)$ be a metric space and let $f : X \to \mathbb{R}$ be a function. For $\emptyset \ne A \subseteq X$, the oscillation of $f$ on $A$ is

$$\omega(f, A) = \sup_{x\in A} f(x) - \inf_{x\in A} f(x) = \sup_{x,y\in A} [f(x) - f(y)] \ge 0$$

Note that if $A \subseteq B$ then

$$\omega(f, A) \le \omega(f, B)$$

For $x_0 \in X$, the oscillation of $f$ at $x_0$ is given by

$$\omega(f, x_0) = \inf_{\delta>0} \omega(f, B_\delta(x_0))$$

**Proposition 10.6**

Let $(X, d)$ be a metric space and let $f : X \to \mathbb{R}$ be a function. Then $f$ is continuous at a point $x_0 \in X$ if and only if $\omega(f, x_0) = 0$.

*Proof.* " $\implies$ " Fix $\varepsilon > 0$. As $f$ is continuous at $x_0$, $\exists \delta > 0$ s.t. $|f(x) - f(x_0)| < \frac{\varepsilon}{4}$ $\forall x \in B_\delta(x_0)$.

$$\begin{aligned} &\implies |f(x) - f(y)| \le |f(x) - f(x_0)| + |f(x_0) - f(y)| < \frac{\varepsilon}{2} \quad \forall x, y \in B_\delta(x_0) \\ &\implies \omega(f, B_\delta(x_0)) = \sup_{x,y \in B_\delta(x_0)} [f(x) - f(y)] \le \frac{\varepsilon}{2} < \varepsilon \\ &\implies \omega(f, x_0) \le \omega(f, B_\delta(x_0)) < \varepsilon \end{aligned}$$

As $\varepsilon > 0$ was arbitrary, $\omega(f, x_0) = 0$.
" $\impliedby$ " Fix $\varepsilon > 0$. Then $\omega(f, x_0) = 0 < \varepsilon$ implies $\exists \delta > 0$ s.t. $\omega(f, B_\delta(x_0)) < \varepsilon$

$$\begin{aligned} &\implies |f(x) - f(y)| < \varepsilon \qquad \forall x, y \in B_\delta(x_0) \\ &\implies |f(x) - f(x_0)| < \varepsilon \qquad \forall x \in B_\delta(x_0) \end{aligned}$$

So $f$ is continuous at $x_0$. □

**Lemma 10.7**

Let $(X, d)$ be a metric space and let $f : X \to \mathbb{R}$ be a function. Then for any $\alpha > 0$,

$$\{x \in X : \omega(f, x) < \alpha\} \text{ is open in } X$$

*Proof.* Fix $\alpha > 0$ and let $A = \{x \in X : \omega(f, x) < \alpha\}$. Fix $x_0 \in A \implies \omega(f, x_0) = \inf_{\delta > 0} \omega(f, B_\delta(x_0)) < \alpha$.

$$\implies \exists \delta > 0 \text{ s.t. } \omega(f, B_\delta(x_0)) < \alpha$$

**Claim 10.4.** $B_\delta(x_0) \subseteq A$ (which implies $x_0 \in \mathring{A}$ and so $A = \mathring{A}$).

Let $x \in B_\delta(x_0)$. Then $r = \delta - d(x, x_0) > 0$ and $B_r(x) \subseteq B_\delta(x_0)$

$$\begin{aligned} &\implies \omega(f, B_r(x)) \le \omega(f, B_\delta(x_0)) < \alpha \\ &\implies \omega(f, x) \le \omega(f, B_r(x)) < \alpha \implies x \in A \end{aligned}$$

□

**Remark 10.8.** Let $(X, d)$ be a metric space and let $f : X \to \mathbb{R}$ be a function. Then

$$\begin{aligned} \{x \in X : f \text{ is continuous at } x\} &= \{x \in X : \omega(f, x) = 0\} \\ &= \bigcap_{n \ge 1} \underbrace{\left\{x \in X : \omega(f, x) < \frac{1}{n}\right\}}_{= G_n} \end{aligned}$$

By the lemma, $G_n = \mathring{G}_n$ $\forall n \ge 1$. Also, $G_{n+1} \subseteq G_n$ $\forall n \ge 1$. This observation allows us to prove that there are no functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous at every rational point and discontinuous at every irrational point.

# §11 Lec 11: Apr 21, 2021

## §11.1 Oscillation of a Function (Cont'd)

Recall from last lecture that there are no functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous at every rational point and discontinuous at every irrational point.

*Proof.* (Sketch) Assume, towards a contradiction, that $f : \mathbb{R} \to \mathbb{R}$ is such a function. Then

$$\mathbb{Q} = \{x \in \mathbb{R} : \ f \text{ is continuous at } x\} = \bigcap_{n \geq 1} G_n \text{ with } G_n \text{ open in } \mathbb{R}$$

Note $\forall n \geq 1$, $Q \subseteq G_n$

$$\begin{aligned} &\implies \mathbb{R} = \overline{\mathbb{Q}} \subseteq \overline{G_n} \subseteq \mathbb{R} \\ &\implies \overline{G_n} = \mathbb{R} \text{ i.e. } G_n \text{ is dense in } \mathbb{R} \end{aligned}$$

Let $\{q_n\}_{n\geq 1}$ be an enumeration of $\mathbb{Q}$. For each $n \geq 1$, let $H_n = \mathbb{R} \backslash \{q_n\} = (-\infty, q_n) \cup (q_n, \infty)$. Note $H_n$ is open and dense ($\overline{H_n} = \mathbb{R}$) in $\mathbb{R}$. Also

$$\bigcap_{n\geq 1} H_n = \mathbb{R} \setminus \mathbb{Q}$$

So

$$\bigcap_{n\geq 1} G_n \cap \bigcap_{n \geq 1} H_n = \mathbb{Q} \cap \mathbb{R} \setminus \mathbb{Q} = \emptyset$$

This contradicts the following property of $\mathbb{R}$:

**Exercise 11.1.** If $\{A_n\}_{n\geq 1}$ is a countable collection of open and dense subsets of $\mathbb{R}$, then

$$\overline{\bigcap_{n\geq 1} A_n} = \mathbb{R}$$

Apply this exercise with $\{A_n : n \geq 1\} = \{G_n : n \geq 1\} \cup \{H_n : n \geq 1\}$. □

## §11.2 Weierstrass Approximation Theorem

**Theorem 11.1** (Weierstrass Approximation)

Fix $a, b \in \mathbb{R}$ with $a < b$. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then, there exists a sequence of polynomials $\{P_n\}_{n\geq 1}$ with $\deg P_n \leq n \ \forall n \geq 1$ s.t.

$$P_n \xrightarrow[n\to\infty]{u} f \quad \text{on} \quad [a, b]$$

*Proof.* First, we reduce to the case when $[a, b]$ is $[0, 1]$. Let $\phi : [0, 1] \to [a, b]$, $\phi(t) = a + t(b - a)$. Note $\phi$ is a continuous, bijective function with the inverse

$$\phi^{-1} : [a, b] \to [0, 1], \quad \phi^{-1}(x) = \frac{x - a}{b - a} \text{ continuous}$$

As $f : [a, b] \to \mathbb{R}$ is continuous, $f \circ \phi : [0, 1] \to \mathbb{R}$ is continuous.
If $\{P_n\}_{n\geq 1}$ is a sequence of polynomials with $\deg P_n \leq n$ s.t.

$$P_n \xrightarrow[n\to\infty]{u} f \circ \phi \text{ on } [0,1]$$

then $P_n \circ \phi^{-1} \xrightarrow[n\to\infty]{u} f$ on $[a, b]$. Indeed,

$$\sup_{x\in[a,b]} \left|\left(P_n \circ \phi^{-1}\right)(x) - f(x)\right| \underset{x=\phi(t)}{=} \underbrace{\sup_{t\in[0,1]} |P_n(t) - (f \circ \phi)(t)|}_{\xrightarrow[n\to\infty]{} 0}$$

Therefore, we may assume $f : [0, 1] \to \mathbb{R}$ is continuous. Define the Bernstein polynomials via

$$P_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k} \qquad \deg P_n \leq n$$

Note that if $f$ is a constant, say $f(x) = c \ \forall x \in [0, 1]$ then

$$P_n(x) = c\sum_{k=0}^{n}\binom{n}{k} x^k(1-x)^{n-k} = c\,(x+1-x)^n = c \quad \forall x \in [0,1]\ \forall n \geq 1$$

We want to show $P_n \xrightarrow[n\to\infty]{u} f$ on $[0, 1]$. Fix $x \in [0, 1]$. Consider

$$\begin{aligned} |f(x) - P_n(x)| &= \left| f(x)\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k} - \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}\right| \\ &= \left|\sum_{k=0}^{n}\left[f(x) - f\left(\frac{k}{n}\right)\right]\binom{n}{k}x^k(1-x)^{n-k}\right| \\ &\leq \sum_{k=0}^{n}\left|f(x) - f\left(\frac{k}{n}\right)\right|\binom{n}{k}x^k(1-x)^{n-k} \end{aligned}$$

To estimate the sum we use the following

- when $\frac{k}{n}$ is close to $x$, we use the continuity of $f$.
- when $\frac{k}{n}$ is far from $x$, we use the fact that $x \overset{g}{\mapsto} x^k(1-x)^{n-k}$ has a local maximum at $x = \frac{k}{n}$.

$$\begin{aligned} g'(x) &= kx^{k-1}(1-x)^{n-k} - (n-k)x^k(1-x)^{n-k-1} \\ &= x^{k-1}(1-x)^{n-k-1}\{k(1-x) - (n-k)x\} \\ &= x^{k-1}(1-x)^{n-k-1}\{k - nx\} \\ &= \begin{cases} > 0 & \text{if } x < \frac{k}{n} \\ = 0 & \text{if } x = \frac{k}{n} \\ < 0 & \text{if } x > \frac{k}{n} \end{cases} \end{aligned}$$

$f : [0,1] \to \mathbb{R}$ is continuous $\implies$ $f$ is uniformly continuous. Fix $\varepsilon > 0$. Then $\exists \delta > 0$ s.t.

$$|f(x) - f(y)| < \varepsilon \quad \text{whenever} \quad x, y \in [0,1], \quad |x - y| < \delta$$

$f : [0,1] \to \mathbb{R}$ is continuous $\implies$ $f$ is bounded. Let $M > 0$ be s.t.

$$|f(x)| \le M \qquad \forall x \in [0,1]$$

We estimate

$$\begin{aligned}
|f(x) - P_n(x)| &\le \sum_{\substack{0 \le k \le n \\ \left|x - \frac{k}{n}\right| < \delta}} \underbrace{\left|f(x) - f\left(\frac{k}{n}\right)\right|}_{<\varepsilon} \binom{n}{k} x^k (1-x)^{n-k} \\
&+ \sum_{\substack{0 \le k \le n \\ \left|x - \frac{k}{n}\right| \ge \delta}} \underbrace{\left|f(x) - f\left(\frac{k}{n}\right)\right|}_{\le 2M} \binom{n}{k} x^k (1-x)^{n-k} \\
&\le \varepsilon \sum_{0 \le k \le n} \binom{n}{k} x^k (1-x)^{n-k} + 2M \sum_{0 \le k \le n} \frac{\left(x - \frac{k}{n}\right)^2}{\delta^2} \binom{n}{k} x^k (1-x)^{n-k} \\
&\le \varepsilon + \frac{2M}{n^2 \delta^2} \sum_{k=0}^{n} (nx - k)^2 \binom{n}{k} x^k (1-x)^{n-k}
\end{aligned}$$

Observe that

$$\begin{aligned}
\sum_{k=0}^{n} (nx - k)^2 \binom{n}{k} x^k (1-x)^{n-k} &= n^2 x^2 \underbrace{\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}}_{=1} \\
&- 2nx \sum_{k=0}^{n} k \cdot \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} + \sum_{k=0}^{n} k^2 \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k}
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{k=0}^{n} k \cdot \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} &= x \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} x^{k-1} (1-x)^{n-k} \\
&= nx \underbrace{\sum_{l=0}^{n-1} \frac{(n-1)!}{l!(n-1-l)!} x^l (1-x)^{n-1-l}}_{=(x+1-x)^{n-1}} \\
&= nx
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{n} k^2 \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} &= nx \sum_{k=1}^{n} \frac{k(n-1)!}{(k-1)!(n-k)!} x^{k-1}(1-x)^{n-k} \\
&= nx \sum_{k=1}^{n} \frac{(k-1+1)(n-1)!}{(k-1)!(n-k)!} x^{k-1}(1-x)^{n-k} \\
&= n(n-1)x^2 \sum_{k=2}^{n} \frac{(n-2)!}{(k-2)!(n-k)!} x^{k-2}(1-x)^{n-k} \\
&+ nx \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} x^{k-1}(1-x)^{n-k} \\
&= n(n-1)x^2 + nx
\end{aligned}$$

So

$$\begin{aligned}
\sum_{k=0}^{n} (nx-k)^2 \binom{n}{k} x^k (1-x)^{n-k} &= n^2x^2 - 2n^2x^2 + n(n-1)x^2 + nx \\
&= nx(1-x)
\end{aligned}$$

We get

$$\begin{aligned}
|f(x) - P_n(x)| &\le \varepsilon + \frac{2M}{n^2\delta^2} \cdot nx(1-x) \\
&\le \varepsilon + \frac{2M}{n\delta^2} \sup_{x\in[0,1]} x(1-x) \\
&\le \varepsilon + \frac{M}{2\delta^2 n} < 2\varepsilon
\end{aligned}$$

provided $n > \frac{M}{2\delta^2\varepsilon}$. So $P_n \xrightarrow[n\to\infty]{u} f$ on $[0,1]$. □

# §12 Lec 12: Apr 23, 2021

## §12.1 Weierstrass Approximation Theorem (Cont'd)

**Corollary 12.1**

Let $M > 0$. Then there exists a sequence of polynomials $\{P_n\}_{n\geq 1}$ s.t.

$$\begin{cases} \deg P_n \leq n & \forall n \geq 1 \\ P_n(0) = 0 & \forall n \geq 1 \\ P_n \xrightarrow[n\to\infty]{u} |x| \text{ on } [-M, M] \end{cases}$$

*Proof.* Let $f : [-M, M] \to \mathbb{R}$, $f(x) = |x|$. Then $f$ is continuous and $[-M, M]$ compact. By Weierstrass Approximation, $\exists \{Q_n\}_{n\geq 1}$ sequence of polynomials s.t.

$$\begin{cases} \deg Q_n \leq n & \forall n \geq 1 \\ Q_n \xrightarrow[n\to\infty]{u} f \text{ on } [-M, M] \end{cases}$$

Note $Q_n \xrightarrow[n\to\infty]{u} f \implies Q_n(0) \xrightarrow[n\to\infty]{} f(0) = 0$.
Let $P_n(x) = Q_n(x) - Q_n(0)$. Then

$$\begin{cases} \deg P_n \leq n & \forall n \geq 1 \\ P_n(0) = 0 & \forall n \geq 1 \end{cases}$$

For $x \in [-M, M]$,

$$|P_n(x) - f(x)| \leq |Q_n(x) - f(x)| + |Q_n(0)| \leq d(Q_n, f) + |Q_n(0)|$$
$$\implies d(P_n, f) \leq d(Q_n, f) + |Q_n(0)| \xrightarrow[n\to\infty]{} 0$$

□

## §12.2 Stone-Weierstrass Theorem

**Definition 12.2** (Algebra) — Let $(X, d)$ be a metric space and let

$$\mathcal{A} \subseteq \{f : X \to \mathbb{R}(\text{or } \mathbb{C});\ f \text{ is a function}\}$$

We say that $\mathcal{A}$ is an algebra if

1. $f + g \in \mathcal{A} \qquad \forall f, g \in \mathcal{A}$.
2. $fg \in \mathcal{A} \qquad \forall f, g \in \mathcal{A}$
3. $\lambda f \in \mathcal{A} \qquad \forall f \in \mathcal{A}\ \forall \lambda \in \mathbb{R}(\text{or } \mathbb{C})$

We say that the algebra $\mathcal{A}$ separates points if whenever $x, y \in X$ with $x \neq y$ then $\exists f \in \mathcal{A}$ s.t. $f(x) \neq f(y)$.
We say that the algebra $\mathcal{A}$ vanishes at no point in $X$ if $\forall x \in X\ \exists f \in \mathcal{A}$ s.t. $f(x) \neq 0$.

**Lemma 12.3**

Let $(X, d)$ be a compact metric space and let $\mathcal{A} \subseteq C(X)$ be an algebra. Then its closure $\overline{\mathcal{A}}$ with respect to the uniform topology is also an algebra.

*Proof.* Let $f, g \in \mathcal{A}$. Then

$$\begin{cases} \exists f_n \in \mathcal{A} \text{ s.t. } f_n \xrightarrow[n\to\infty]{u} f \text{ on } X \\ \exists g_n \in \mathcal{A} \text{ s.t. } g_n \xrightarrow[n\to\infty]{u} g \text{ on } X \end{cases}$$

$$\left.\begin{aligned} & d\left(f_n + g_n, f+g\right) \le d(f_n, f) + d(g_n, g) \xrightarrow[n\to\infty]{} 0 \\ & f_n + g_n \in \mathcal{A} \text{ (because } \mathcal{A} \text{ is an algebra)} \end{aligned}\right\} \implies f + g \in \overline{\mathcal{A}}$$

Similarly, for $\lambda \in \mathbb{R}$,

$$\left.\begin{aligned} & d\left(\lambda f_n, \lambda f\right) \le |\lambda| d\left(f_n, f\right) \xrightarrow[n\to\infty]{} 0 \\ & \lambda f_n \in \mathcal{A} \text{ (because } \mathcal{A} \text{ is an algebra)} \end{aligned}\right\} \implies \lambda f \in \overline{\mathcal{A}}$$

Then

$$\begin{aligned} d\left(f_n g_n, fg\right) &= \sup_{x\in X} |f_n(x) g_n(x) - f(x) g(x)| \\ &\le \sup_{x\in X} \left[|f_n(x) - f(x)|\,|g_n(x)| + |f(x)|\,|g_n(x) - g(x)|\right] \\ &\le d(f_n, f) \sup_{x\in X} |g_n(x)| + d(g_n, g) \sup_{x\in X} |f(x)| \end{aligned}$$

By Weierstrass,

$$\left.\begin{aligned} & f_n \xrightarrow[n\to\infty]{u} f \text{ on } X \\ & f_n \in C(X) \end{aligned}\right\} \implies \left.\begin{aligned} & f \in C(X) \\ & X \text{ compact} \end{aligned}\right\} \implies \exists M > 0 \text{ s.t. } \sup_{x\in X} |f(x)| \le M$$

Similarly, $g \in C(X) \implies \exists M_2 > 0$ s.t. $\sup_{x\in X} |g(x)| \le M_2$

$$d(g_n, 0) \le d(g_n, g) + d(g, 0) \le 1 + M_2 \qquad \forall n \ge n_1$$

Let $M_3 = \max\left\{1 + M_2, \underbrace{d(g_1, 0)}_{<\infty}, \ldots, \underbrace{d(g_{n_1}, 0)}_{<\infty}\right\}$. So $d(g_n, 0) \le M_3 \, \forall n \ge 1$. Thus

$$\left.\begin{aligned} & d\left(f_n g_n, fg\right) \le d(f_n, f) \cdot M_3 + d(g_n, g) \cdot M_1 \xrightarrow[n\to\infty]{} 0 \\ & f_n g_n \in \mathcal{A} \text{ (since } \mathcal{A} \text{ is an algebra)} \end{aligned}\right\} \implies f \cdot g \in \overline{\mathcal{A}}$$ ∎

**Lemma 12.4**

Let $(X, d)$ be a compact metric space and let $\mathcal{A} \subseteq C(X)$ be an algebra that separates points and vanishes at no point in $X$. Then

$$\forall \alpha, \beta \in \mathbb{R} \quad \forall x_1, x_2 \in X \text{ s.t. } x_1 \ne x_2 \quad \exists f \in \mathcal{A} \text{ s.t. } \begin{cases} f(x_1) = \alpha \\ f(x_2) = \beta \end{cases}$$

*Proof.* Fix $\alpha, \beta \in \mathbb{R}$. Fix $x_1, x_2 \in X$ s.t. $x_1 \neq x_2$. We would like

$$f(x) = \alpha \cdot \frac{u(x)}{u(x_1)} + \beta \cdot \frac{v(x)}{v(x_1)}$$

for $u, v \in \mathcal{A}$ s.t.

$$\begin{aligned} u(x_1) \neq 0 \quad &\text{and} \quad u(x_2) = 0 \\ v(x_1) = 0 \quad &\text{and} \quad v(x_2) \neq 0 \end{aligned}$$

Then $f \in \mathcal{A}$ (because $\mathcal{A}$ is an algebra) is the desired function.

As $\mathcal{A}$ separates points, $\exists g \in \mathcal{A}$ s.t. $g(x_1) \neq g(x_2)$.

As $\mathcal{A}$ vanishes at no point in $X$,

$$\begin{cases} \exists h \in \mathcal{A} \text{ s.t } h(x_1) \neq 0 \\ \exists k \in \mathcal{A} \text{ s.t. } k(x_2) \neq 0 \end{cases}$$

Then, we define

$$\begin{aligned} u(x) &= [g(x) - g(x_2)] \cdot h(x) \in \mathcal{A} \\ v(x) &= [g(x) - g(x_1)] \cdot k(x) \in \mathcal{A} \end{aligned}$$

□

**Theorem 12.5** (Stone-Weierstrass)

Let $(X, d)$ be a compact metric space and let $\mathcal{A} \subseteq C(X)$ be an algebra that separates points and vanishes no point in $X$. Then $\mathcal{A}$ is dense in $C(X)$, i.e., $\overline{\mathcal{A}} = C(X) = \{f : X \to \mathbb{R};\ f \text{ continuous}\}$.

*Proof.* Want to show $\forall f \in C(X)\ \forall \varepsilon > 0\ \exists g \in \mathcal{A}$ s.t. $d(f, g) < \varepsilon$.

**Step 1**: If $f \in \overline{\mathcal{A}}$ then $|f| \in \overline{\mathcal{A}}$. Let $f \in \overline{\mathcal{A}} \implies \exists f_n \in \mathcal{A}$ s.t.

$$\left.\begin{aligned} &f_n \xrightarrow[n\to\infty]{u} f \text{ on } X \\ &f_n \in C(X) \end{aligned}\right\} \implies f \in C(X)$$

As $X$ is compact, $\exists M > 0$ s.t. $|f(x)| \leq M\ \forall x \in X$. By the previous Corollary 12.1, $\exists \{P_n\}_{n\geq 1}$ sequence of polynomials with $\deg P_n \leq n\ \forall n \geq 1$ s.t.

$$\begin{cases} P_n \xrightarrow[n\to\infty]{u} |x| \text{on } [-M, M] \\ P_n(0) = 0 \end{cases} \implies P_n(f) \xrightarrow[n\to\infty]{u} |f| \text{ on } X$$

If $P_n(x) = \sum_{k=1}^n c_k x^k$ then $P_n(f) = \sum_{k=1}^n c_k f^k \in \mathcal{A}$ which implies $|f| \in \overline{\mathcal{A}}$.

**Step 2**: If $f, g \in \overline{\mathcal{A}}$ then $\max\{f, g\}$, $\min\{f, g\} \in \overline{\mathcal{A}}$.

$$\begin{aligned} \max\{f, g\} &= \frac{f+g}{2} + \frac{|f-g|}{2} \in \overline{\mathcal{A}} \\ \min\{f, g\} &= \frac{f+g}{2} - \frac{|f-g|}{2} \in \overline{\mathcal{A}} \end{aligned}$$

**Step 3**: $\forall f \in C(X)$, $\forall x \in X$, $\forall \varepsilon > 0$, $\exists g \in \overline{\mathcal{A}}$ s.t.

$$g(x) = f(x) \quad \text{and} \quad g(y) > f(y) - \varepsilon \quad \forall y \in X$$

Continue in the next lecture. □

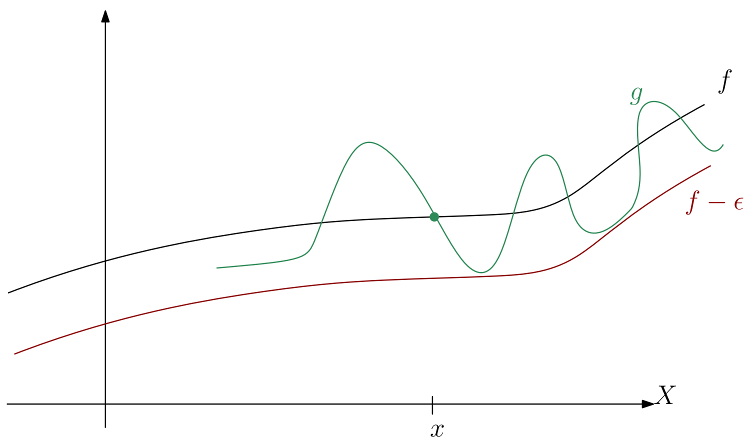
# §13 Lec 13: Apr 26, 2021

## §13.1 Stone-Weierstrass Theorem (Cont'd)

We continue with the proof of Stone-Weierstrass from lecture 12. Recall that we are at step 3 so far.

*Proof.* **Step 3**: For any $f \in C(X)$, $x \in X$, $\varepsilon > 0$, there exists $g \in \overline{\mathcal{A}}$ s.t.

$$\begin{cases} g(x) = f(x) \\ g(y) > f(y) - \varepsilon \qquad \forall y \in X \end{cases}$$




For any $y \in X$, there exists $h_y \in \overline{\mathcal{A}}$ s.t.

$$h_y(x) = f(x)$$
$$h_y(y) = f(y)$$

As $h_y \in \overline{\mathcal{A}}$, $h_y$ is continuous. Thus, $h_y - f$ is continuous at $y$. So $\exists \delta_y > 0$ s.t. $|h_y(z) - f(z)| < \varepsilon$, $\forall z \in B_{\delta_y}(y)$. In particular,

$$h_y(z) > f(z) - \varepsilon \qquad \forall z \in B_{\delta_y}(y)$$

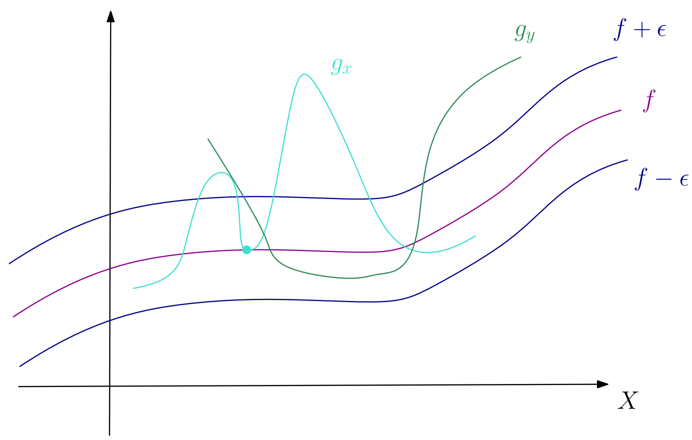
Note that

$$\left.\begin{matrix} X = \bigcup_{y \in X} B_{\delta_y}(y) \\ X \text{ compact} \end{matrix}\right\} \implies \exists N \geq 1 \text{ and } \exists y_1, \ldots, y_N \in X$$

s.t. $X = \bigcup_{n=1}^{N} B_{\delta_n}(y_n)$ where $\delta_n = \delta_{y_n}$.
Take $g = \max\{h_{y_1}, \ldots, h_{y_N}\}$ (by step 2). By construction, $g(x) = f(x)$. Also if $y \in X$, $\exists 1 \leq n \leq N$ s.t. $y \in B_{\delta_n}(y_n)$. So

$$g(y) \geq h_{y_n}(y) > f(y) - \varepsilon$$

**Step 4**: For all $f \in C(X)$ and $\varepsilon > 0$, $\exists g \in \overline{\mathcal{A}}$ s.t. $d(f, g) < \varepsilon$. Fix $f \in C(X)$, $\varepsilon > 0$




For $x \in X$, let $g_x \in \overline{\mathcal{A}}$ be the function given by step 3. In particular, $g_x(x) = f(x)$,

$$g_x(y) > f(y) - \varepsilon \qquad \forall y \in X$$

As $g_x \in \overline{\mathcal{A}}$, the function $g_x - f$ is continuous at $x$. So $\exists \delta_x > 0$ s.t. $|g_x(y) - f(y)| < \varepsilon$, $\forall y \in B_{\delta_x}(x)$. In particular,

$$g_x(y) < f(y) + \varepsilon \qquad \forall y \in B_{\delta_x}(x)$$

Note

$$\left.\begin{aligned} &X = \textstyle\bigcup_{x\in X} B_{\delta_x}(x) \\ &X \text{ compact} \end{aligned}\right\} \implies \exists N \geq 1 \text{ and } \exists x_1, \ldots, x_N \in X \text{ s.t.}$$

$X = \bigcup_{n=1}^{N} B_{\delta_n}(x_n)$ where $\delta_n = \delta_{x_n}$.
Take $g = \min\{g_{x_1}, \ldots, g_{x_N}\} \in \overline{\mathcal{A}}$ (by step 2).
For $y \in X$, $\exists 1 \leq n \leq N$ s.t. $y \in B_{\delta_n}(x_n)$ and so

$$g(y) \leq g_{x_n}(y) < f(y) + \varepsilon$$

Moreover, as $g_{x_n}(y) > f(y) - \varepsilon$, $\forall y \in X$, $\forall 1 \leq n \leq N$, we have

$$g(y) > f(y) - \varepsilon \qquad \forall y \in X$$

This shows $C(X) \subseteq \overline{\overline{\mathcal{A}}} = \overline{\mathcal{A}} \subseteq C(X)$. □

## §13.2 Differentiation

**Definition 13.1** (Limit) — Let $(X, d_X), (Y, d_Y)$ be metric spaces, let $\emptyset \neq A \subseteq X$, let $f : A \to Y$. For $x_0 \in A'$ and $y_0 \in Y$ we write

$$f \xrightarrow[x\to x_0]{} y_0 \qquad \text{or} \qquad \lim_{x\to x_0} f(x) = y_0$$

if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $d_Y(f(x), y_0) < \varepsilon$ whenever $0 < d_X(x, x_0) < \delta$.
Equivalently, $\lim_{x\to x_0} f(x) = y_0$ if

$$\lim_{n\to\infty} f(x_n) = y_0 \text{ for every sequence } \{x_n\}_{n\geq 1} \subseteq A \setminus \{x_0\} \text{ s.t. } x_n \xrightarrow[n\to\infty]{d_X} x_0$$

Note also that if $x_0 \in A' \cap A$ then $f$ is continuous at $x_0 \iff \lim_{x \to x_0} f(x) = f(x_0)$.

**Exercise 13.1.** Let $(X, d)$ be a metric space, $\emptyset \neq A \subseteq X$, $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be functions. Assume that at a point $a \in A'$ we have

$$\lim_{x \to x_0} f(x) = \alpha \quad \text{and} \quad \lim_{x \to x_0} g(x) = \beta$$

Then

1. $\lim_{x \to x_0} (\lambda f(x)) = \lambda \alpha$, $\lambda \in \mathbb{R}$
2. $\lim_{x \to x_0} (f(x) + g(x)) = \alpha + \beta$
3. $\lim_{x \to x_0} (f(x) g(x)) = \alpha \cdot \beta$
4. If $\beta \neq 0$ then $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\alpha}{\beta}$

**Definition 13.2** (Differentiability) — Let $I$ be an open interval and let $f : I \to \mathbb{R}$ be a function. We say that $f$ is differentiable at $a \in I$ if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \text{ exists and is finite}$$

in which case we denote it $f'(a)$.

**Example 13.3**

Fix $n \geq 1$ and let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^n$. For $a \in \mathbb{R}$ and $x \neq a$

$$\begin{aligned} \frac{f(x) - f(a)}{x - a} &= \frac{x^n - a^n}{x - a} \\ &= x^{n-1} + x^{n-2} a + \ldots + a^{n-1} \underset{x \to a}{\longrightarrow} n a^{n-1} \end{aligned}$$

So $f$ is differentiable at $a$ and $f'(a) = n a^{n-1}$.

**Theorem 13.4**

Let $I$ be an open interval and let $f : I \to \mathbb{R}$ be differentiable at $a \in I$. Then $f$ is continuous at $a$.

*Proof.* For $x \in I \setminus \{a\}$, we write

$$f(x) = \underbrace{\frac{f(x) - f(a)}{x - a}}_{\underset{x \to a}{\longrightarrow} f'(a)} \cdot \underbrace{(x - a)}_{\underset{x \to a}{\longrightarrow} 0} + \underbrace{f(a)}_{\underset{x \to a}{\longrightarrow} f(a)} \underset{x \to a}{\longrightarrow} f(a)$$

□

**Theorem 13.5**

Let $I$ be an open interval and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be two functions differentiable at $a \in I$. Then

1. $\forall \lambda \in \mathbb{R}$, $\lambda f$ is differentiable at $a$ and
$$(\lambda f)'(a) = \lambda f'(a)$$
2. $f + g$ is differentiable at $a$ and
$$(f+g)'(a) = f'(a) + g'(a)$$
3. $f \cdot g$ is differentiable at $a$ and
$$(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a)$$
4. $\frac{f}{g}$ is differentiable at $a$ if $g(a) \neq 0$ and
$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}$$

*Proof.* For $x \neq a$

1. Consider
$$\frac{\lambda f(x) - \lambda f(a)}{x-a} = \lambda \cdot \frac{f(x)-f(a)}{x-a} \xrightarrow[x\to a]{} \lambda f'(a)$$
2. Consider
$$\frac{(f(x)+g(x)) - (f(a)+g(a))}{x-a} = \frac{f(x)-f(a)}{x-a} + \frac{g(x)-g(a)}{x-a} \xrightarrow[x\to a]{} f'(a) + g'(a)$$
3. Consider
$$\underbrace{\frac{f(x)-f(a)}{x-a}}_{\xrightarrow[x\to a]{} f'(a)} \cdot \underbrace{g(x)}_{\xrightarrow[x\to a]{} g(a)} + \underbrace{f(a)}_{\xrightarrow[x\to a]{} f(a)} \cdot \underbrace{\frac{g(x)-g(a)}{x-a}}_{\xrightarrow[x\to a]{} g'(a)} \xrightarrow[x\to a]{} f'(a)g(a) + f(a)g'(a)$$
4. Consider
$$\frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x-a} \underbrace{\frac{f(x)-f(a)}{x-a}}_{\xrightarrow[x\to a]{} f'(a)} \cdot \underbrace{\frac{1}{g(x)}}_{\xrightarrow[x\to a]{} \frac{1}{g(a)}} + f(a) \cdot \underbrace{\frac{g(a)-g(x)}{x-a}}_{\xrightarrow[x\to a]{} -g'(a)} \cdot \underbrace{\frac{1}{g(x)}}_{\xrightarrow[x\to a]{} \frac{1}{g(a)}} \cdot \frac{1}{g(a)}$$
$$\xrightarrow[x\to a]{} \frac{f'(a)}{g(a)} - \frac{g'(a)}{g^2(a)} f(a)$$

□

# §14 Lec 14: Apr 28, 2021

## §14.1 Chain Rule

**Theorem 14.1** (Chain Rule)

Let $I$ and $J$ be two open intervals and let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be two functions. Assume that $f$ is differentiable at $a \in I$ and that $g$ is differentiable at $f(a) \in J$. Then $g \circ f$ is well defined on a neighborhood of $a$, $g \circ f$ is differentiable at $a$, and

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$$

*Proof.* Consider:

$$\left.\begin{matrix} f(a) \in J \\ J \text{ is open} \end{matrix}\right\} \implies \exists \varepsilon > 0 \text{ s.t. } (f(a) - \varepsilon, f(a) + \varepsilon) \subseteq J$$

$f$ is differentiable at $a \implies f$ is continuous at $a \implies \exists \delta > 0$ s.t. $f((a-\delta, a+\delta) \cap I) \subseteq (f(a) - \varepsilon, f(a) + \varepsilon)$. As $a \in I$ and $I$ is open, shrinking $\delta$ if necessary, me may assume that $(a - \delta, a + \delta) \subseteq I$.

Then $g \circ f$ is well-defined on $(a - \delta, a + \delta)$.

$$\underbrace{(a-\delta, a+\delta)}_{\subseteq I} \xrightarrow{f} \underbrace{(f(a) - \varepsilon, f(a) + \varepsilon)}_{\subseteq J} \xrightarrow{g} \mathbb{R}$$

<u>Caution</u>: The following argument does not work

$$\frac{g(f(x)) - g(f(a))}{x - a} = \underbrace{\frac{g(f(x)) - g(f(a))}{f(x) - f(a)}}_{\xrightarrow{x \to a} g'(f(a))} \cdot \underbrace{\frac{f(x) - f(a)}{x - a}}_{\xrightarrow{x \to a} f'(a)}$$

because $f$ is continuous at $a \implies f(x) \xrightarrow{x \to a} f(a)$

Instead, we argue as follows: Define $h : J \to \mathbb{R}$,

$$h(y) = \begin{cases} \frac{g(y) - g(f(a))}{y - f(a)}, & \text{if } y \in J \setminus \{f(a)\} \\ g'(f(a)), & \text{if } y = f(a) \end{cases}$$

As $g$ is differentiable at $f(a)$, $h$ is continuous at $f(a)$. Moreover, we can write

$$g(y) - g(f(a)) = h(y) \cdot (y - f(a)) \qquad \forall y \in J$$

For $x \in (a - \delta, a + \delta) \implies f(x) \in J$. So for $x \in (a - \delta, a + \delta) \setminus \{a\}$,

$$\frac{g(f(x)) - g(f(a))}{x - a} = \underbrace{h(f(x))}_{\xrightarrow{x \to a} h(f(a))} \cdot \underbrace{\frac{f(x) - f(a)}{x - a}}_{\xrightarrow{x \to a} f'(a)}$$

So $\lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a} = h(f(a)) f'(a) = g'(f(a)) \cdot f'(a)$. ∎

**Lemma 14.2**

Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. If $f$ is increasing then $f'(x) \geq 0 \, \forall x \in (a, b)$ or decreasing then $f'(x) \leq 0 \, \forall x \in (a, b)$.

*Proof.* Assume $f$ is increasing (if $f$ is decreasing, replace $f$ by $-f$ in what follows). Fix $x \in (a, b)$ and let $\{x_n\}_{n \geq 1}$ be an increasing from $(a, b)$ with $\lim_{n\to\infty} x_n = x$.
Then $f'(x) = \lim_{n\to\infty} \frac{f(x_n)-f(x)}{x_n - x} \geq 0$ where $f(x_n) - f(x) \leq 0$ and $x_n - x < 0$. □

**Theorem 14.3**

Let $f : (a, b) \to \mathbb{R}$ be a function. Assume that $x_0 \in (a, b)$ is a point of local maximum/minimum for $f$. Assume also that $f$ is differentiable at $x_0$. Then $f'(x_0) = 0$.

*Proof.* Assume that $x_0$ is a point of local maximum for $f$ (if $x_0$ is a point of local minimum, replace $f$ by $-f$ in what follows).

Then $\exists \delta > 0$ s.t. $f(x) \leq f(x_0) \quad \forall x \in (x_0 - \delta, x_0 + \delta) \cap (a, b)$. For $x_n \in (x_0 - \delta, x_0) \cap (a, b)$ s.t. $x_n \underset{n\to\infty}{\longrightarrow} x_0$, we have

$$f'(x_0) = \lim_{n\to\infty} \frac{f(x_n) - f(x_0) \leq 0}{x_n - x_0 < 0} \geq 0$$

On the other hand, for $y_n \in (x_0, x_0 + \delta) \cap (a, b)$ s.t. $y_n \underset{n\to\infty}{\longrightarrow} x_0$, we have

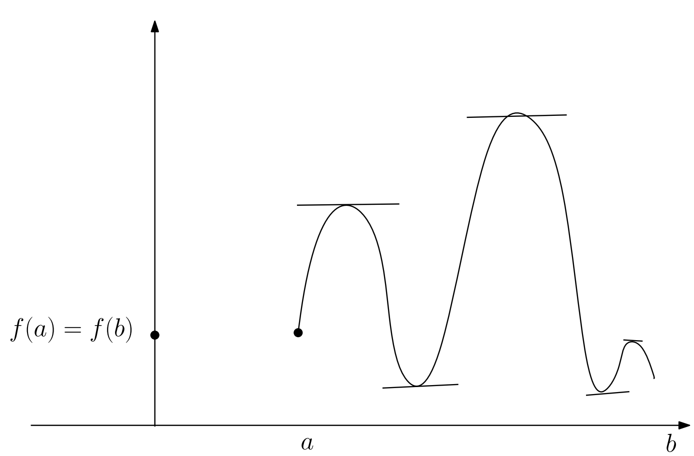
$$f'(x_0) = \lim_{n\to\infty} \frac{f(y_n) - f(x_0) \leq 0}{y_n - x_0 > 0} \leq 0$$

Thus, we get $f'(x_0) = 0$. □

## §14.2 Mean Value Theorem

**Theorem 14.4** (Rolle)

Let $f : [a, b] \to \mathbb{R}$ be a function which is continuous on the $[a, b]$, differentiable on $(a, b)$, and s.t. $f(a) = f(b)$. Then there exists (at least one) $x \in (a, b)$ s.t. $f'(x) = 0$.



*Proof.* Consider:

$$\left.\begin{array}{l} f:[a,b]\to\mathbb{R} \text{ continuous} \\ [a,b] \text{ compact} \end{array}\right\} \implies \exists x_0, y_0 \in [a,b]$$

s.t.

$$f(x_0) = \sup_{x\in[a,b]} f(x) \quad \text{and} \quad f(y_0) = \inf_{x\in[a,b]} f(x)$$

So $f(y_0) \le f(x) \le f(x_0) \quad \forall x \in [a,b]$.
**Case 1:** We have

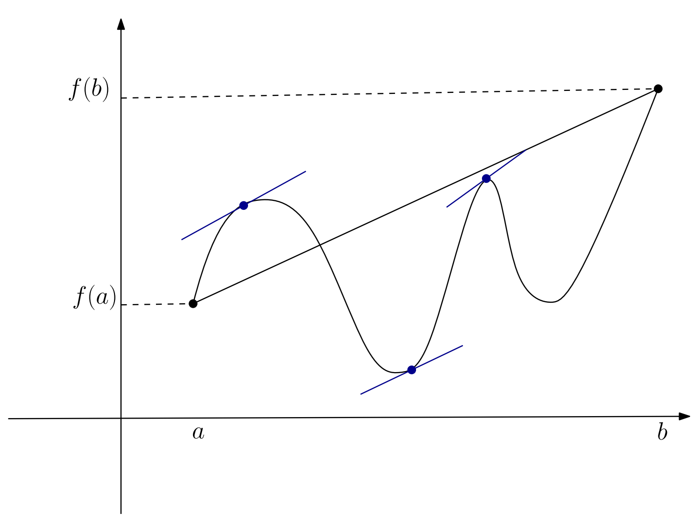
$$\left.\begin{array}{l} \{x_0, y_0\} \subseteq \{a,b\} \\ f(a) = f(b) \end{array}\right\} \implies f(x_0) = f(y_0) \implies f \text{ constant} \implies f'(x) = 0 \,\forall x \in (a,b)$$

**Case 2:** $\{x_0, y_0\} \nsubseteq \{a,b\} \implies x_0 \notin \{a,b\}$ or $y_0 \notin \{a,b\}$. Say $x_0 \notin \{a,b\} \implies x_0 \in (a,b)$. By Theorem 14.3, we get $f'(x_0) = 0$. ☐

**Theorem 14.5** (Mean Value)
Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)$. Then there exists (at least one) $y \in (a,b)$ s.t.

$$f'(y) = \frac{f(b)-f(a)}{b-a}$$

**Remark 14.6.** The Mean Value Theorem implies Rolle's Theorem. We will see from the proof that Rolle's Theorem implies the Mean Value Theorem, so the two are equivalent.




*Proof.* We define $l:[a,b]\to\mathbb{R}$ where

$$l(x) = \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$$

Note that $l$ is continuous on $[a, b]$, differentiable on $(a, b)$, and

$$l'(x) = \frac{f(b) - f(a)}{b - a} \qquad \forall x \in (a, b)$$

Let $g : [a, b] \to \mathbb{R}$, $g(x) = f(x) - l(x)$. Then $g$ is continuous on $[a, b]$, differentiable on $(a, b)$, and $g(a) = 0 = g(b)$. Then Rolle's implies that $\exists y \in (a, b)$ s.t.

$$g'(y) = 0 \implies f'(y) - l'(y) = 0 \implies f'(y) = \frac{f(b) - f(a)}{b - a}$$ □

**Corollary 14.7**
If $f : (a, b) \to \mathbb{R}$ is differentiable and $f'(x) = 0 \, \forall x \in (a, b)$, then $f$ is a constant.

*Proof.* Assume $f$ is not a constant. Then $\exists a < x_1 < x_2 < b$ s.t.

$$f(x_1) \neq f(x_2)$$

Then $f$ is continuous on $[x_1, x_2]$, differentiable on $(x_1, x_2)$. By Mean Value, $\exists y \in (x_1, x_2)$ s.t.

$$f'(y) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} \neq 0$$

Contradiction! □

**Corollary 14.8**
If $f, g : (a, b) \to \mathbb{R}$ are differentiable s.t. $f'(x) = g'(x) \, \forall x \in (a, b)$, then $\exists c \in \mathbb{R}$ s.t.

$$f(x) = g(x) + c \qquad \forall x \in (a, b)$$

# §15 Lec 15: Apr 30, 2021

## §15.1 Mean Value Theorem (Cont'd)

**Theorem 15.1**

Let $f : [a, b] \to \mathbb{R}$, $g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$. Then there exists (at least one) $c \in (a, b)$ s.t.

$$f'(c)\,[g(b) - g(a)] = g'(c)\,[f(b) - f(a)]$$

**Remark 15.2.** Taking $g(x) = x$ we recover the Mean Value theorem. In fact, the two results are equivalent, as can be seen from the proof.

*Proof.* We define $h : [a, b] \to \mathbb{R}$

$$h(x) = f(x)\,[g(b) - g(a)] - g(x)\,[f(b) - f(a)]$$

Note that $h$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Moreover,

$$\left.\begin{aligned} h(a) &= f(a)\,[g(b) - g(a)] - g(a)\,[f(b) - f(a)] = f(a)g(b) - g(a)f(b) \\ h(b) &= f(b)\,[g(b) - g(a)] - g(b)\,[f(b) - f(a)] = -f(b)g(a) + g(b)f(a) \end{aligned}\right\} \implies h(a) = h(b)$$

By Rolle's theorem, $\exists c \in (a, b)$ s.t $h'(c) = 0$. □

**Corollary 15.3**

Let $f : (a, b) \to \mathbb{R}$ be differentiable.

1. If $f'(x) > 0\ \forall x \in (a, b)$ then $f$ is strictly increasing.
2. If $f'(x) \geq 0\ \forall x \in (a, b)$ then $f$ is increasing.
3. If $f'(x) < 0\ \forall x \in (a, b)$ then $f$ is strictly decreasing.
4. If $f'(x) \leq 0\ \forall x \in (a, b)$ then $f$ is decreasing.

*Proof.* We only present the details for (1).
Fix $a < x_1 < x_2 < b$. $f$ is differentiable on $(a, b) \implies f$ is continuous on $[x_1, x_2]$ and differentiable on $(x_1, x_2)$. By the Mean Value theorem, $\exists c \in (x_1, x_2)$ s.t.

$$0 < f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \implies f(x_1) < f(x_2)$$

As $a < x_1 < x_2 < b$ were arbitrary, $f$ is strictly increasing. □

**Example 15.4**

The derivative of a differentiable function need not be continuous

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$f$ is continuous on $\mathbb{R} \setminus \{0\}$. To see that it's continuous at 0,

$$|f(x) - f(0)| = \left| x^2 \sin \frac{1}{x} \right| \leq x^2 \underset{x \to 0}{\longrightarrow} 0 \tag{*}$$

$f$ is differentiable on $\mathbb{R} \setminus \{0\}$. To see that it's differentiable at 0, we compute

$$x \neq 0 : \quad \frac{f(x) - f(0)}{x - 0} = x \sin \frac{1}{x} \underset{x \to 0}{\longrightarrow} 0 \quad \text{(as in (*))}$$

So $f'(0) = 0$. Thus,

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} + x^2 \cos \frac{1}{x} \cdot \frac{-1}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases} = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$f'$ is continuous on $\mathbb{R} \setminus \{0\}$ (not continuous at 0). While $\lim_{x \to 0} 2x \sin \frac{1}{x} = 0$, for each $\lambda \in [-1, 1]$, there exists $x_n(\lambda) \underset{n \to \infty}{\longrightarrow} 0$ s.t. $\cos \frac{1}{x_n(\lambda)} = \lambda$. Nevertheless, the derivative of a differentiable function has the Darboux property.

**Theorem 15.5** (Intermediate Value for Derivatives)

Let $f : (a, b) \to \mathbb{R}$ be differentiable. Then $f'$ has the Darboux property, that is, if $a < x_1 < x_2 < b$ and $\lambda$ lies between $f'(x_1)$ and $f'(x_2)$, then there exists $c \in (x_1, x_2)$ s.t.

$$f'(c) = \lambda$$

*Proof.* Let $g : (a, b) \to \mathbb{R}$, $g(x) = f(x) - \lambda x$. $g$ is differentiable on $(a, b) \implies g$ is continuous on $(a, b)$. Fix $a < x_1 < x_2 < b$ and assume without loss of generality

$$f'(x_1) < \lambda < f'(x_2)$$

Then

$$g'(x_1) = f'(x_1) - \lambda < 0$$
$$g'(x_2) = f'(x_2) - \lambda > 0$$

$g$ is continuous on $[x_1, x_2]$

$$\implies \exists c \in [x_1, x_2] \text{ s.t. } g(c) = \inf_{x \in [x_1, x_2]} g(x)$$

If we can prove that $c \in (x_1, x_2)$ then $g'(c) = 0$. To see that $c \neq x_1$ we argue as follows:

$$0 > g'(x_1) = \lim_{x \to x_1} \frac{g(x) - g(x_1)}{x - x_1} \implies \exists \delta_1 > 0$$

s.t. if $0 < |x - x_1| < \delta_1$ then

$$\frac{g(x) - g(x_1)}{x - x_1} < 0$$

In particular, for $x \in (x_1, x_1 + \delta_1)$ we have

$$\frac{g(x) - g(x_1)}{\underbrace{x - x_1}_{>0}} < 0 \implies g(x) < g(x_1)$$

$$\implies g \text{ cannot attain its minimum at } x_1$$

Similarly,

$$0 < g'(x_2) = \lim_{x \to x_2} \frac{g(x) - g(x_2)}{x - x_2} \implies \exists \delta_2 > 0$$

s.t. if $0 < |x - x_2| < \delta_2$ then

$$\frac{g(x) - g(x_2)}{x - x_2} > 0$$

In particular, if $x \in (x_2 - \delta_2, x_2)$ then

$$\frac{g(x) - g(x_2)}{\underbrace{x - x_2}_{<0}} \implies g(x) < g(x_2)$$

$$\implies g \text{ cannot attain its minimum at } x_2$$

□

## §15.2 Derivative of Inverse Functions

**Theorem 15.6**

Let $I$ be an open interval and let $f : I \to \mathbb{R}$ be continuous and injective. Then $f(I) = J$ is an interval and $f : I \to J$ is bijective. If $f$ is differentiable at $x_0 \in I$ and $f'(x_0) \neq 0$ then $f^{-1} : J \to I$ is differentiable at $y_0 = f(x_0)$ and

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'\left(f^{-1}(y_0)\right)}$$

*Proof.* The proof uses the following two exercises:

**Exercise 15.1.** Let $I$ be an interval and let $f : I \to \mathbb{R}$ be continuous and injective. Then $f$ is strictly monotone.

**Exercise 15.2.** Let $I$ be an interval and let $f : I \to \mathbb{R}$ be strictly increasing and so that $f(I)$ is an interval. Then $f$ is continuous.

Using exercise 1, we find that $f$ is strictly monotone. Assume $f$ is strictly increasing $\implies f^{-1}$ is strictly increasing.
Using exercise 2 with $g = f^{-1} : J \to I$, we find that $f^{-1}$ is continuous.

**Claim 15.1.** $J$ is an open interval.

Assume, towards a contradiction, that $\inf J \in J = f(I) \implies \exists a \in I$ s.t. $f(a) = \inf J$.

$$\left.\begin{array}{l} I \text{ open } \implies \exists \delta > 0 \text{ s.t. } (a-\delta, a+\delta) \subseteq I \\ f \text{ is strictly increasing} \end{array}\right\} \implies J = f(I) \ni f\left(a - \frac{\delta}{2}\right) < f(a) = \inf J$$

Contradiction!
Similarly, one can show that $\sup J \notin J$

$$\left.\begin{array}{l} f \text{ is diff at } x_0 \implies f'(x_0) = \lim \frac{f(x)-f(x_0)}{x-x_0} \\ f'(x_0) \neq 0 \text{ and } f(x) \neq f(x_0) \quad \forall x \neq x_0 \end{array}\right\} \implies$$

$$\implies \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)}$$

$$\implies \forall \varepsilon > 0 \quad \exists \delta > 0 \text{ s.t. } 0 < |x - x_0| < \delta \implies \left| \frac{x - x_0}{f(x) - f(x_0)} - \frac{1}{f'(x_0)} \right| < \varepsilon$$

$f^{-1}$ is continuous at $y_0 \implies \exists \eta > 0$ s.t. $0 < |y - y_0| < \eta$ implies

$$0 < \left| f^{-1}(y) - f^{-1}(y_0) \right| < \delta$$

So for $0 < |y - y_0| < \eta$ we get

$$\left| \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} - \frac{1}{f'(x_0)} \right| < \varepsilon$$

which implies

$$\left(f^{-1}\right)'(y_0) = \lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{1}{f'(x_0)}$$

□

# §16 Lec 16: May 3, 2021

## §16.1 L'Hopital Rule

**Definition 16.1** (Existence of Limit) — Let $-\infty \le a < b \le \infty$ and let $f : (a,b) \to \mathbb{R}$ be a function. For $c \in (a,b) \cup \{a\}$ we write

$$\lim_{x \to c^+} f(x) = L \in \mathbb{R} \cup \{\pm\infty\}$$

if for every sequence $\{x_n\}_{n\ge1} \subseteq (c,b)$ s.t. $\lim_{n\to\infty} x_n = c$ we have

$$\lim_{n\to\infty} f(x_n) = L$$

For $c \in (a,b) \cup \{b\}$ we write

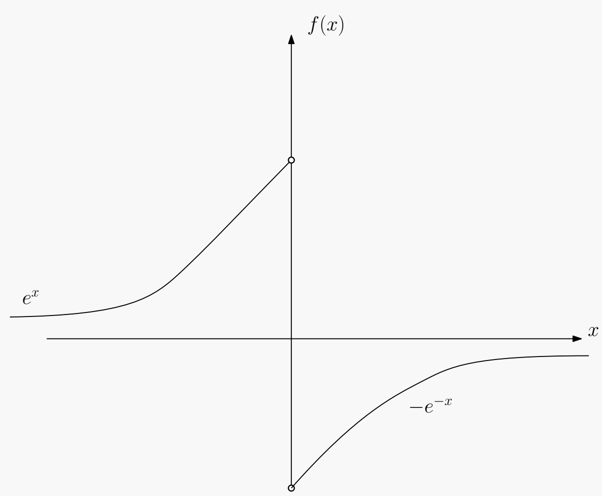
$$\lim_{x \to c^-} f(x) = M \in \mathbb{R} \cup \{\pm\infty\}$$

if for every sequence $\{x_n\}_{n\ge1} \subseteq (a,c)$ s.t. $\lim_{n\to\infty} x_n = c$ we have

$$\lim_{n\to\infty} f(x_n) = M$$

**Remark 16.2.** In general, if $c \in (a,b)$ we have

$$f(c) \ne \lim_{x\to c^-} f(x) \ne \lim_{x \to c^+} f(x) \ne f(c)$$



**Theorem 16.3** (L'Hopital)
Let $-\infty \le a < b \le \infty$ and let $f, g : (a, b) \to \mathbb{R}$ be differentiable. Assume that $g'(x) \neq 0$ $\forall x \in (a, b)$ and that

$$\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R} \cup \{\pm\infty\}$$

Assume also that either

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0 \tag{1}$$

or

$$\lim_{x\to a^+} |g(x)| = \infty \tag{2}$$

Then

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$$

**Remark 16.4.** $\lim_{x\to a^+}$ in the theorem can be replaced by $\lim_{x\to b^-}$ or by $\lim_{x\to c}$ for some $c \in (a, b)$.

*Proof.* We'll present the details for $L \in \mathbb{R}$. We'll prove

**Claim 16.1.** $\forall \varepsilon > 0$ $\exists \delta_1(\varepsilon) > 0$ s.t.

$$\frac{f(x)}{g(x)} < L + \varepsilon \qquad \forall x \in (a, a + \delta_1)$$

**Claim 16.2.** $\forall \varepsilon > 0$ $\exists \delta_2(\varepsilon) > 0$ s.t.

$$L - \varepsilon < \frac{f(x)}{g(x)} \qquad \forall x \in (a, a + \delta_2)$$

Then taking $\delta(\varepsilon) = \min\{\delta_1(\varepsilon), \delta_2(\varepsilon)\}$ we get

$$\left|\frac{f(x)}{g(x)} - L\right| < \varepsilon \qquad \forall x \in (a, a + \delta)$$

$\implies \lim_{x\to a^+} \frac{f(x)}{g(x)} = L$.
Note: If $L = -\infty$ then it suffices to prove Claim 1 with $L + \varepsilon$ replaced by $M < 0$.
If $L = \infty$ then it suffices to prove Claim 2 with $L - \varepsilon$ replaced by $M > 0$.
By assumption, $g'(x) \neq 0$ $\forall x \in (a, b)$. As $g$ is differentiable on $(a, b)$, $g'$ has the Darboux property. So either $g'(x) < 0$ $\forall x \in (a, b)$ or $g'(x) > 0$ $\forall x \in (a, b)$.
Assume $g'(x) < 0$ $\forall x \in (a, b) \implies g$ strictly decreasing on $(a, b)$. In case 1,

$$\lim_{x\to a^+} g(x) = 0$$

As $g$ is strictly decreasing, we get

$$g(x) < 0 \qquad \forall x \in (a, b)$$

In case 2,

$$\lim_{x\to a^+} |g(x)| = \infty$$

As $g$ is strictly decreasing, we get

$$\lim_{x\to a^+} g(x) = \infty$$

and so $\exists c \in (a,b)$ s.t. $g(x) > 0$ $\forall x \in (a,c)$ (**). In particular, in both cases $g(x) \neq 0$ $\forall x \in (a,c)$. We prove claim 1:

Fix $\varepsilon > 0$. As $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$, $\exists \delta_1(\varepsilon) > 0$ s.t.

$$\frac{f'(x)}{g'(x)} < L + \frac{\varepsilon}{2} \qquad \forall x \in (a, a+\delta_1)$$

Fix $a < x < y < \min(a+\delta_1, c)$. By (an equivalent formulation of) Mean Value theorem, $\exists z \in (x,y)$ s.t.

$$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(z)}{g'(z)} < L + \frac{\varepsilon}{2} \tag{*}$$

In case 1, take the limit $x \to a^+$ in (*) to get

$$\frac{f(y)}{g(y)} \le L + \frac{\varepsilon}{2} < L + \varepsilon \qquad \forall a < y < \min(a+\delta_1, c)$$

In case 2, we write

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(y)}{g(x)-g(y)} \cdot \frac{g(x)-g(y)}{g(x)} + \frac{f(y)}{g(x)}$$

By (**) we have $g(x) > g(y) > 0 \implies \frac{g(x)-g(y)}{g(x)} > 0$. So

$$\begin{aligned}
\frac{f(x)}{g(x)} &< \left(L + \frac{\varepsilon}{2}\right) \frac{g(x)-g(y)}{g(x)} + \frac{f(y)}{g(x)} \\
&= \left(L + \frac{\varepsilon}{2}\right)\left(1 - \frac{g(y)}{g(x)}\right) + \frac{f(y)}{g(x)} \\
&= L + \frac{\varepsilon}{2} + \frac{f(y) - \left(L + \frac{\varepsilon}{2}\right) g(y)}{g(x)}
\end{aligned}$$

For $y$ fixed, $\lim_{x\to a^+} \frac{f(y)-\left(L+\frac{\varepsilon}{2}\right)g(y)}{g(x)} = 0$

$$\implies \exists \tilde{\delta_1}(\varepsilon) > 0 \text{ s.t. } \left| \frac{f(y) - \left(L + \frac{\varepsilon}{2}\right) g(y)}{g(x)} \right| < \frac{\varepsilon}{2} \qquad \forall x \in \left(a, a + \tilde{\delta_1}\right)$$

In particular,

$$\frac{f(x)}{g(x)} < L + \varepsilon \qquad \forall a < x < \min\left\{a + \delta_1, a + \tilde{\delta_1}, c\right\}$$

**Exercise 16.1.** Prove claim 2. □

## §16.2 Taylor's Theorem

**Definition 16.5** (Taylor Expansion) — Let $I$ be an open interval and let $f : I \to \mathbb{R}$ be differentiable of any order. For $x_0 \in I$, the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

is called the Taylor expansion of $f$ about $x_0$. For $n \geq 1$, we define the remainder

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

**Theorem 16.6** (Taylor)

Let $n \geq 1$ and assume $f : (a, b) \to \mathbb{R}$ is $n$ times differentiable. Let $x_0 \in (a, b)$. Then for any $x \in (a, b) \setminus \{x_0\}$ there exists $y$ between $x$ and $x_0$ s.t.

$$R_n(x) = \frac{f^{(n)}(y)}{n!}(x - x_0)^n$$

In particular,

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n)}(y)}{n!}(x - x_0)^n$$

*Proof.* Fix $x \in (a, b) \setminus \{x_0\}$. Define $M \in \mathbb{R}$ to be the unique solution to the equation

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + M \cdot \frac{(x - x_0)^n}{n!}$$

We want to show that there exists $y$ between $x$ and $x_0$ s.t.

$$M = f^{(n)}(y)$$

Let $g : (a, b) \to \mathbb{R}$

$$g(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(t - x_0)^k - M \cdot \frac{(t - x_0)^n}{n!}$$

Note $g$ is $n$ times differentiable. For $1 \leq l \leq n - 1$,

$$\begin{aligned} g^{(l)}(t) &= f^{(l)}(t) - \sum_{k \geq l}^{n-1} \frac{f^{(k)}(x_0)}{(k - l)!}(t - x_0)^{k-l} - M\frac{(t - x_0)^{n-l}}{(n - l)!} \\ g^{(n)}(t) &= f^{(n)}(t) - M \end{aligned}$$

In particular, if $0 \leq l \leq n - 1$,

$$g^{(l)}(x_0) = f^{(l)}(x_0) - f^{(l)}(x_0) = 0$$

Also $g(x) = 0$ by contradiction.
$g$ is continuous on $[x, x_0]$, differentiable on $(x, x_0)$ and

$$g(x) = g(x_0) = 0 \implies \exists x_1 \in (x, x_0) \text{ s.t. } g'(x_1) = 0$$

By Rolle's theorem,

$$\begin{aligned} \exists x_2 \in (x_1, x_0) &\quad \text{s.t.} \quad g''(x_2) = 0 \\ &\vdots \\ \exists x_n \in (x_{n-1}, x_0) &\quad \text{s.t.} \quad g^{(n)}(x_n) = 0 \end{aligned}$$

Set $y = x_n$. □

# §17 Lec 17: May 5, 2021

## §17.1 Taylor's Theorem (Cont'd)

**Corollary 17.1**

Fix $a > 0$ and let $f : (-a, a) \to \mathbb{R}$ be a function differentiable of any order. Assume that all derivatives of $f$ are uniformly bounded on $(-a, a)$, that is,

$$\exists M > 0 \text{ s.t. } \left|f^{(n)}(x)\right| \le M \quad \forall x \in (-a, a), \quad \forall n \ge 1$$

Then

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k \xrightarrow[n\to\infty]{u} 0 \text{ on } (-a, a)$$

*Proof.* Fix $x \in (-a, a) \setminus \{0\}$. By Taylor, there exists $y$ between $x$ and 0 s.t.

$$\begin{aligned} R_n(x) &= \frac{f^{(n)}(y)}{n!} x^n \\ \implies |R_n(x)| &\le M \frac{|x|^n}{n!} \le M \frac{a^n}{n!} \\ \implies \sup_{x\in(-a,a)} |R_n(x)| &\le M \cdot \frac{a^n}{n!} \xrightarrow[n\to\infty]{} 0 \end{aligned}$$

□

**Example 17.2**

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos x$

$$f^{(n)}(x) = \begin{cases} -\sin x, & n = 1 + 4k \\ -\cos x, & n = 2 + 4k \\ \sin x, & n = 3 + 4k \\ \cos x, & n = 4k \end{cases} \qquad \text{for } k \ge 0$$

So $\left|f^{(n)}(x)\right| \le 1 \ \forall x \in \mathbb{R} \ \forall n \ge 0$. We get

$$f(x) = u - \lim_{N\to\infty} \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n \quad \text{on } (-a, a) \text{ for any } a > 0$$

Let $n = 2l$

$$\begin{aligned} &\implies f^{(n)}(0) = \begin{cases} -1, & \text{if } l \text{ odd} \\ 1, & \text{if } l \text{ even} \end{cases} = (-1)^l \\ &\implies f(X) = \sum_{n\ge0} \frac{f^{(n)}(0)}{n!} x^n = \sum_{l\ge0} \frac{(-1)^l}{(2l)!} x^{2l} \end{aligned}$$

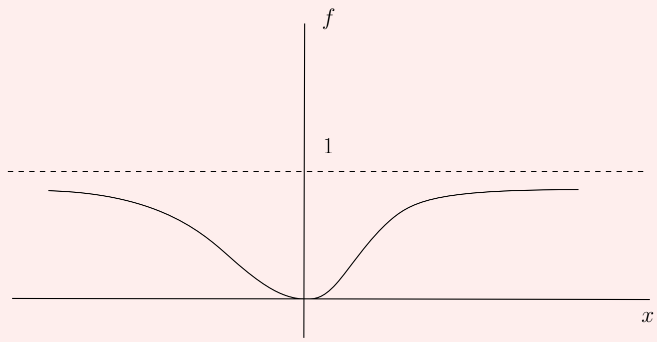
A similar argument gives

$$\sin x = \sum_{n\geq 0} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

**Example 17.3**

$f : \mathbb{R} \to \mathbb{R}$ where

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$




Note $f$ is differentiable of any order on $\mathbb{R}$. Clearly, this holds on $\mathbb{R} \setminus \{0\}$. In fact, for $x \in \mathbb{R} \setminus \{0\}$,

$$f^{(n)}(x) = P_n\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}$$

where

$$P_n\left(\frac{1}{x}\right) = \left(\frac{2}{x^3}\right)^n + \dots$$

To see that $f$ is differentiable at 0 we compute

$$\lim_{x\to 0^+} \frac{f(x)}{x} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{e^{\frac{1}{x^2}}} = \lim_{t\to\infty} \frac{t}{e^{t^2}} = \lim_{t\to\infty} \frac{1}{2te^{t^2}} = 0$$

Similarly,

$$\lim_{x\to 0^-} \frac{f(x)}{x} = \lim_{t\to -\infty} \frac{t}{e^{t^2}} = 0$$

Proceeding inductively, we can prove that $f$ is differentiable of any order at 0 and

$$f^{(n)}(0) = 0$$

We consider

$$\lim_{x\to 0^+} \frac{f^{(n)}(x)}{x} = \lim_{x\to 0^+} \frac{P_n\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}}{x} \lim_{t\to\infty} \frac{tP_n(t)}{e^{t^2}} = 0$$

and

$$\lim_{x\to 0^-} \frac{f^{(n)}(x)}{x} = 0$$

**Example 17.4** (Cont'd from above)
Thus,

$$\sum_{n\geq 0} \frac{f^{(n)}(0)}{n!}x^n \equiv 0$$

At leading order as $x \to 0$,

$$f^{(n)}(x) \sim 2^n \cdot \left(\frac{1}{x^2}\right)^{\frac{3n}{2}} e^{-\frac{1}{x^2}} \sim 2^n e^{-\frac{1}{x^2}+\frac{3n}{2}\ln\frac{1}{x^2}}$$

The function $g : (0,\infty) \to \mathbb{R}$, $g(t) = -t + \frac{3n}{2}\ln t$ achieves its maximum at

$$g'(t) = 0 \iff -1 + \frac{3n}{2t} = 0 \iff t = \frac{3n}{2}$$

So $f^{(n)}\left(\sqrt{\frac{2}{3n}}\right) \sim 2^n e^{-\frac{3n}{2}+\frac{3n}{2}\ln\frac{3n}{2}} \sim 2^n e^{\frac{3n}{2}\ln\left(\frac{3n}{2e}\right)} \sim 2^n\left(\frac{3n}{2e}\right)^{\frac{3n}{2}} \underset{n\to\infty}{\longrightarrow} \infty.$

**Theorem 17.5**
Assume that $f_n : [a,b] \to \mathbb{R}$ are continuous on $[a,b]$ and differentiable on $(a,b)$. Assume also that

1. $\{f_n'\}_{n\geq 1}$ converges uniformly on $(a,b)$
2. $\{f_n\}_{n\geq 1}$ converges at some $x_0$ in $[a,b]$

Then $\{f_n\}_{n\geq 1}$ converges uniformly on $[a,b]$ to some function $f$. Moreover, $f$ is differentiable on $(a,b)$ and

$$f'(x) = \lim_{n\to\infty} f_n'(x) \qquad \forall x \in (a,b)$$

**Remark 17.6.** We can restate the conclusion as follows:

$$\lim_{y\to x}\lim_{n\to\infty}\frac{f_n(y)-f_n(x)}{y-x} = \lim_{y\to x}\frac{f(y)-f(x)}{y-x} = f'(x) = \lim_{n\to\infty}\lim_{y\to x}\frac{f_n(y)-f_n(x)}{y-x}$$

*Proof.* Let's prove that $\{f_n\}_{n\geq 1}$ converges uniformly on $[a,b]$. Fix $\varepsilon > 0$.
$\{f_n'\}_{n\geq 1}$ converges uniformly on $(a,b)$ which implies $\{f_n'\}_{n\geq 1}$ is uniformly Cauchy on $(a,b)$ which also implies $\exists n_1(\varepsilon) \in \mathbb{N}$ s.t.

$$\left|f_n'(x) - f_m'(x)\right| < \varepsilon \quad \forall n,m \geq n_1(\varepsilon) \quad \forall x \in (a,b)$$

Also, we know that $\{f_n(x_0)\}_{n\geq 1}$ converges which means $\{f_n(x_0)\}$ is Cauchy which implies $\exists n_2(\varepsilon) \in \mathbb{N}$ s.t.

$$|f_n(x_0) - f_m(x_0)| < \varepsilon \quad \forall n,m \geq n_2(\varepsilon)$$

For $x \in [a,b] \setminus \{x_0\}$,

$$|f_n(x) - f_m(x)| \leq |f_n(x_0) - f_m(x_0)| + |[f_n(x) - f_m(x)] - [f_n(x_0) - f_m(x_0)]|$$

By the Mean Value theorem, there exists $y$ between $x$ and $x_0$ s.t.

$$|\,[f_n(x) - f_m(x)] - [f_n(x_0) - f_m(x_0)]\,| = \left|f_n'(y) - f_m'(y)\right| |x - x_0| < \varepsilon(b-a)$$

So for $n, m \geq n(\varepsilon) = \max\{n_1(\varepsilon), n_2(\varepsilon)\}$ we get

$$\begin{aligned} &|f_n(x) - f_m(x)| \leq |f_n(x_0) - f_m(x_0)| + \varepsilon(b-a) \leq \varepsilon(1+b-a) \\ \implies\ & sup_{x\in[a,b]} |f_n(x) - f_m(x)| \leq \varepsilon(1+b-a) \qquad \forall n, m \geq n(\varepsilon) \end{aligned}$$

So $\{f_n\}_{n\geq 1}$ are uniformly Cauchy on $[a,b]$ and so converge to a function $f = \lim_{n\to\infty} f_n$. It remains to show that $f$ is differentiable on $(a,b)$ and

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

which we will prove in the next lecture. □

# §18 Lec 18: May 7, 2021

## §18.1 Taylor's Theorem (Cont'd)

*Proof.* (Cont'd from lecture 17) Fix $x \in (a,b)$. We want to show that $f$ is differentiable at $x$ and

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

We define

$$g : [a,b] \setminus \{x\} \to \mathbb{R}, \quad g(y) = \frac{f(y) - f(x)}{y - x}$$
$$g_n : [a,b] \setminus \{x\} \to \mathbb{R}, \quad g_n(y) = \frac{f_n(y) - f_n(x)}{y - x}$$

Since $f_n \xrightarrow[n\to\infty]{u} f$ we have

$$\lim_{n\to\infty} g_n(y) = g(y)$$

Since $f_n$ is differentiable at $x$,

$$\lim_{y\to x} g_n(y) = f_n'(x)$$

Let $L(x) = \lim_{n\to\infty} f_n'(x)$. We want to show that

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \text{ s.t. } |g(y) - L(x)| < \varepsilon \text{ whenever } 0 < |y - x| < \delta \ y \in [a,b]$$

Fix $\varepsilon > 0$. By the triangle inequality,

$$|g(y) - L(x)| \le |g(y) - g_n(y)| + \left|g_n(y) - f_n'(x)\right| + \left|f_n'(x) - L(x)\right|$$

We have $\{f_n'\}_{n\ge1}$ converges uniformly on $(a,b) \implies \{f_n'\}_{n\ge1}$ is uniformly Cauchy on $(a,b)$ $\implies \exists n_1(\varepsilon) \in \mathbb{N}$ s.t.

$$\left|f_n'(z) - f_m'(z)\right| < \varepsilon \qquad \forall n,m \ge n_1(\varepsilon) \quad \forall z \in (a,b) \tag{1}$$

Letting $m \to \infty$ we get

$$\left|f_n'(z) - L(z)\right| \le \varepsilon \qquad \forall n \ge n_1(\varepsilon) \quad \forall z \in (a,b)$$

For $y \in [a,b] \setminus \{x\}$, by the Mean Value theorem, we can find a point $z$ between $x$ and $y$ so that

$$\begin{aligned}|g_n(y) - g_m(y)| &= \left|\frac{f_n(y) - f_n(x)}{y - x} - \frac{f_m(y) - f_m(x)}{y - x}\right| \\ &= \frac{|[f_n(y) - f_m(y)] - [f_n(x) - f_m(x)]|}{|y - x|} \\ &= \left|f_n'(z) - f_m'(z)\right| \overset{(1)}{<} \varepsilon \quad \forall n,m \ge n_1(\varepsilon)\end{aligned}$$

Letting $m \to \infty$ we find

$$|g_n(y) - g(y)| \le \varepsilon \qquad \forall n \ge n_1(\varepsilon) \quad \forall y \in [a,b] \setminus \{x\} \tag{3}$$

Fix $n \geq n_1(\varepsilon)$. As $f_n$ is differentiable at $x$ we find $\delta = \delta(\varepsilon, n) > 0$ s.t.

$$\left|g_n(y) - f_n'(x)\right| < \varepsilon \qquad \forall 0 < |y-x| < \delta \quad y \in [a,b] \tag{4}$$

Thus for this $n \geq n_1(\varepsilon)$ and $0 < |y-x| < \delta$ we have

$$\begin{aligned} |g(y) - L(x)| &\leq |g(y) - g_n(y)| + \left|g_n(y) - f_n'(x)\right| + \left|f_n'(x) - L(x)\right| \\ \text{by (2), (3), (4)} &\leq 3\varepsilon \end{aligned}$$

□

**Example 18.1**

$f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{x}{1+nx^2}$, $f_n$ is differentiable and

$$f_n'(x) = \frac{1}{1+nx^2} - \frac{x \cdot 2nx}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}$$

Now

$$f_n \xrightarrow[n\to\infty]{u} f \equiv 0$$

$$f_n'(x) \xrightarrow[n\to\infty]{} \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

Note that $f_n'$ do not converge uniformly since their limit is not continuous.

$$\lim_{n\to\infty} \lim_{y\to 0} \frac{f_n(y) - f_n(0)}{y-0} = \lim_{n\to\infty} f_n'(0) = 1$$

but

$$\lim_{y\to 0} \lim_{n\to\infty} \frac{f_n(y) - f_n(0)}{y-0} = \lim_{y\to 0} 0 = 0$$

## §18.2 Darboux Integral

**Definition 18.2** (Partition) — Let $f : [a,b] \to \mathbb{R}$ be a bounded function. If $S \subseteq [a,b]$ we denote

$$M(f;S) = \sup_{x\in S} f(x) \quad \text{and} \quad m(f;S) = \inf_{x\in S} f(x)$$

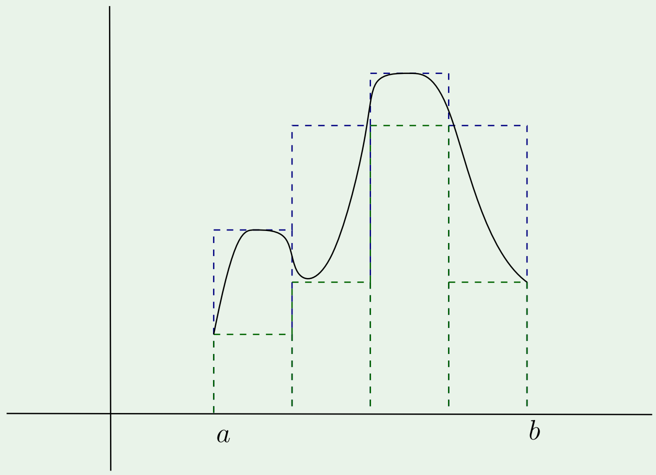
A partition of $[a,b]$ is a finite ordered set $P \subseteq [a,b]$. We write

$$P = \{a = t_0 < t_1 < \ldots < t_n = b\}$$

for some $n \geq 1$.

**Definition 18.3** (Darboux Sum) — The upper Darboux sum of $f$ with respect to $P$ is

$$U(f;P) = \sum_{k=1}^{n} M\left(f;[t_{k-1},t_k]\right)(t_k - t_{k-1})$$




The lower Darboux sum of $f$ with respect to $P$ is

$$L(f;P) = \sum_{k=1}^{n} m\left(f;[t_{k-1},t_k]\right)(t_k - t_{k-1})$$

Note that

$$m\left(f;[a,b]\right)(b-a) \le L(f;P) \le U(f;P) \le M\left(f;[a,b]\right)(b-a)$$

So

$$\{L(f;P) : P \text{ partition of } [a,b]\} \text{ is bounded above}$$
$$\{U(f;P) : P \text{ partition of } [a,b]\} \text{ is bounded below}$$

**Definition 18.4** (Darboux Integral) — The upper Darboux integral of $f$ on $[a,b]$ is

$$U(f) = \inf\{U(f;P) : P \text{ partition of } [a,b]\}$$

The lower Darboux integral of $f$ on $[a,b]$ is

$$L(f) = \sup\{L(f;P) : P \text{ partition of } [a,b]\}$$

We say that $f$ is Darboux integrable on $[a,b]$ if $U(f) = L(f)$. In this case we write

$$\int_a^b f(x)\,dx = U(f) = L(f)$$

**Example 18.5**

Let $f : [0, M] \to \mathbb{R}$, $f(x) = x^3$. Then $f$ is Darboux integrable.
Let $P = \{0 = t_0 < \ldots < t_n = M\}$ be a partition of $[0, M]$ and

$$\begin{aligned} U(f;P) &= \sum_{k=1}^{n} M\left(f;[t_{k-1},t_k]\right)(t_k - t_{k-1}) \\ &= \sum_{k=1}^{n} t_k^3 \left(t_k - t_{k-1}\right) \end{aligned}$$

Similarly,

$$L(f;P) = \sum_{k=1}^{n} m\left(f;[t_{k-1},t_k]\right)(t_k - t_{k-1}) = \sum_{k=1}^{n} t_{k-1}^3 \left(t_k - t_{k-1}\right)$$

Take $t_k = \frac{kM}{n}$ $0 \le k \le n$. Then

$$\begin{aligned} U(f;P) &= \sum_{k=1}^{n} \left(\frac{kM}{n}\right)^3 \cdot \frac{M}{n} = \frac{M^4}{n^4} \sum_{k=1}^{n} k^3 = \frac{M^4}{n^4}\left[\frac{n(n+1)^2}{2}\right] \xrightarrow[n\to\infty]{} \frac{M^4}{4} \\ L(f;P) &= \sum_{k=1}^{n} \left(\frac{(k-1)M}{n}\right)^3 \cdot \frac{M}{n} = \frac{M^4}{n^4} \sum_{k=0}^{n-1} k^3 = \frac{M^4}{n^4}\left[\frac{n(n-1)^2}{2}\right] \xrightarrow[n\to\infty]{} \frac{M^4}{4} \end{aligned}$$

So, $U(f) \le \frac{M^4}{4}$ and $L(f) \ge \frac{M^4}{4}$ and we will show that $L(f) \le U(f)$ which imply $U(f) = L(f) = \frac{M^4}{4}$. So $f$ is Darboux integrable and $\int_0^M f(x)\,dx = \frac{M^4}{4}$.

**Example 18.6**

Given

$$f : [0,1] \to \mathbb{R}, \quad f(x) = \begin{cases} 1, & x \in [0,1] \cap \mathbb{Q} \\ 0, & x \in [0,1] \setminus \mathbb{Q} \end{cases}$$

$f$ is not Darboux integrable. For any partition $P$, $U(f;P) = 1$ and $L(f;P) = 0$ which implies $U(f) = 1$ and $L(f) = 0$.

# §19 Lec 19: May 10, 2021

## §19.1 Darboux Integral (Cont'd)

Recall: If $f : [a, b] \to \mathbb{R}$ bounded

$$P = \{a = t_0 < \ldots < t_n = b\} \text{ partition of } [a, b]$$

then

$$U(f; P) = \sum_{k=1}^{n} M\left(f; [t_{k-1}, t_k]\right)\left(t_k - t_{k-1}\right)$$
$$L(f; P) = \sum_{k=1}^{n} m\left(f; [t_{k-1}, t_k]\right)\left(t_k - t_{k-1}\right)$$

are the upper and lower Darboux sum associated with $P$, respectively $f$ is Darboux integrable if $U(f) = L(f)$ where

$$U(f) = \inf_P U(f; P) \quad \text{and} \quad L(f) = \sup_P L(f; P)$$

**Proposition 19.1**

Let $f : [a, b] \to \mathbb{R}$ be two bounded and let $P$ and $Q$ be partitions of $[a, b]$ s.t. $P \subseteq Q$. Then

$$L(f; p) \leq L(f; Q) \leq U(f; Q) \leq U(f; P)$$

*Proof.* We will prove the third inequality. The first inequality follows from a similar argument. Arguing by induction, it suffices to prove the claim when the partition $Q$ contains exactly one extra point compared to the partition $P$. Let

$$P = \{a = t_0 < t_1 < \ldots < t_n = b\}$$
$$Q = \{a = t_0 < \ldots < t_{l-1} < s < t_l < \ldots < t_n = b\}$$

for some $1 \leq l \leq n$.

$$U(f; Q) = \sum_{k=1}^{l-1} M\left(f; [t_{k-1}, t_k]\right)\left(t_k - t_{k-1}\right) + M\left(f; [t_{l-1}, s]\right)\left(s - t_{l-1}\right) + M\left(f; [s, t_l]\right)\left(t_l - s\right)$$
$$+ \sum_{k=l+1}^{n} M\left(f; [t_{k-1}, t_k]\right)\left(t_k - t_{k-1}\right)$$

Clearly,

$$M\left(f; [t_{l-1}, s]\right) \leq M\left(f; [t_{l-1}, t_l]\right)$$
$$M\left(f; [s, t_l]\right) \leq M\left(f; [t_{l-1}, t_l]\right)$$

So

$$U(f; Q) \leq \sum_{k=1}^{n} M\left(f; [t_{k-1}, t_k]\right)\left(t_k - t_{k-1}\right) = U(f; P) \qquad \square$$

**Corollary 19.2**

Let $f : [a, b] \to \mathbb{R}$ be bounded and let $P, Q$ be two partitions of $[a, b]$. Then

$$L(f; P) \leq U(f; Q)$$

Consequently,

$$L(f) \leq U(f)$$

*Proof.* Consider the partition $P \cup Q$. We have

$$\begin{aligned} L(f;P) \leq L(f; P \cup Q) &\leq U(f; P \cup Q) \leq U(f;Q) \\ \implies L(f) &= \sup_P L(f;P) \leq U(f;Q) \\ \implies L(f) &\leq \inf_Q U(f;Q) = U(f) \end{aligned}$$

□

**Theorem 19.3**

Let $f : [a, b] \to \mathbb{R}$ be bounded. Then $f$ is Darboux integrable if and only if

$$\forall \varepsilon > 0 \quad \exists P \text{ partitions of } [a,b] \quad \ni \quad U(f;P) - L(f;P) < \varepsilon$$

*Proof.* " $\impliedby$ " Fix $\varepsilon > 0$. Then there exists $P$ partition of $[a, b]$ s.t. $U(f; P) - L(f; P) < \varepsilon$

$$\implies U(f) \leq U(f;P) < L(f;P) + \varepsilon \leq L(f) + \varepsilon$$

$$\implies \left.\begin{array}{r} U(f) < L(f) + \varepsilon \\ \varepsilon > 0 \text{ was arbitrary} \end{array}\right\} \implies \left.\begin{array}{r} U(f) \leq L(f) \\ L(f) \leq U(f) \end{array}\right\} \implies U(f) = L(f)$$

$$\implies f \text{ is Darboux integrable}$$

" $\implies$ " Fix $\varepsilon > 0$, $f$ is Darboux integrable implies

$$U(f) = L(f)$$

Then

$$\begin{aligned} U(f) = \inf_P U(f;P) &\implies \exists P_1 \text{ partition of } [a,b] \text{ s.t. } U(f;P_1) < U(f) + \frac{\varepsilon}{2} \\ L(f) = \sup_P L(f;P) &\implies \exists P_2 \text{ partition of } [a,b] \text{ s.t. } L(f;P_2) > L(f) - \frac{\varepsilon}{2} \end{aligned}$$

Consider the partition $P_1 \cup P_2$. Then

$$L(f;P_2) \leq L(f; P_1 \cup P_2) \leq U(f; P_1 \cup P_2) \leq U(f; P_1)$$

So

$$U(f; P_1 \cup P_2) - L(f; P_1 \cup P_2) < U(f) + \frac{\varepsilon}{2} - \left(L(f) - \frac{\varepsilon}{2}\right) = \varepsilon$$

□

**Definition 19.4** (Mesh) — Let $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ be a partition of $[a, b]$. The mesh of $P$ is given by

$$\text{mesh}(P) = \max_{1 \le k \le n} (t_k - t_{k-1})$$

**Theorem 19.5**

Let $f : [a, b] \to \mathbb{R}$ be bounded. Then $f$ is Darboux integrable if and only if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \text{ s.t. if } P \text{ is a partition of } [a, b] \text{ with mesh}(P) < \delta$$

then

$$U(f; P) - L(f; P) < \varepsilon$$

*Proof.* " $\Longleftarrow$ " By the previous theorem, it suffices to show that $\forall \delta > 0$ $\exists P$ partition of $[a, b]$ with $\text{mesh}(P) < \delta$. For $\delta > 0$, let $P = \{a = t_0 < \ldots < t_n = b\}$ where

$$t_k = a + k \cdot \frac{\delta}{2} \quad \text{for} \quad 0 \le k \le \lfloor \frac{2(b-a)}{\delta} \rfloor = n - 1$$

and $t_n = b$. Clearly,

$$\text{mesh}(P) = \frac{\delta}{2} < \delta$$

" $\Longrightarrow$ " Fix $\varepsilon > 0$. By the previous theorem, as $f$ is Darboux integrable, there exists a partition $P_0 = \{a = s_0 < \ldots < s_m = b\}$ of $[a, b]$ s.t.

$$U(f; P_0) - L(f; P_0) < \frac{\varepsilon}{2}$$

Let $0 < \delta < \text{mesh}(P_0)$ to be chosen later and let $P = \{a = t_0 < \ldots < t_n = b\}$ be a partition of $[a, b]$ with $\text{mesh}(P) < \delta$

$$\begin{aligned} U(f; P) - L(f; P) &\le U(f; P) - U(f; P_0) + U(f; P_0) - L(f; P_0) + L(f; P_0) - L(f; P) \\ &\le \frac{\varepsilon}{2} + U(f; P) - U(f; P_0) + L(f; P_0) - L(f; P) \end{aligned}$$

Consider the partition $P \cup P_0$. Then

$$U(f; P) - U(f; P_0) \le U(f; P) - U(f; P \cup P_0)$$

As $\text{mesh}(P) < \delta < \text{mesh}(P_0)$, there must be at most one point from $P_0$ in each $[t_{k-1}, t_k]$. Only subintervals $[t_{k-1}, t_k]$ with an $s_j \in P_0 \cap [t_{k-1}, t_k]$ contribute to $U(f; P) - U(f; P_0 \cup P)$. There are only $m$ many such intervals. The contribution of one such interval to $U(f; P) - U(f; P_0 \cup P)$ is

$$M(f; [t_{k-1}, t_k])(t_k - t_{k-1}) - M(f; [t_{k-1}, s_j])(s_j - t_{k-1}) - M(f; [s_j, t_k])(t_k - s_j)$$

As $f$ is bounded, $\exists M > 0$ s.t. $|f(x)| \le M$ $\forall x \in [a, b]$. Note

$$\begin{aligned} &M(f; [t_{k-1}, t_k]) \le M \\ &M(f; [t_{k-1}, s_j]) \ge -M; \qquad M(f; [s_j, t_k]) \ge -M \end{aligned}$$

So

$$M\left(f;\left[t_{k-1},t_{k}\right]\right)\left(t_{k}-t_{k-1}\right)-M\left(f;\left[t_{k-1},s_{j}\right]\right)\left(s_{j}-t_{k-1}\right)-M\left(f;\left[s_{j},t_{k}\right]\right)\left(t_{k}-s_{j}\right)$$

which is smaller than or equal to

$$M\left(t_{k}-t_{k-1}\right)-\left(-M\right)\left[\left(s_{j}-t_{k-1}\right)+\left(t_{k}-s_{j}\right)\right]=2M\left(t_{k}-t_{k-1}\right)<2M\cdot\operatorname{mesh}(P)$$

Thus

$$U(f;P)-U(f;P_0)<m\cdot 2M\cdot\operatorname{mesh}(P)$$

Similarly,

$$L(f;P_0)-L(f;P)<m\cdot 2M\cdot\operatorname{mesh}(P)$$

which requires

$$4Mm\cdot\operatorname{mesh}(P)<\frac{\varepsilon}{2}\iff\operatorname{mesh}(P)<\frac{\varepsilon}{8Mm}$$

Thus, $\delta<\min\left\{\frac{\varepsilon}{8Mm},\operatorname{mesh}(P_0)\right\}$. □

# §20 Lec 20: May 12, 2021

## §20.1 Riemann Integral

**Definition 20.1** (Riemann Sum) — Let $f : [a, b] \to \mathbb{R}$ be a function and let $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ be a partition of $[a, b]$. A Riemann sum of $f$ associated to $P$ is a sum of the form

$$S = \sum_{k=1}^{n} f(x_k)\,(t_k - t_{k-1}) \qquad \text{where } x_k \in [t_{k-1}, t_k] \quad \forall 1 \le k \le n$$

*Note*: If $S$ is a Riemann sum associated with a partition $P$ of $[a, b]$ then

$$L(f; P) \le S \le U(f; P)$$

**Definition 20.2** (Riemann Integrable) — We say that $f$ is Riemann integrable if $\exists r \in \mathbb{R}$ s.t. $\forall \varepsilon > 0\ \exists \delta > 0$ s.t.

$$|S - r| < \varepsilon$$

for any Riemann sum $S$ of $f$ associated with a partition $P$ with $\operatorname{mesh}(P) < \delta$. Then $r$ is called the Riemann integral of $f$ and we write

$$r = \mathcal{R}\int_a^b f(x)\,dx$$

**Lemma 20.3**

If $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then $f$ is bounded.

*Proof.* Let $r = \mathcal{R}\int_a^b f(x)\,dx$. Taking $\varepsilon = 1$ we find $\delta > 0$ s.t. $|S - r| < 1$ for any Riemann sum $S$ of $f$ associated to a partition $P$ with $\operatorname{mesh}(P) < \delta$.

Let $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ with $\operatorname{mesh}(P) < \delta$. Fix $1 \le k \le n$. Fix $x_l \in [t_{l-1}, t_l]$ for $1 \le l \le n$, $l \ne k$. For $x \in [t_{k-1}, t_k]$ we have

$$\left| \sum_{l \ne k} f(x_l)\,(t_l - t_{l-1}) + f(x)\,(t_k - t_{k-1}) - r \right| < 1$$

$$\left.\begin{array}{l} \frac{r-1-\sum_{l\ne k} f(x_l)(t_l - t_{l-1})}{t_k - t_{k-1}} < f(x) < \frac{1+r-\sum_{l\ne k} f(x_l)(t_l - t_{l-1})}{t_k - t_{k-1}} \\ x \in [t_{k-1}, t_k] \text{ is arbitrary} \end{array}\right\} \implies$$

$$\implies \left.\begin{array}{l} f \text{ is bounded on } [t_{k-1}, t_k] \\ 1 \le k \le n \text{ is arbitrary} \end{array}\right\} \implies f \text{ is bounded on } [a, b]$$

□

**Theorem 20.4**

Let $f : [a, b] \to \mathbb{R}$. The following are equivalent

1. $f$ is Riemann integrable.
2. $f$ is bounded and Darboux integrable.

If either conditions holds, then the integrals agree.

*Proof.* 2) $\implies$ 1) Fix $\varepsilon > 0$.

$f$ is Darboux integrable $\implies$ $\exists \delta > 0$ s.t. $U(f; P) - L(f; P) < \varepsilon$ for any partition $P$ with mesh$(P) < \delta$. Let $P$ be a partition of $[a, b]$ with mesh$(P < \delta)$. If $S$ is a Riemann sum of $f$ associated to $P$, then

$$\left.\begin{array}{l} S \le U(f;P) < L(f;P) + \varepsilon \le L(f) + \varepsilon = \int_a^b f(x)\,dx + \varepsilon \\ S \ge L(f;P) > U(f;P) - \varepsilon \ge U(f) - \varepsilon = \int_a^b f(x)\,dx - \varepsilon \end{array}\right\} \implies \left| s - \int_a^b f(x)dx \right| < \varepsilon$$

By definition, $f$ is Riemann integrable and $\mathcal{R}\int_a^b f(x)dx = \int_a^b f(x)dx$.

1) $\implies$ 2) By the previous lemma, $f$ is bounded. Fix $\varepsilon > 0$. Let $r = \mathcal{R}\int_a^b f(x)dx$. Then $\exists \delta > 0$ s.t.

$$|S - r| < \frac{\varepsilon}{2}$$

for any Riemann sum of $f$ associated with a partition of $P$ with mesh$(P) < \delta$. Fix $P = \{a = t_0 < t_1 < \ldots . < t_n = b\}$ be a partition with (mesh$(P) < \delta$. There exist $x_k, y_k \in [t_{k-1}, t_k]$ s.t.

$$f(x_k) > M\left(f; [t_{k-1}, t_k]\right) - \frac{\varepsilon}{2(b-a)}$$
$$f(y_k) < m\left(f; [t_{k-1}, t_k]\right) + \frac{\varepsilon}{2(b-a)}$$

Then

$$\begin{aligned} S_1 = \sum_{k=1}^n f(x_k)\left(t_k - t_{k-1}\right) &> U(f;P) - \frac{\varepsilon}{2(b-a)} \sum_{k=1}^n (t_k - t_{k-1}) \\ &= U(f;P) - \frac{\varepsilon}{2} \\ S_2 = \sum_{k=1}^n f(y_k)\left(t_k - t_{k-1}\right) &< L(f;P) + \frac{\varepsilon}{2(b-a)} \sum_{k=1}^n (t_k - t_{k-1}) \\ &= L(f;P) + \frac{\varepsilon}{2} \end{aligned}$$

However, $|S_1 - r| < \frac{\varepsilon}{2}$ and $|S_2 - r| < \frac{\varepsilon}{2}$. So

$$\left.\begin{array}{l} U(f;P) - \frac{\varepsilon}{2} < S_1 < r + \frac{\varepsilon}{2} \implies U(f) \le U(f;P) < r + \varepsilon \\ r - \frac{\varepsilon}{2} < S_2 < L(f;P) + \frac{\varepsilon}{2} \implies r - \varepsilon < L(f;P) \le L(f) \end{array}\right\} \implies$$

$$\implies \left.\begin{array}{l} r - \varepsilon < L(f) \le U(f) < r + \varepsilon \\ \varepsilon > 0 \text{ arbitrary} \end{array}\right\} \implies f \text{ is Darboux integrable and } \int_a^b f(x)\,dx = r$$

□

**Theorem 20.5**

Let $f : [a, b] \to \mathbb{R}$ be monotonic. Then $f$ is integrable.

*Proof.* Assume $f$ is increasing. Then

$$f(a) \le f(x) \le f(b) \qquad \forall x \in [a, b]$$

So $f$ is bounded.
Let $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ with $\text{mesh}(P) < \delta$ for $\delta$ to be chosen later. Then

$$\begin{aligned} U(f; P) - L(f; P) &= \sum_{k=1}^{n} \left[M\left(f; [t_{k-1}, t_k]\right) - m\left(f; [t_{k-1}, t_k]\right)\right](t_k - t_{k-1}) \\ &= \sum_{k=1}^{n} \left[f(t_k) - f(t_{k-1})\right](t_k - t_{k-1}) \\ &\le \text{mesh}(P) \sum_{k=1}^{n} \left[f(t_k) - f(t_{k-1})\right] \\ &< \delta \cdot [f(b) - f(a)] \end{aligned}$$

Taking $\delta < \frac{\varepsilon}{f(b) - f(a) + 1}$ we see that $f$ is Darboux integrable. □

**Theorem 20.6**

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then $f$ is integrable.

*Proof.* We have

$$\left.\begin{array}{l} f : [a, b] \to \mathbb{R} \text{ continuous} \\ [a, b] \text{ compact} \end{array}\right\} \implies f \text{ is bounded}$$

Fix $\varepsilon > 0$. As $f$ is continuous on $[a, b]$ compact, $f$ is uniformly continuous. So $\exists \delta > 0$ s.t.

$$|f(x) - f(y)| < \frac{\varepsilon}{b - a} \qquad \forall x, y \in [a, b] \text{ with } |x - y| < \delta$$

Let $P = \{a = t_0 < \ldots < t_n = b\}$ with $\text{mesh}(P) < \delta$.

$$U(f; P) - L(f; P) = \sum_{k=1}^{n} \left[M\left(f; [t_{k-1}, t_k]\right) - m\left(f; [t_{k-1}, t_k]\right)\right](t_k - t_{k-1})$$

$f$ continuous on $[t_{k-1}, t_k]$ compact implies $\exists x_k, y_k \in [t_{k-1}, t_k]$ s.t.

$$\begin{aligned} f(x_k) &= M\left(f; [t_{k-1}, t_k]\right) \\ f(y_k) &= m\left(f; [t_{k-1}, t_k]\right) \end{aligned}$$

So

$$\begin{aligned} U(f; P) - L(f; P) &= \sum_{k=1}^{n} \left[f(x_k) - f(y_k)\right](t_k - t_{k-1}) \\ &< \sum_{k=1}^{n} \frac{\varepsilon}{b - a}(t_k - t_{k-1}) = \varepsilon \end{aligned}$$

Then $f$ is Darboux integrable. □

**Theorem 20.7**

Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable.

1. For any $\alpha \in \mathbb{R}$, $\alpha f$ is Riemann integrable and
$$\int_a^b (\alpha f)(x)\,dx = \alpha \int_a^b f(x)\,dx$$
2. $f + g$ is Riemann integrable and
$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

*Proof.* 1. If $\alpha = 0$ this is clear. Assume $\alpha > 0$. For any $S \subseteq [a, b]$
$$M(\alpha f; S) = \alpha M(f; S)$$
$$m(\alpha f; S) = \alpha m(f; S)$$

For by partition $P$ of $[a, b]$,
$$U(\alpha f; P) = \alpha U(f; P) \implies U(\alpha f) = \sup_P U(\alpha f; P)$$
$$= \sup_P [\alpha \cdot U(f; P)]$$
$$= \alpha \sup_P U(f; P) = \alpha U(f)$$

Similarly,
$$L(\alpha f) = \alpha L(f)$$
$$L(f) = U(f)$$

$\implies$ $\alpha f$ is Darboux integrable and $\int_a^b (\alpha f)(x)dx = \alpha \int_a^b f(x)dx$. □

# §21 Lec 21: May 14, 2021

## §21.1 Riemann Integral (Cont'd)

Recall from last lecture, we have the following theorem,

**Theorem 21.1**
Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable.

1. For any $\alpha \in \mathbb{R}$, $\alpha f$ is Riemann integrable and
$$\int_a^b (\alpha f)(x)\, dx = \alpha \int_a^b f(x)\, dx$$
2. $f + g$ is Riemann integrable and
$$\int_a^b (f+g)(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$$

*Proof.* 1. Last time we proved the result for $\alpha \geq 0$. Assume $\alpha < 0$. For $S \subseteq [a, b]$, we have
$$M(\alpha f; S) = \alpha m(f; S) \quad \text{and} \quad m(\alpha f; S) = \alpha M(f; S)$$
If $P$ is a partition of $[a, b]$,
$$U(\alpha f; P) = \alpha L(f; P) \quad \text{and} \quad L(\alpha f; P) = \alpha U(f; P)$$
Thus,
$$\left.\begin{aligned} &U(\alpha f) = \inf_P U(\alpha f; P) = \inf_P \alpha L(f; P) = \alpha \sup_P L(f; P) = \alpha L(f) \\ &L(\alpha f) = \ldots = \alpha U(f) \\ &f \text{ is Riemann integrable} \implies f \text{ bounded and } L(f) = U(f) = \textstyle\int_a^b f(x)\, dx \end{aligned}\right\} \implies$$
$$\implies \alpha f \text{ is bounded and } L(\alpha f) = U(\alpha f) = \alpha \int_a^b f(x)\, dx$$
$$\implies \alpha f \text{ is Riemann integrable and } \int_a^b (\alpha f)(x)\, dx = \alpha \int_a^b f(x)\, dx$$

2. As $f, g$ are Riemann integrable, $f + g$ is bounded and $f, g$ are Darboux integrable. Fix $\varepsilon > 0$. Then, $f$ is Darboux integrable implies $\exists P_1$ partition of $[a, b]$ s.t.
$$U(f; P_1) - L(f; P_1) < \frac{\varepsilon}{2}$$
$g$ is Darboux integrable implies $\exists P_2$ partition of $[a, b]$ s.t.
$$U(g; P_2) - L(g; P_2) < \frac{\varepsilon}{2}$$

Let $P = P_1 \cup P_2$. Then, we have

$$U(f;P) - L(f;P) < \frac{\varepsilon}{2} \quad \text{and} \quad U(g;P) - L(g;P) < \frac{\varepsilon}{2}$$

For $S \subseteq [a,b]$,

$$M(f+g;S) \le M(f;S) + M(g;S)$$
$$m(f+g;S) \ge m(f;S) + m(g;S)$$

So

$$\left.\begin{aligned} U(f+g;P) &\le U(f;P) + U(g;P) \\ L(f+g;P) &\ge L(f;P) + L(g;P) \end{aligned}\right\} \implies$$

$$\implies U(f+g;P) - L(f+g;P) \le U(f;P) - L(f;P) + U(g;P) - L(g;P) < \varepsilon$$

$$\implies \left.\begin{aligned} &f+g \text{ is Darboux integrable} \\ &f+g \text{ is bounded} \end{aligned}\right\} \implies f+g \text{ is Riemann integrable}$$

Moreover,

$$\begin{aligned} U(f+g) \le U(f+g;P) &\le U(f;P) + U(g;P) \\ &< L(f;P) + L(g;P) + \varepsilon \\ &\le L(f) + L(g) + \varepsilon = \int_a^b f(x)dx + \int_a^b g(x)dx + \varepsilon \end{aligned}$$

Similarly,

$$\begin{aligned} L(f+g) \ge L(f+g;P) &\ge L(f;P) + L(g;P) \\ &> U(f;P) + U(g;P) - \varepsilon \\ &\ge U(f) + U(g) - \varepsilon = \int_a^b f(x)dx + \int_a^b g(x)dx - \varepsilon \end{aligned}$$

Let $\varepsilon \to 0$, we get

$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

□

**Theorem 21.2**

Let $f, g : [a,b] \to \mathbb{R}$ be Riemann integrable. Assume $f(x) \le g(x)\ \forall x \in [a,b]$. Then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

*Proof.* By the previous theorem, $h : [a,b] \to \mathbb{R}$, $h = g - f$ is Riemann integrable. Moreover, since $h \ge 0$, we have

$$\int_a^b h(x)\,dx = L(h) = \sup_P L(h;P) \ge 0$$

which implies

$$0 \le \int_a^b h(x)\,dx = \int_a^b (g-f)(x)\,dx = \int_a^b g(x)\,dx - \int_a^b f(x)\,dx$$

□

**Theorem 21.3**

Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Then $|f|$ is Riemann integrable and

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

*Proof.* Let $f$ is Riemann integrable. Then, $f$ is bounded and Darboux integrable. So $|f|$ is bounded. For $S \subseteq [a, b]$ we have

$$\begin{aligned}
M(|f|; S) - m(|f|; S) &= \sup_{x\in S} |f(x)| - \inf_{y\in S} |f(y)| \\
&= \sup_{x\in S} |f(x)| + \sup_{y\in S} -|f(y)| \\
&= \sup_{x,y\in S} \{|f(x)| - |f(y)|\} \\
&\le \sup_{x,y\in S} |f(x) - f(y)| \\
&= \sup_{x,y\in S} \{f(x) - f(y)\} \\
&= \sup_{x\in S} f(x) - \inf_{y\in S} f(y) \\
&= M(f; S) - m(f; S)
\end{aligned}$$

So for any partition $P$ of $[a, b]$ we have

$$U(|f|; P) - L(|f|; P) \le U(f; P) - L(f; P)$$

$f$ Darboux integrable $\implies \forall \varepsilon > 0\ \exists P$ partition of $[a, b]$ s.t.

$$\begin{aligned}
&U(f; P) - L(f; P) < \varepsilon \\
\implies &\forall \varepsilon > 0\, \exists P \text{ partition of } [a, b] \text{ s.t. } U(|f|; P) - L(|f|; P) < \varepsilon \\
\implies &\left.\begin{array}{l} |f| \text{ is Darboux integrable} \\ |f| \text{ is bounded} \end{array}\right\} \implies |f| \text{ is Riemann integrable}
\end{aligned}$$

We have

$$-|f(x)| \le f(x) \le |f(x)| \qquad \forall x \in [a, b]$$

By the previous theorem,

$$-\int_a^b |f(x)|\,dx = \int_a^b -|f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx$$

which implies

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

□

**Theorem 21.4**

Let $f : [a, b] \to \mathbb{R}$ be a function and let $a < c < b$. Assume $f$ is Riemann integrable on $[a, c]$ and on $[c, b]$. Then $f$ is Riemann integrable on $[a, b]$ and

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

*Proof.* $f$ is Riemann integrable on $[a, c]$ and on $[c, b]$

$$\implies f \text{ bounded on } [a, c] \text{ and on } [c, b]$$
$$\implies f \text{ bounded on } [a, b]$$

Fix $\varepsilon > 0$. As $f$ is Riemann integrable on $[a, c]$, $f$ is Darboux integrable on $[a, c]$

$$\implies \exists P_1 \text{ partition of } [a, c] \text{ s.t. } U_a^c(f; P_1) - L_a^c(f; P_1) < \frac{\varepsilon}{2}$$

Similarly, as $f$ is Riemann integrable on $[c, b] \implies f$ Darboux integrable on $[c, b]$

$$\implies \exists P_2 \text{ partition of } [c, b] \text{ s.t. } U_c^b(f; P_2) - L_c^b(f; P_2) < \frac{\varepsilon}{2}$$

Let $P = P_1 \cup P_2$ partition on $[a, b]$ and

$$U(f; P) = U_a^c(f; P_1) + U_c^b(f; P_2)$$
$$L(f; P) = L_a^c(f; P_1) + L_c^b(f; P_2)$$

So

$$U(f; P) - L(f; P) < \frac{\varepsilon}{2}$$

Therefore, as $f$ is Darboux integrable and bounded on $[a, b]$, $f$ is Riemann integrable on $[a, b]$. Moreover,

$$\begin{aligned} U(f) \le U(f; P) = U_a^c(f; P_1) + U_c^b(f; P_2) &< L_a^c(f; P_1) + L_c^b(f; P_2) + \varepsilon \\ &\le \int_a^c f(x)\,dx + \int_c^b f(x)\,dx + \varepsilon \end{aligned}$$

Similarly,

$$L(f) \ge \int_a^c f(x)\,dx + \int_c^b f(x)\,dx - \varepsilon$$

Since $\varepsilon > 0$ is arbitrary,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

□

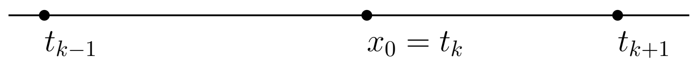
**Lemma 21.5**

Let $f, g : [a, b] \to \mathbb{R}$ be functions s.t. $f$ is Riemann integrable and $g(x) = f(x)$ except at finitely many points in $[a, b]$. Then $g$ is Riemann integrable and

$$\int_a^b g(x)\,dx = \int_a^b f(x)\,dx$$

*Proof.* Arguing by induction, we may assume that there exists exactly one point $x_0 \in [a, b]$ s.t. $f(x_0) \neq g(x_0)$. Let $B > 0$ s.t. $|f(x)| \leq B$ and $|g(x)| \leq B$ $\forall x \in [a, b]$. Let $P = \{a = t_0 < \ldots < t_n = b\}$. We consider

$$U(f; P) - U(g; P)$$
$$L(f; P) - L(g; P)$$




The largest contribution occurs when $x_0 = t_k$ for some $1 \leq k \leq n - 1$.

$$\begin{aligned} |M(f; [t_{k-1}, t_k]) - M(g; [t_{k-1}, t_k])| &\leq [B - (-B)](t_k - t_{k-1}) \\ &\leq 2B \text{ mesh}(P) \\ \implies |U(f; P) - U(g; P)| &\leq 4B \text{ mesh}(P) \end{aligned}$$

Similarly,

$$\begin{aligned} |m(f; [t_{k-1}, t_k]) - m(g; [t_{k-1}, t_k])| &\leq 2B \text{ mesh}(P) \\ \implies |L(f; P) - L(g; P)| &\leq 4B \text{ mesh}(P) \end{aligned}$$

Thus,

$$\begin{aligned} U(g; P) - L(g; P) &\leq U(f; P) - L(f; P) + |U(f; P) - U(g; P)| \\ &\quad + |L(f; P) - L(g; P)| \\ &\leq U(f; P) - L(f; P) + 8B \text{ mesh}(P) \end{aligned}$$

$f$ Darboux integrable $\implies \forall \varepsilon > 0\ \exists \delta > 0$ s.t.

$$U(f; P) - L(f; P) < \frac{\varepsilon}{2} \qquad \forall P \text{ partition with mesh}(P) < \delta$$

Choose $\delta$ even smaller if necessary so that

$$8B\delta < \frac{\varepsilon}{2} \iff \delta < \frac{\varepsilon}{16B}$$

Then $U(g; P) - L(g; P) < \varepsilon$ for all $P$ partition with mesh$(P) < \delta$.

$$\left.\begin{aligned} &g \text{ is Darboux integrable} \\ &g \text{ bounded} \end{aligned}\right\} \implies g \text{ is Riemann integrable}$$

**Exercise 21.1.** Show $\int_a^b g(x)\, dx = \int_a^b f(x)\, dx$. □

# §22 Lec 22: May 17, 2021

## §22.1 Riemann Integral (Cont'd)

**Definition 22.1** (Piecewise Monotone) — We say that a function $f : [a, b] \to \mathbb{R}$ is piecewise monotone if there exists a partition $P = \{a = t_0 < \ldots < t_n = b\}$ s.t. $f$ is monotone on $(t_{k-1}, t_k)$ for each $1 \le k \le n$.

**Definition 22.2** (Piecewise Continuous) — We say that $f : [a, b] \to \mathbb{R}$ is piecewise continuous if there exists a partition $P = \{a = t_0 < \ldots < t_n = b\}$ s.t. $f$ is uniformly continuous on $(t_{k-1}, t_k)$ for each $1 \le k \le n$.

**Theorem 22.3**

Let $f : [a, b] \to \mathbb{R}$ be a function that satisfies

1. $f$ is bounded and piecewise monotone.

   **or**

2. $f$ is piecewise continuous.

Then $f$ is Riemann integrable.

*Proof.* Let $P = \{a = t_0 < \ldots < t_n = b\}$ be a partition of $[a, b]$ s.t. 1) $f$ is monotone **or** 2) $f$ is uniformly continuous on $(t_{k-1}, t_k)$ $\forall 1 \le k \le n$.

If $f$ is monotone on $(t_{k-1}, t_k)$, then $f$ can be extended to a monotone function on $f_k$ on $[t_{k-1}, t_k]$. For example, if $f$ is increasing on $(t_{k-1}, t_k)$ we define

$$f_k(t) = \begin{cases} \inf_{t \in (t_{k-1}, t_k)} f(t), & t = t_{k-1} \\ f(t), & t \in (t_{k-1}, t_k) \\ \sup_{t \in (t_{k-1}, t_k)} f(t), & t = t_k \end{cases}$$

As $f_k$ is monotone on $[t_{k-1}, t_k]$, $f_k$ is Riemann integrable on $[t_{k-1}, t_k]$. As $f$ differs from $f_k$ at most two points, $f$ is Riemann integrable on $[t_{k-1}, t_k]$ and

$$\int_{t_{k-1}}^{t_k} f(t)\, dt = \int_{t_{k-1}}^{t_k} f_k(t)\, dt$$

If $f$ is uniformly continuous on $(t_{k-1}, t_k)$, then $f$ admits a continuous extension $f_k$ to $[t_{k-1}, t_k]$. Then $f_k$ is Riemann integrable on $[t_{k-1}, t_k]$ and so $f$ is Riemann integrable on $[t_{k-1}, t_k]$ and

$$\int_{t_{k-1}}^{t_k} f(t)\, dt = \int_{t_{k-1}}^{t_k} f_k(t)\, dt$$

By the last theorem from last lecture, we conclude that $f$ is Riemann integrable on $[a, b]$ and

$$\int_a^b f(t)\,dt = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f(t)\,dt \qquad \square$$

**Theorem 22.4** (Intermediate Value Property for Integrals)
Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exists $c \in [a, b]$ s.t.

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx$$

*Proof.* $f$ is continuous on $[a, b]$ compact which implies there exist $x_0, y_0 \in [a, b]$ s.t.

$$\begin{cases} f(x_0) = \inf_{x\in[a,b]} f(x) \\ f(y_0) = \sup_{x\in[a,b]} f(x) \end{cases}$$

So

$$(b-a)f(x_0) = \int_a^b f(x_0)\,dx \le \int_a^b f(x)\,dx \le \int_a^b f(y_0)\,dx = (b-a)f(y_0)$$

$$\left.\begin{array}{r} \implies f(x_0) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le f(y_0) \\ f \text{ is continuous} \implies f \text{ has the Darboux property} \end{array}\right\} \implies$$

$\implies \exists c$ between $x_0$ and $y_0$ s.t. $f(c) = \frac{1}{b-a}\int_a^b f(x)dx$. $\square$

## §22.2 Fundamental Theorem of Calculus

**Definition 22.5** (Riemann Integrable – "Extension") — We say that a function $f : (a, b) \to \mathbb{R}$ is Riemann integrable on $[a, b]$ if every extension of $f$ to $[a, b]$ is Riemann integrable. In this case, $\int_a^b f(t)dt$ does not depend on the values of the extension at $a$ and at $b$.

**Theorem 22.6** (Fundamental Theorem of Calculus Part II)
Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$. If $f'$ is Riemann integrable on $[a, b]$ then

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

*Proof.* Fix $\varepsilon > 0$. As $f'$ is Riemann integrable on $[a, b]$, $\exists P = \{a = t_0 < \ldots < t_n = b\}$ s.t.

$$U(f'; P) - L(f'; P) < \varepsilon$$

where $f$ is continuous on $[t_{k-1}, t_k]$ and differentiable on $(t_{k-1}, t_k)$. So, by the Mean Value theorem, $\exists x_k \in (t_{k-1}, t_k)$ s.t.

$$f'(x_k) = \frac{f(t_k) - f(t_{k-1})}{t_k - t_{k-1}}$$

In particular,

$$\sum_{k=1}^{n} f'(x_k)(t_k - t_{k-1}) = \sum_{k=1}^{n} [f(t_k) - f(t_{k-1})] = f(b) - f(a)$$

is a Riemann sum of $f'$ associated to the partition $P$. Moreover,

$$\left.\begin{aligned} L(f';P) \le f(b) - f(a) \le U(f';P) < L(f';P) + \varepsilon \\ L(f';P) \le \textstyle\int_a^b f'(x)\,dx \le U(f';P) < L(f';P) + \varepsilon \end{aligned}\right\} \implies$$

$$\left.\begin{aligned} &\implies \left| \textstyle\int_a^b f'(x)\,dx - [f(b) - f(a)] \right| < 2\varepsilon \\ &\varepsilon > 0 \text{ was arbitrary} \end{aligned}\right\} \implies \int_a^b f'(x)\,dx = f(b) - f(a)$$ □

**Theorem 22.7** (Integration by Parts)
Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$. If $f'$ and $g'$ are Riemann integrable on $[a, b]$, then

$$\int_a^b f(x)g'(x)\,dx + \int_a^b f'(x)g(x)\,dx = f(b)g(b) - f(a)g(a)$$

*Proof.* By Exc 1 from Hw 8, the product of two Riemann integrable functions is Riemann integrable. In particular, $f'g$ and $fg'$ are Riemann integrable. Let $h : [a, b] \to \mathbb{R}$, $h(x) = f(x)g(x)$. We have $h$ is continuous on $[a, b]$, differentiable on $(a, b)$ and

$$h'(x) = f'(x)g(x) + f(x)g'(x)$$

$h'$ is Riemann integrable on $[a, b]$. By Fundamental Theorem of Calculus Part II,

$$\int_a^b h'(x)\,dx = h(b) - h(a)$$

$$\implies \int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a)$$ □

**Theorem 22.8** (Fundamental Theorem of Calculus Part I)
Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. For $x \in [a, b]$, we define

$$F(x) = \int_a^x f(t)\,dt$$

Then $F$ is continuous on $[a, b]$. Moreover, if $f$ is continuous at a point $x_0 \in (a, b)$, then $F$ is differentiable at $x_0$ and

$$F'(x_0) = f(x_0)$$

*Proof.* For $a \leq x < y \leq b$,

$$\begin{aligned} F(y) - F(x) &= \int_a^y f(t)\,dt - \int_a^x f(t)\,dt \\ &= \int_a^x f(t)\,dt + \int_x^y f(t)\,dt - \int_a^x f(t)\,dt \\ &= \int_x^y f(t)\,dt \end{aligned}$$

$f$ is Riemann integrable $\implies$ $f$ is bounded $\implies$ $\exists M > 0$ s.t.

$$|f(x)| \leq M \qquad \forall x \in [a,b]$$

So

$$|F(y) - F(x)| \leq \int_x^y |f(t)|\,dt \leq M\,|y - x|$$

This shows $F$ is uniformly continuous on $[a,b]$. For each $\varepsilon > 0$ if $|y - x| < \frac{\varepsilon}{M}$ then

$$|F(y) - F(x)| < \varepsilon$$

Assume $f$ is continuous at $x_0 \in (a,b)$. For $x \in [a,b] \setminus \{x_0\}$,

$$\begin{aligned} \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) &= \frac{1}{x - x_0} \int_{x_0}^x f(t)\,dt - f(x_0) \\ &= \frac{1}{x - x_0} \int_{x_0}^x f(t)\,dt - \frac{1}{x - x_0} \int_{x_0}^x f(x_0)\,dt \\ &= \frac{1}{x - x_0} \int_{x_0}^x \left[ f(t) - f(x_0) \right]\,dt \end{aligned}$$

Fix $\varepsilon > 0$. As $f$ is continuous at $x_0$, $\exists \delta > 0$ s.t.

$$|f(x) - f(x_0)| < \varepsilon \qquad \forall\, |x - x_0| < \delta \quad x \in [a,b]$$

So for $x \in [a,b]$ with $0 < |x - x_0| < \delta$,

$$\begin{aligned} \left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| &\leq \frac{1}{|x - x_0|} \int_{x_0}^x |f(t) - f(x_0)|\,dt \\ &< \frac{1}{|x - x_0|} \int_{x_0}^x \varepsilon\,dt = \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $F$ is differentiable at $x_0$ and $F'(x_0) = f(x_0)$. □

# §23 Lec 23: May 19, 2021

## §23.1 Change of Variables

**Theorem 23.1** (Change of Variables)
Let $J$ be an open interval in $\mathbb{R}$ and let $u : J \to \mathbb{R}$ be differentiable with $u'$ continuous on $J$. Let $I$ be an open interval in $\mathbb{R}$ s.t. $u(J) \subseteq I$ and let $f : I \to \mathbb{R}$ be continuous. Then $f \circ u : J \to \mathbb{R}$ is continuous and for any $a, b \in J$ with $a < b$ we have

$$\int_a^b f\left(u(x)\right) \cdot u'(x)\, dx = \int_{u(a)}^{u(b)} f(y)\, dy$$

*Proof.* As $f \circ u$ and $u'$ are continuous on $[a, b]$, the function $x \mapsto (f \circ u)(x) \cdot u'(x)$ is continuous on $[a, b]$ and so it's Riemann integrable on $[a, b]$.

Fix $c \in I$ and consider $F(x) = \int_c^x f(t)dt$. By Fundamental Theorem of Calculus Part I, $F$ is differentiable on $I$ (because $f$ is continuous on $I$) and $F'(x) = f(x)\ \forall x \in I$. Consider $x \mapsto (F \circ u)(x)$ is differentiable on $J$ and

$$(F \circ u)'(x) = f\left(u(x)\right) \cdot u'(x) \qquad \forall x \in J$$

By the Fundamental Theorem of Calculus Part II,

$$\int_a^b (F \circ u)'(x)\, dx = (F \circ u)(b) - (F \circ u)(a)$$

which implies

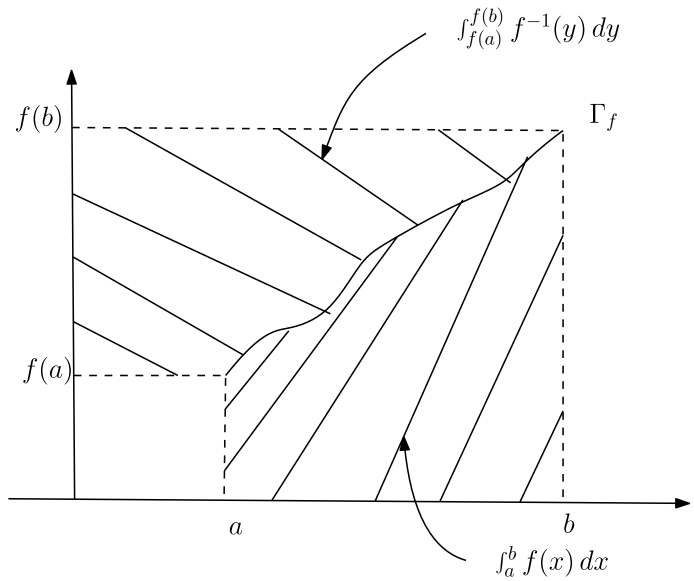
$$\implies \int_a^b f\left(u(x)\right) \cdot u'(x)\, dx = \int_c^{u(b)} f(y)\, dy - \int_c^{u(a)} f(y)\, dy = \int_{u(a)}^{u(b)} f(y)\, dy \qquad \square$$

**Exercise 23.1.** Let $I$ be an open interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be injective and differentiable with $f'$ continuous on $I$. Then $J = f(I)$ is an open interval and $f^{-1} : J \to I$ is differentiable.

Then for any $a, b \in I$ with $a < b$ we have

$$\int_a^b f(x)\, dx + \int_{f(a)}^{f(b)} f^{-1}(y)\, dy = bf(b) - af(a)$$

*Proof.* Consider:




$$\Gamma_f = \{(x, f(x)) : a \leq x \leq b\} = \left\{\left(f^{-1}(y), y\right) : y \text{ between } f(a) \text{ and } f(b)\right\}$$

We perform a change of variables:

$$\int_{f(a)}^{f(b)} f^{-1}(y)\,dy = \int_a^b f^{-1}\left(f(x)\right) f'(x)\,dx$$

where $y = f(x)$ and $dy = f'dx$

$$\begin{aligned}\int_a^b f^{-1}\left(f(x)\right) f'(x)\,dx &= \int_a^b xf'(x)\,dx \\ &= xf(x)\Big|_{x=a}^{x=b} - \int_a^b f(x)\,dx \\ &= bf(b) - af(a) - \int_a^b f(x)\,dx\end{aligned}$$ □

**Theorem 23.2**

Let $f_n : [a, b] \to \mathbb{R}$ be Riemann integrable s.t. $f_n \xrightarrow[n\to\infty]{u} f$ on $[a, b]$. Then $f$ is Riemann integrable and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx = \int_a^b f(x)\,dx$$

*Proof.* For $n \geq 1$ let $d_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|$. As $f_n \xrightarrow[n\to\infty]{u} f$ on $[a, b]$ we have $d_n \xrightarrow[n\to\infty]{} 0$. In particular, $f_n(x) - d_n \leq f(x) \leq f_n(x) + d_n$ for all $x \in [a, b]$ (and thus $f$ is bounded). For any partition $P$ of $[a, b]$, we have

$$\begin{cases} U(f_n; P) - d_n(b-a) \leq U(f; P) \leq U(f_n; P) + d_n(b-a) \\ L(f_n; P) - d_n(b-a) \leq L(f; P) \leq L(f_n; P) + d_n(b-a) \end{cases}$$

So

$$U(f;P) - L(f;P) \le U(f_n;P) - L(f_n;P) + 2d_n(b-a)$$

Fix $\varepsilon > 0$. As $d_n \underset{n\to\infty}{\longrightarrow} 0$, $\exists n_\varepsilon \in \mathbb{N}$ s.t.

$$d_n < \frac{\varepsilon}{4(b-a)} \qquad \forall n \ge n_\varepsilon$$

Then for each $n \ge n_\varepsilon$ (fixed) there exists a partition $P = P(\varepsilon, n)$ of $[a,b]$ s.t.

$$U(f_n;P) - L(f_n;P) < \frac{\varepsilon}{2}$$

For $n \ge n_\varepsilon$ and $P = P(\varepsilon, n)$ as above we get

$$U(f;P) - L(f;P) < \varepsilon$$

As $\varepsilon > 0$ is arbitrary, this shows that $f$ is Riemann integrable (since it's Darboux integrable and bounded). Moreover,

$$\begin{aligned}\int_a^b f(x)\,dx \le U(f;P) &\le U(f_n;P) + d_n(b-a)\\ &< L(f_n;P) + \frac{\varepsilon}{2} + \frac{\varepsilon}{4}\\ &\le \int_a^b f_n(x)\,dx + \frac{3\varepsilon}{4}\end{aligned}$$

Similarly,

$$\begin{aligned}\int_a^b f(x)\,dx \ge L(f;P) &\ge L(f_n;P) - d_n(b-a)\\ &> U(f_n;P) - \frac{\varepsilon}{2} - \frac{\varepsilon}{4}\\ &\ge \int_a^b f_n(x)\,dx - \frac{3\varepsilon}{4}\end{aligned}$$

Thus,

$$\implies \left|\int_a^b f(x)\,dx - \int_a^b f_n(x)\,dx\right| < \frac{3\varepsilon}{4} \qquad \forall n \ge n_\varepsilon$$

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

□

## §23.2 Lebesgue Criterion

**Definition 23.3** (Zero Outer Measure) — A set $A \subseteq \mathbb{R}$ is said to have zero outer measure if for every $\varepsilon > 0$ there exists a countable collection of open intervals $\{(a_n, b_n)\}_{n\ge 1}$ s.t.

$$\begin{cases} A \subseteq \bigcup_{n\ge 1}(a_n, b_n) \\ \sum_{n\ge 1}(b_n - a_n) < \varepsilon \end{cases}$$

**Remark 23.4.**
1. If $A \subseteq \mathbb{R}$ has zero outer measure and $B \subseteq A$, then $B$ has zero outer measure.
2. If $\{A_n\}_{n\geq 1}$ is a sequence of zero outer measure sets, then $\bigcup_{n\geq 1} A_n$ has zero outer measure.
3. If $A$ is a set that is at most countable, then $A$ has zero outer measure.

*Proof.* 2. Fix $\varepsilon > 0$. For each $n \geq 1$, let $\left\{\left(a_m^{(n)}, b_m^{(n)}\right)\right\}_{m\geq 1}$ be open intervals s.t.

$$\begin{cases} A_n \subseteq \bigcup_{m\geq 1} \left(a_m^{(n)}, b_m^{(n)}\right) \\ \sum_{n\geq 1} \left(b_m^{(n)} - a_m^{(n)}\right) < \frac{\varepsilon}{2^n} \end{cases}$$

Then $\left\{\left(a_m^{(n)}, b_m^{(n)}\right)\right\}_{m,n\geq 1}$ is a countable collection of open intervals s.t.

$$\begin{cases} \bigcup_{n\geq 1} A_n \subseteq \bigcup_{n,m\geq 1} \left(a_m^{(n)}, b_m^{(n)}\right) \\ \sum_{n\geq 1}\sum_{m\geq 1} \left(b_m^{(n)} - a_m^{(n)}\right) < \sum_{n\geq 1} \frac{\varepsilon}{2^n} = \varepsilon \end{cases}$$

□

**Theorem 23.5** (Lebesgue Criterion)
Let $f : [a, b] \to \mathbb{R}$ be bounded. Then $f$ is Riemann integrable if and only if the set

$$\mathscr{D}_f = \{x \in [a, b] : f \text{ is not continuous at } x\}$$

has zero outer measure.

*Proof.* We have

$$\mathscr{D}_f = \{x \in [a, b] : \omega(f, x) = 0\}$$

where

$$\begin{aligned} \omega(f, x) &= \inf_{\delta>0} \omega\left(f, B_\delta(x)\right) \\ &= \inf_{\delta>0} \left[\sup_{y\in B_\delta(x)} f(y) - \inf_{y\in B_\delta(x)} f(y)\right] \\ &= \inf_{\delta>0} \left[M\left(f; B_\delta(x)\right) - m\left(f; B_\delta(x)\right)\right] \end{aligned}$$

Then

$$\begin{aligned} \mathscr{D}_f &= \{x \in [a, b] : \omega(f, x) > 0\} \\ &= \bigcup_{n\geq 1} \underbrace{\left\{x \in [a, b] : \omega(f, x) \geq \frac{1}{n}\right\}}_{:=F_n} \end{aligned}$$

Key Observation: If $P = \{a = t_0 < \ldots < t_n = b\}$ then

$$\begin{aligned} U(f;P) - L(f;P) &= \sum_{k=1}^{n} \left[ M\left(f;[t_{k-1},t_k]\right) - m\left(f;[t_{k-1},t_k]\right)\right]\left(t_k - t_{k-1}\right) \\ &= \sum_{k=1}^{n} \omega\left(f;[t_{k-1},t_k]\right)\left(t_k - t_{k-1}\right) \end{aligned}$$

We will continue with this proof in the next lecture. □

# §24 Lec 24: May 21, 2021

## §24.1 Lebesgue Criterion (Cont'd)

*Proof.* (Cont'd) " $\implies$ " Assume that $f$ is Riemann integrable. We denote

$$\begin{aligned}\mathscr{D}_f &= \{x \in [a,b] : \omega(f,x) > 0\} \\ &= \bigcup_{n \geq 1} \left\{x \in [a,b] : \omega(f,x) \geq \frac{1}{n}\right\}\end{aligned}$$

For $n \geq 1$, let $F_n = \left\{x \in [a,b] : \omega(f,x) \geq \frac{1}{n}\right\}$. To show that $\mathscr{D}_f$ has zero outer measure, it suffices to prove that $F_n$ has zero outer measure for all $n \geq 1$.

Fix $N \geq 1$ and $\varepsilon > 0$. As $f$ is Riemann integrable, there exists a partition $P = \{a = t_0 < \ldots < t_n = b\}$ s.t.

$$U(f;P) - L(f;P) < \frac{\varepsilon}{N}$$

Let $I = \{1 \leq k \leq n : F_N \cap (t_{k-1}, t_k) \neq \emptyset\}$. Then

$$F_N \subseteq \bigcup_{k \in I} (t_{k-1}, t_k) \cup P$$

As $P$ is finite, it has zero outer measure. Thus, it suffices to show that

$$\sum_{k \in I} (t_k - t_{k-1}) < \varepsilon$$

Then,

$$\begin{aligned}\frac{\varepsilon}{N} > U(f;P) - L(f;P) &= \sum_{k=1}^{n} \left[M\left(f; [t_{k-1}, t_k]\right) - m\left(f; [t_{k-1}, t_k]\right)\right](t_k - t_{k-1}) \\ &\geq \sum_{k \in I} \omega\left(f; [t_{k-1}, t_k]\right)(t_k - t_{k-1}) \\ &\geq \frac{1}{N} \sum_{k \in I} (t_k - t_{k-1})\end{aligned}$$

which implies

$$\sum_{k \in I} (t_k - t_{k-1}) < \varepsilon$$

" $\impliedby$ " Assume that $\mathscr{D}_f$ has zero outer measure.

$$f \text{ bounded} \implies \exists M > 0 \text{ s.t. } |f(x)| \leq M \qquad \forall x \in [a,b]$$

Fix $\varepsilon > 0$ and let $\alpha > 0$ to be chosen later. Consider

$$\left.\begin{aligned}&F_\alpha = \{x \in [a,b] : \omega(f,x) \geq \alpha\} \subseteq \mathscr{D}_f \\ &\mathscr{D}_f \text{ has zero outer measure}\end{aligned}\right\} \implies F_\alpha \text{ has zero outer measure}$$

$$\implies \exists \{(a_n, b_n)\}_{n \geq 1} \text{ s.t. } \begin{cases} F_\alpha \subseteq \bigcup_{n \geq 1} (a_n, b_n) \\ \sum_{n \geq 1} (b_n - a_n) < \varepsilon \end{cases}$$

Let $A = [a,b] \setminus F_\alpha$. For any $x \in A$, $\omega(f,x) < \alpha \implies \exists (c_x, d_x)$ neighborhood of $x$ s.t.

$$\omega(f;[c_x,d_x]) < \alpha$$

So

$$\left.\begin{aligned}&[a,b] = F_\alpha \cup A \subseteq \textstyle\bigcup_{n\ge1}(a_n,b_n) \cup \bigcup_{x\in A}(c_x,d_x)\\ &[a,b] \text{ is compact}\end{aligned}\right\}$$

which implies there exists $n_0 \in \mathbb{N}$ and $J \subseteq A$ finite s.t.

$$[a,b] \subseteq \bigcup_{k=1}^{n_0}(a_k,b_k) \cup \bigcup_{x\in J}(c_x,d_x)$$

Let $P$ be a partition of $[a,b]$ formed by the points

$$\left(\{a,b\} \cup \bigcup_{k=1}^{n_0}\{a_x,b_x\} \cup \bigcup_{x\in J}\{c_x,d_x\}\right) \cap [a,b]$$

Say $P = \{a = t_0 < \ldots < t_n = b\}$. For any $1 \le l \le n$, we have

$$[t_{l-1},t_l] \subseteq [a_k,b_k] \text{ for some } 1 \le k \le n_0$$

or

$$[t_{l-1},t_l] \subseteq [c_x,d_x] \text{ for some } x \in J$$

Let

$$\begin{aligned}I_1 &= \{1 \le l \le n : [t_{l-1},t_l] \subseteq [a_k,b_k] \text{ for some } 1 \le k \le n_0\}\\ I_2 &= \{1,\ldots,n\} \setminus I_1\end{aligned}$$

Note that

$$\sum_{l\in I_1}(t_l - t_{l-1}) \le \sum_{k=1}^{n_0}(b_k - a_k) < \varepsilon$$
$$l \in I_2,\ \omega(f;[t_{l-1},t_l]) \le \omega(f;[c_x,d_x]) < \alpha$$

Then,

$$\begin{aligned}U(f;P) - L(f;P) &= \sum_{l=1}^{n}\left[M(f;[t_{l-1},t_l]) - m(f;[t_{l-1},t_l])\right](t_l - t_{l-1})\\ &= \sum_{l\in I_1}\left[M(f;[t_{l-1},t_l]) - m(f;[t_{l-1},t_l])\right](t_l - t_{l-1})\\ &+ \sum_{l\in I_2}\omega(f;[t_{l-1},t_l])(t_l - t_{l-1})\end{aligned}$$

Notice that

$$\sum_{l\in I_1}\left[M(f;[t_{l-1},t_l]) - m(f;[t_{l-1},t_l])\right](t_l - t_{l-1}) \le 2M\sum_{l\in I_1}(t_l - t_{l-1}) < 2M\varepsilon$$

So

$$\sum_{l\in I_2} \omega\left(f;[t_{l-1},t_l]\right)(t_l - t_{l-1}) < \alpha \sum_{l\in I_2}(t_l - t_{l-1}) \le \alpha \sum_{l=1}^{n}(t_l - t_{l-1}) = \alpha(b-a)$$

Choose $\alpha < \frac{\varepsilon}{b-a}$ to get

$$U(f;P) - L(f;P) < 2M\varepsilon + \varepsilon$$

As $\varepsilon$ is arbitrary, this shows that $f$ is Darboux integrable, and thus Riemann integrable. □

## §24.2 Improper Riemann Integrals

**Definition 24.1** (Locally Riemann Integrable) — Let $-\infty < a < b \le \infty$. We say that $f : [a,b) \to \mathbb{R}$ is locally Riemann integrable if $f$ is integrable on $[a,c]$ for any $c \in (a,b)$.

**Definition 24.2** (Improper Riemann Integral) — Let $-\infty < a < b \le \infty$ and $f : [a,b) \to \mathbb{R}$ is locally Riemann integrable. In addition,

$$\lim_{c\to b} \int_a^c f(x)\,dx \text{ exists in } \mathbb{R}$$

We denote it $\int_a^b f(x)dx$ and we call it the improper Riemann integral of $f$. In this case we say that the improper Riemann integral of $f$ converges. If

$$\lim_{c\to b} \int_a^c f(x)\,dx = \pm\infty$$

then we write $\int_a^b f(x)dx = \pm\infty$ and we say that the improper Riemann integral of $f$ diverges to $\pm\infty$.

**Remark 24.3.** One can make a similar definition if $-\infty \le a < b < \infty$ and $f : (a,b] \to \mathbb{R}$ or if $-\infty \le a < b \le \infty$ and $f : (a,b) \to \mathbb{R}$.

**Theorem 24.4**

Let $-\infty < a < b < \infty$ and let $f : [a,b) \to \mathbb{R}$ be locally Riemann integrable and bounded. Then the improper Riemann integral $\int_a^b f(x)dx$ converges. Moreover, any extension $\tilde{f} : [a,b] \to \mathbb{R}$ of $f$ to $[a,b]$ is Riemann integrable and

$$\int_a^b \tilde{f}(x)\,dx = \int_a^b f(x)\,dx$$

*Proof.* Let $\tilde{f} : [a, b] \to \mathbb{R}$ be an extension of $f$ to $[a, b]$. As $f$ is bounded, $\exists M > 0$ s.t.

$$\left|\tilde{f}(x)\right| \leq M \qquad \forall x \in [a, b]$$

For $c \in (a, b)$,

$$U_a^b(\tilde{f}) = U_a^c(\tilde{f}) + U_c^b(\tilde{f}) = \int_a^c f(x)\,dx + U_c^b(\tilde{f}) \tag{*}$$

$$L_a^b(\tilde{f}) = L_a^c(\tilde{f}) + L_c^b(\tilde{f}) = \int_a^c f(x)\,dx + L_c^b(\tilde{f})$$

$$\left.\begin{aligned} &\implies U_a^b(\tilde{f}) - L_a^b(\tilde{f}) = U_c^b(\tilde{f}) - L_c^b(\tilde{f}) \\ &U_c^b(\tilde{f}) \leq M(b-c) \\ &\left|L_c^b(\tilde{f})\right| \leq M(b-c) \end{aligned}\right\} \implies U_a^b(\tilde{f}) - L_a^b(\tilde{f}) \leq \underbrace{2M(b-c)}_{\overset{c\to b}{\longrightarrow} 0}$$

This shows that $\tilde{f}$ is Riemann integrable. Moreover, by (*),

$$\int_a^b \tilde{f}(x)\,dx = \lim_{c \to b} \int_a^c f(x)\,dx$$

Thus, the improper Riemann integral of $f$ converges and

$$\int_a^b f(x)\,dx = \int_a^b \tilde{f}(x)\,dx$$

□

# §25 Lec 25: May 24, 2021

## §25.1 Improper Riemann Integrals (Cont'd)

**Proposition 25.1**

Let $-\infty < a < b \le \infty$ and let $f, g : [a, b) \to \mathbb{R}$ be locally Riemann integrable s.t. the improper Riemann integrals of $f$ and $g$ converge. Then

1. For any $\alpha \in \mathbb{R}$, the improper Riemann integral of $\alpha f$ converges and

$$\int_a^b (\alpha f)(x)\,dx = \alpha \int_a^b f(x)\,dx$$

2. The improper Riemann integral of $f + g$ converges and

$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

*Proof.* 1. Consider:

$$\begin{aligned}\mathbb{R} \ni \alpha \int_a^b f(x)\,dx = \alpha \lim_{c\to b} \int_a^c f(x)\,dx &= \lim_{c\to b} \alpha \int_a^c f(x)\,dx \\ (f \text{ is locally Riemann integrable}) &= \lim_{c\to b} \int_a^c (\alpha f)(x)\,dx\end{aligned}$$

So the improper Riemann integral of $\alpha f$ converges and

$$\int_a^b (\alpha f)(x)\,dx = \lim_{c\to b} \int_a^c (\alpha f)(x)\,dx = \alpha \int_a^b f(x)\,dx$$

2. Consider:

$$\begin{aligned}\mathbb{R} \ni \int_a^b f(x)\,dx + \int_a^b g(x)\,dx &= \lim_{c\to b} \int_a^c f(x)\,dx + \lim_{c\to b} \int_a^c g(x)\,dx \\ &= \lim_{c\to b} \left[\int_a^c f(x)\,dx + \int_a^c g(x)\,dx\right] \\ &= \lim_{c\to b} \int_a^c \left[f(x) + g(x)\right]\,dx\end{aligned}$$

So the improper Riemann integral of $f + g$ converges and

$$\int_a^b (f+g)(x)\,dx = \lim_{c\to b} \int_a^c (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

□

**Remark 25.2.** If $f, g : [a, b] \to \mathbb{R}$ are Riemann integrable functions, then

- $|f|$ is Riemann integrable.
- $f \cdot g$ is Riemann integrable.

However, if $f, g : [a, b)$ are locally integrable functions s.t. the improper Riemann integrals of $f$ and $g$ converge, then

- the improper Riemann integral of $|f|$ need not converge.
- the improper Riemann integral of $f \cdot g$ need not converge.

**Example 25.3**

Let $f, g : (0, 1] \to \mathbb{R}$, $f(x) = g(x) = \frac{1}{\sqrt{x}}$. The improper Riemann integral of $f$ converges

$$\int_c^1 f(x)\, dx = \int_c^1 \frac{1}{\sqrt{x}}\, dx = 2\sqrt{x}\Big|_{x=c}^{x=1} = 2 - 2\sqrt{c} \underset{c\to 0}{\longrightarrow} 2$$

The improper Riemann integral of $f \cdot g$ does not converge

$$\int_c^1 f(x)g(x)\, dx = \int_c^1 \frac{1}{x}\, dx = \ln x\Big|_{x=c}^{x=1} = -\ln c \underset{c\to 0}{\longrightarrow} \infty$$

More generally, we can take $f, g : (0, 1] \to \mathbb{R}$

$$f(x) = \frac{1}{x^\alpha}, \quad g(x) = \frac{1}{x^\beta} \quad \text{with} \quad 0 < \alpha, \beta < 1 \quad \text{and} \quad \alpha + \beta \geq 1$$

**Lemma 25.4** (Cauchy Criterion)

Let $-\infty < a < b \leq \infty$. Let $f : [a, b) \to \mathbb{R}$ be locally integrable. Then the improper Riemann integral of $f$ converges if and only if

$$\forall \varepsilon > 0 \quad \exists c_\varepsilon \in (a, b) \text{ s.t. } \left|\int_{c_1}^{c_2} f(x)\, dx\right| < \varepsilon \quad \forall c_\varepsilon < c_1 < c_2 < b$$

*Proof.* " $\implies$ " Assume that the improper Riemann integral of $f$ converges. Let

$$\alpha = \int_a^b f(x)\, dx \in \mathbb{R}$$

We have

$$\alpha = \lim_{c\to b} \int_a^c f(x)\, dx$$

Then $\forall \varepsilon > 0\ \exists c_\varepsilon \in (a, b)$ s.t.

$$\left|\alpha - \int_a^c f(x)\, dx\right| < \frac{\varepsilon}{2} \qquad \forall c_\varepsilon < c < b$$

For $c_\varepsilon < c_1 < c_2 < b$ we have

$$\begin{aligned}\left|\int_{c_1}^{c_2} f(x)\,dx\right| &= \left|\int_a^{c_2} f(x)\,dx - \int_a^{c_1} f(x)\,dx\right| \\ &\le \left|\int_a^{c_2} f(x)\,dx - \alpha\right| + \left|\alpha - \int_a^{c_1} f(x)\,dx\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

" $\Longleftarrow$ " Fix $\varepsilon > 0$ and let $c_\varepsilon \in (a,b)$ s.t.

$$\left|\int_{c_1}^{c_2} f(x)\,dx\right| < \varepsilon \qquad \forall c_\varepsilon < c_1 < c_2 < b$$

Let $\{c_n\}_{n\ge1} \subseteq (a,b)$ s.t. $c_n \underset{n\to\infty}{\longrightarrow} b$. Then $\exists n_\varepsilon \in \mathbb{N}$ s.t. $c_\varepsilon < c_n < b$ for all $n \ge n_\varepsilon$. In particular,

$$\begin{aligned}&\left|\int_a^{c_m} f(x)\,dx - \int_a^{c_n} f(x)\,dx\right| = \left|\int_{c_n}^{c_m} f(x)\,dx\right| < \varepsilon \qquad n,m \ge n_\varepsilon \\ &\implies \left\{\int_a^{c_n} f(x)\,dx\right\}_{n\ge1} \subseteq \mathbb{R} \text{ is Cauchy and so convergent}\end{aligned}$$

Let $\alpha = \lim_{n\to\infty} \int_a^{c_n} f(x)\,dx$. To prove that the Riemann integral of $f$ converges, we need to show that $\alpha$ does not depend on $\{c_n\}_{n\ge1}$. Let $\{d_n\}_{n\ge1} \subseteq (a,b)$ s.t. $\lim_{n\to\infty} d_n = b$. Consider

$$x_n = \begin{cases} c_k & \text{if } n = 2k \\ d_k & \text{if } n = 2k-1 \end{cases} \qquad \text{for } k \ge 1$$

Then $x_n \underset{n\to\infty}{\longrightarrow} b$. From the same argument used for the sequence $\{c_n\}_{n\ge1}$, we conclude that $\left\{\int_a^{x_n} f(x)\,dx\right\}_{n\ge1}$ is Cauchy and so convergent. So

$$\begin{aligned}\lim_{n\to\infty}\int_a^{x_{2n}} f(x)\,dx &= \lim_{n\to\infty}\int_a^{x_{2n-1}} f(x)\,dx \\ \alpha = \lim_{n\to\infty}\int_a^{c_n} f(x)\,dx &= \lim_{n\to\infty}\int_a^{d_n} f(x)\,dx\end{aligned}$$

□

**Theorem 25.5** (Abel Criterion)

Let $-\infty < a < b \le \infty$ and let $f, g : [a,b) \to \mathbb{R}$ be locally integrable. Assume that $g$ is decreasing and $\lim_{x\to b} g(x) = 0$. Assume also that there exists $M > 0$ s.t.

$$\left|\int_a^c f(x)\,dx\right| \le M \qquad \forall a < c < b$$

Then the improper Riemann integral of $f \cdot g$ converges.

**Remark 25.6.** Compare this with the series version

$$\left.\begin{array}{l}\{a_n\}_{n\geq 1} \text{ is decreasing with } \lim_{n\to\infty} a_n = 0 \\ \exists M > 0 \text{ s.t. } \left|\sum_{k=1}^{n} b_k\right| \leq M \quad \forall n \geq 1\end{array}\right\} \implies \sum_{n\geq 1} a_n b_n \text{ converges}$$

*Proof.* We'll use the Cauchy Criterion. Fix $\varepsilon > 0$.

$$\lim_{x\to b} g(x) = 0 \implies \exists c_\varepsilon \in (a,b) \text{ s.t. } |g(x)| < \varepsilon \quad \forall c_\varepsilon < x < b$$

Fix $c_\varepsilon < c_1 < c_2 < b$ and consider $\int_{c_1}^{c_2} f(x)g(x)dx$. Using exercise #6 in HW8, we can find $x_0 \in [c_1, c_2]$ s.t.

$$\begin{aligned}\int_{c_1}^{c_2} f(x)g(x)\,dx &= g(c_1)\int_{c_1}^{x_0} f(x)\,dx + g(c_2)\int_{x_0}^{c_2} f(x)\,dx \\ &= g(c_1)\left[\int_a^{x_0} f(x)\,dx - \int_a^{c_1} f(x)\,dx\right] \\ &+ g(c_2)\left[\int_a^{c_2} f(x)\,dx - \int_a^{x_0} f(x)\,dx\right]\end{aligned}$$

which implies

$$\begin{aligned}\left|\int_{c_1}^{c_2} f(x)g(x)\,dx\right| &\leq g(c_1)\left[\left|\int_a^{x_0} f(x)\,dx\right| + \left|\int_a^{c_1} f(x)\,dx\right|\right] \\ &+ g(c_2)\left[\left|\int_a^{c_2} f(x)\,dx\right| + \left|\int_a^{x_0} f(x)\,dx\right|\right] \\ &< 4M\varepsilon\end{aligned}$$

As $c_\varepsilon < c_1, c_2 < b$ are arbitrary and $\varepsilon > 0$ is arbitrary, we conclude that the improper Riemann integral of $fg$ converges. □

# §26 Lec 26: May 26, 2021

## §26.1 Improper Riemann Integrals (Cont'd)

**Exercise 26.1.** Show that the improper Riemann integral

$$\int_0^\infty \frac{\sin x}{x}\,dx \quad \text{converges}$$

but the improper Riemann integral

$$\int_0^\infty \left|\frac{\sin x}{x}\right| dx \quad \text{does not converge}$$

*Proof.* To show that $\int_0^\infty \frac{\sin x}{x} dx$ converges, we have to prove that

$$\lim_{M\to\infty} \int_0^M \frac{\sin x}{x}\,dx \quad \text{exists in } \mathbb{R}$$

Note that

$$x \mapsto \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

is continuous on on $[0,\infty)$ and so it is Riemann integrable on $[0, M]$ for each $M > 0$. For $M > 1$, we write

$$\int_0^M \frac{\sin x}{x}\,dx = \underbrace{\int_0^1 \frac{\sin x}{x}\,dx}_{\in\mathbb{R}} + \int_1^M \frac{\sin x}{x}\,dx$$

Note that $f, g : [1,\infty) \to \mathbb{R}$, $f(x) = \sin x$ and $g(x) = \frac{1}{x}$ are continuous and so Riemann integrable on $[1, M]$ $\forall M > 1$. Also,

- $g$ is decreasing and $\lim_{x\to\infty} g(x) = 0$
- In addition,

$$\left|\int_1^M \sin x\,dx\right| = |\cos 1 - \cos M| \le 2 \qquad \forall M > 1$$

So by the Abel Criterion, the improper Riemann integral $\int_1^\infty \frac{\sin(x)}{x} dx$ converges. Moreover,

$$\begin{aligned}\int_0^\infty \frac{\sin x}{x}\,dx = \lim_{M\to\infty}\int_0^M \frac{\sin x}{x}\,dx &= \int_0^1 \frac{\sin x}{x}\,dx + \lim_{M\to\infty}\int_1^M \frac{\sin x}{x}\,dx \\ &= \int_0^1 \frac{\sin x}{x}\,dx + \int_1^\infty \frac{\sin x}{x}\,dx\end{aligned}$$

Let's show that the improper Riemann integral $\int_0^\infty \frac{|\sin x|}{x} dx$ diverges to $\infty$. We'll use that

$$|\sin x| \ge \frac{1}{2} \quad \text{on} \quad \left[k\pi + \frac{\pi}{6}, k\pi + \frac{5\pi}{6}\right]$$

for all $k \geq 0$. So

$$\begin{aligned}
\int_0^\infty \frac{|\sin x|}{x}\,dx &\geq \sum_{k\geq 0} \int_{k\pi+\frac{\pi}{6}}^{k\pi+\frac{5\pi}{6}} \frac{|\sin x|}{x}\,dx \\
&\geq \sum_{k\geq 0} \frac{1}{2}\cdot \frac{1}{k\pi + \frac{5\pi}{6}} \cdot \left[\left(k\pi + \frac{5\pi}{6}\right) - \left(k\pi + \frac{\pi}{6}\right)\right] \\
&\geq \sum_{k\geq 0} \frac{1}{2}\cdot\frac{1}{(k+1)\pi}\cdot\frac{2\pi}{3} = \frac{1}{3}\sum_{k\geq 0}\frac{1}{k+1} = \infty
\end{aligned}$$

□

**Proposition 26.1**

Let $-\infty < a < b \leq \infty$ and let $f : [a, b) \to \mathbb{R}$ be locally Riemann integrable s.t. the improper Riemann integral of $|f|$ converges. Then the improper Riemann integral of $f$ converges and

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

*Proof.* As the improper Riemann integral of $|f|$ converges, by the Cauchy Criterion we have

$$\forall \varepsilon > 0 \quad \exists c_\varepsilon \in (a,b) \text{ s.t. } \int_{c_1}^{c_2} |f(x)|\,dx < \varepsilon \quad \forall c_\varepsilon < c_1 < c_2 < b$$

As $f$ is locally integrable, $f$ is integrable on $[c_1, c_2]$ and

$$\left|\int_{c_1}^{c_2} f(x)\,dx\right| \leq \int_{c_1}^{c_2} |f(x)|\,dx < \varepsilon \qquad \forall c_\varepsilon < c_1 < c_2 < b$$

By the Cauchy Criterion, the improper Riemann integral of $f$ converges. Moreover,

$$\begin{aligned}
\left|\int_a^b f(x)\,dx\right| = \left|\lim_{c\to b}\int_a^c f(x)\,dx\right| &= \lim_{c\to b}\left|\int_a^c f(x)\,dx\right| \\
(f \text{ is locally integrable}) &\leq \lim_{c\to b}\int_a^c |f(x)|\,dx \\
&= \int_a^b |f(x)|\,dx
\end{aligned}$$

□

**Definition 26.2** (Absolute Convergence – Integral) — Let $-\infty < a < b \leq \infty$ and $f : [a, b) \to \mathbb{R}$ be locally integrable. We say that the improper Riemann integral of $f$ converges absolutely if the improper Riemann integral of $|f|$ converges.

**Remark 26.3.**

1. If the improper Riemann integral of $f$ converges absolutely, then it converges.
2. The improper Riemann integral of $f$ converges absolutely if and only if

$$\lim_{c \to b} \int_a^c |f(x)|\, dx \in \mathbb{R} \iff \exists M > 0 \text{ s.t. } \int_a^c |f(x)|\, dx \le M \quad \forall c \in [a, b)$$

3. If $f, g : [a, b) \to \mathbb{R}$ are locally integrable s.t. $|f(x)| \le |g(x)|\ \forall x \in [a, b)$ and the improper Riemann integral of $g$ converges absolutely, then the improper Riemann integral of $f$ converges absolutely.
4. If $f, g : [a, b) \to \mathbb{R}$ are locally integrable and their improper Riemann integrals converge absolutely, then the improper Riemann integral of $f + g$ converges absolutely.
5. If $f, g : [a, b) \to \mathbb{R}$ are locally integrable s.t. $f$ is bounded and the improper Riemann integral of $g$ converges absolutely, then the improper Riemann integral of $f \cdot g$ converges absolutely.

## §26.2 Continuous 1-Periodic Functions

**Definition 26.4** (Convolution) — Let $f, g : \mathbb{R} \to \mathbb{C}$ be continuous functions with period 1, that is,

$$f(x+1) = f(x) \quad \text{and} \quad g(x+1) = g(x) \qquad x \in \mathbb{R}$$

Their convolution $f * g : \mathbb{R} \to \mathbb{C}$ is defined via

$$(f * g)(x) = \int_0^1 f(y) g(x - y)\, dy$$

**Claim 1**:

$$(f * g)(x) = \int_a^{a+1} f(y) g(x-y)\, dy \qquad \forall a \in \mathbb{R},\ \forall x \in \mathbb{R}$$

This is obviously true if $a = k \in \mathbb{Z}$. For $y = k + z$,

$$\begin{aligned} \int_k^{k+1} f(y) g(x-y)\, dy &= \int_0^1 f(k+z) g(x - z - k)\, dz \\ (f\&g \text{ periodic}) &= \int_0^1 f(z) g(x-z)\, dz = (f * g)(x) \end{aligned}$$

Next, decomposing $a = \underbrace{[a]}_{\in\mathbb{Z}} + \underbrace{\{a\}}_{\in[0,1)}$ we see that it suffices to prove the claim for $a \in (0,1)$.

$$\begin{aligned}\int_a^{a+1} f(y)g(x-y)\,dy &= \int_a^1 f(y)g(x-y)\,dy + \int_1^{1+a} f(y)g(x-y)\,dy \\ &= \int_a^1 f(y)g(x-y)\,dy + \int_0^a f(z+1)g(x-z-1)\,dz \\ &= \int_a^1 f(y)g(x-y)\,dy + \int_0^a f(z)g(x-z)\,dz \\ &= \int_0^1 f(y)g(x-y)\,dy = (f*g)(x)\end{aligned}$$

**Claim 2**: $f*g$ is 1-periodic.

$$(f*g)(x+1) = \int_0^1 f(y)g(x+1-y)\,dy = \int_0^1 f(y)g(x-y)\,dy = (f*g)(x)$$

**Claim 3**: $f*g$ is continuous

$$\begin{aligned}|(f*g)(x_1)-(f*g)(x_2)| &= \left|\int_0^1 f(y)\left[g(x_1-y)-g(x_2-y)\right]\,dy\right| \\ &\le \int_0^1 |f(y)|\,|g(x_1-y)-g(x_2-y)|\,dy\end{aligned}$$

$g$ continuous on $[0,2]$ compact $\implies$ $g$ is uniformly continuous on $[0,2]$, and since $g$ is 1-periodic, we conclude that $g$ is uniformly continuous on $\mathbb{R}$. So $\forall\varepsilon>0\ \exists\delta>0$ s.t.

$$|g(x)-g(y)|<\varepsilon \qquad \forall\,|x-y|<\delta$$

$f$ is continuous on $[0,1]$ compact $\implies M>0$ s.t.

$$|f(x)|\le M \qquad \forall x\in[0,1]$$

So

$$|(f*g)(x_1)-(f*g)(x_2)| \le \int_0^1 M\cdot\varepsilon\,dy = M\cdot\varepsilon \quad \forall\,|x_1-x_2|<\delta$$

**Claim 4**: $f*g = g*f$. For $z = x-y$,

$$\begin{aligned}(g*f)(x) = \int_0^1 g(y)f(x-y)\,dy &= -\int_x^{x-1} g(x-z)f(z)\,dz \\ &= \int_{x-1}^x f(y)g(x-y)\,dy \\ &= \int_0^1 f(y)g(x-y)\,dy \\ &= (f*g)(x)\end{aligned}$$

**Claim 5**: For all $\alpha\in\mathbb{C}$,

$$(\alpha f)*g = f*(\alpha g) = \alpha(f*g)$$

**Claim 6**: If $f,g,h$ are continuous, 1-periodic functions,

$$\begin{cases} f*(g+h) = f*g+f*h \\ (f*g)*h = f*(g*h)\end{cases}$$

Left as exercise!

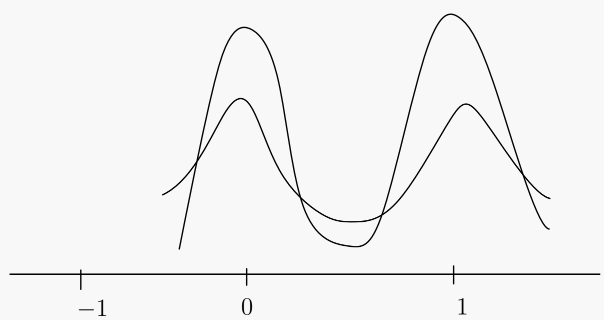
# §27 Lec 27: May 28, 2021

## §27.1 Continuous 1-Periodic Functions (Cont'd)

**Definition 27.1** (Approximation to the Identity) — A sequence of continuous, 1-periodic functions $K_n : \mathbb{R} \to \mathbb{C}$ is called an approximation to the identity if it satisfies the following:

1. $\int_0^1 K_n(x)\, dx = 1 \ \forall n \geq 1$
2. $\exists M > 0$ s.t. $\int_0^1 |K_n(x)|\, dx \leq M \ \forall n \geq 1$
3. $\forall \delta > 0$, $\int_\delta^{1-\delta} |K_n(x)|\, dx \underset{n\to\infty}{\longrightarrow} 0$.

**Remark 27.2.** While 1) says that $K_n$ assigns mass 1 to each period, 3) says that this mass is concentrating at the integers as $n \to \infty$.




**Theorem 27.3**

Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous, 1-periodic function and let $\{K_n\}_{n\geq 1}$ be an approximation to the identity. Then

$$K_n * f \underset{n\to\infty}{\overset{u}{\longrightarrow}} f \text{ on } \mathbb{R}$$

*Proof.* Fix $x \in \mathbb{R}$.

$$\begin{aligned}
(K_n * f)(x) - f(x) &= \int_0^1 K_n(y) f(x-y)\, dy - f(x) \int_0^1 K_n(y)\, dy \\
&= \int_0^1 K_n(y) \left[f(x-y) - f(x)\right] dy \\
\implies |(K_n * f)(x) - f(x)| &\leq \int_0^1 |K_n(y)|\, |f(x-y) - f(x)|\, dy
\end{aligned}$$

$f$ is continuous and 1-periodic $\implies$ $f$ is uniformly continuous.

Let $\varepsilon > 0$. Then $\exists \delta > 0$ s.t. $|f(x) - f(y)| < \varepsilon$ for all $|x - y| < \delta$

$$\begin{aligned}
\int_0^\delta |K_n(y)| \underbrace{|f(x-y) - f(x)|}_{<\varepsilon} \, dy &< \varepsilon \int_0^\delta |K_n(y)| \, dy \\
&\le \varepsilon \int_0^1 |K_n(y)| \, dy \le \varepsilon M \\
\int_{1-\delta}^1 |K_n(y)| \, |f(x-y) - f(x)| \, dy &\overset{y=1+z}{=} \int_{-\delta}^0 |K_n(1+z)| \, |f(x-z-1) - f(x)| \, dz \\
&= \int_{-\delta}^0 |K_n(z)| \underbrace{|f(x-z) - f(x)|}_{<\varepsilon} \, dz \\
&< \varepsilon \int_{-1}^0 |K_n(z)| \, dz \le \varepsilon M \\
\int_\delta^{1-\delta} |K_n(y)| \, |f(x-y) - f(x)| \, dy &\le \int_\delta^{1-\delta} |K_n(y)| \, [|f(x-y)| + |f(x)|] \, dy \\
&\le 2 \sup_{x \in [0,1]} |f(x)| \int_\delta^{1-\delta} |K_n(y)| \, dy
\end{aligned}$$

As $\int_\delta^{1-\delta} |K_n(y)| \, dy \underset{n\to\infty}{\longrightarrow} 0$, $\exists n_\varepsilon \in \mathbb{N}$ s.t.

$$\int_\delta^{1-\delta} |K_n(y)| \, dy < \frac{\varepsilon}{2\|f\|_\infty + 1}$$

So collecting our estimates, we get

$$|(K_n * f)(x) - f(x)| \le 2\varepsilon M + \varepsilon \qquad \forall x \in \mathbb{R}, \ \forall n \ge n_\varepsilon$$

As $\varepsilon > 0$ is arbitrary, we get $K_n * f \overset{u}{\underset{n\to\infty}{\longrightarrow}} f$. □

## §27.2 Fourier Series

**Definition 27.4** (Orthonormal Family) — For $n \in \mathbb{Z}$, let $e_n(x) = e^{2\pi i n x} = \cos(2\pi n x) + i \sin(2\pi n x)$. Note $e_n : \mathbb{R} \to \mathbb{C}$ is continuous, 1-periodic.

$$\int_0^1 e_n(x) \, dx = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

So

$$\int_0^1 e_n(x) \overline{e_m(x)} \, dx = \int_0^1 e_{n-m}(x) \, dx = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$$

$\implies \{e_n\}_{n \ge 1}$ form an orthonormal family.

**Definition 27.5** (Trigonometric Polynomial) — A trigonometric polynomial takes the form

$$\sum_{|n|\leq N} c_n e_n(x)$$

where $c_n \in \mathbb{C}$ for all $|n| \leq N$.

**Definition 27.6** (Fourier Series) — Given a continuous, 1-periodic function $f : \mathbb{R} \to \mathbb{C}$, we define its $n^{\text{th}}$ Fourier coefficient via

$$\hat{f}(n) = \int_0^1 f(x)\overline{e_n(x)}\, dx = \int_0^1 f(x)e^{-2\pi inx}\, dx$$

The Fourier series of $f$ is given by $\sum_{n\in\mathbb{Z}} \hat{f}(n)e_n(x)$.

**Question 27.1.** Can we recover $f$ from its Fourier series?

If $f \in C^2$, then

$$\sum_{n\in\mathbb{Z}} \hat{f}(n)e_n(x) \xrightarrow[n\to\infty]{u} f(x)$$

In 1966, Carleson proved that the Fourier series of an integrable function converges pointwise to $f$ outside a set of measure zero.
For $N \geq 0$, let

$$\begin{aligned} S_N(f)(x) = \sum_{|n|\leq N} \hat{f}(n)e_n(x) &= \sum_{|n|\leq N} \int_0^1 f(y)\overline{e_n(y)}\, dy \cdot e_n(x) \\ &= \sum_{|n|\leq N} \int_0^1 f(y)e_n(x-y)\, dy \\ &= \int_0^1 f(y) \left( \sum_{|n|\leq N} e_n \right) (x-y)\, dy \\ &= \left[ f * \left( \sum_{|n|\leq N} e_n \right) \right] (x) \end{aligned}$$

For $N \geq 0$, let $D_N = \sum_{|n|\leq N} e_n$ denote the Dirichlet Kernel. Note that

$$\int_0^1 D_N(x)\, dx = \sum_{|n|\leq N} \int_0^1 e_n(x)\, dx = 1 \qquad \forall N \geq 0$$

$\{D_N\}_{N\geq 0}$ do not form an approximation to the identity since

$$\int_0^1 |D_N(x)|\, dx \xrightarrow[N\to\infty]{} \infty$$

We have

$$\begin{aligned} D_N &= \sum_{|n|\le N} e_n \\ (e_1 - 1)D_N &= \sum_{n=-N+1}^{N+1} e_n - \sum_{n=-N}^{N} e_n = e_{N+1} - e_{-N} \\ \implies D_N &= \frac{e_{N+1} - e_{-N}}{e_1 - 1} \end{aligned} \tag{1}$$

In addition,

$$\begin{aligned} D_N(x) = \frac{e^{2\pi i(N+1)x} - e^{-2\pi i N x}}{e^{2\pi i x} - 1} &= \frac{e^{\pi i x}\left(e^{2\pi i\left(N+\frac{1}{2}\right)x} - e^{-2\pi i\left(N+\frac{1}{2}\right)x}\right)}{e^{\pi i x}\left(e^{\pi i x} - e^{-\pi i x}\right)} \\ &= \frac{\sin\left(2\pi\left(N+\frac{1}{2}\right)x\right)}{\sin(\pi x)} \end{aligned}$$

Also,

$$\begin{aligned} \int_0^1 |D_N(x)|\,dx \ge \int_0^1 \frac{\left|\sin\left(2\pi\left(N+\frac{1}{2}\right)x\right)\right|}{\pi x}\,dx \\ \underset{y=2\pi\left(N+\frac{1}{2}\right)x}{=} \int_0^{2\pi\left(N+\frac{1}{2}\right)} \frac{|\sin(y)|}{\pi\cdot\frac{y}{2\pi\left(N+\frac{1}{2}\right)}}\cdot\frac{dy}{2\pi\left(N+\frac{1}{2}\right)} \\ = \frac{1}{\pi}\int_0^{2\pi\left(N+\frac{1}{2}\right)} \frac{|\sin(y)|}{y}\,dy \underset{N\to\infty}{\longrightarrow} \infty \end{aligned}$$

The average of the Dirichlet kernels do form an approximation to the identity. For $N \ge 1$, let $F_N = \frac{D_0 + \ldots + D_{N_1}}{N}$ denote the Fejer Kernels. Note that

$$\int_0^1 F_N(x)\,dx = \frac{1}{N}\sum_{k=0}^{N-1}\int_0^1 D_k(x)\,dx = 1 \qquad N \ge 1$$

We will show that $F_N \ge 0$ and so

- $\int_0^1 |F_N(x)|\,dx = \int_0^1 F_N(x)dx = 1\ \forall N \ge 1$
- $\forall \delta > 0$, $\int_\delta^{1-\delta} |F_N(x)|\,dx \underset{N\to\infty}{\longrightarrow} 0$

Consequently, we obtain the following

**Theorem 27.7**

If $f : \mathbb{R} \to \mathbb{C}$ is a continuous, 1-periodic function, then

$$F_N * f \xrightarrow[N\to\infty]{u} f \text{ on } \mathbb{R}$$

if and only if

$$\sigma(f) = \frac{1}{N}\sum_{k=0}^{N-1} S_N(f) \xrightarrow[N\to\infty]{u} f \text{ on } \mathbb{R}$$

**Corollary 27.8**

If $f : \mathbb{R} \to \mathbb{C}$ is a continuous, 1-periodic function, with $\hat{f}(n) = 0\ \forall n \in \mathbb{Z}$, then $f \equiv 0$.

**Corollary 27.9**

Every continuous, 1-periodic function can be approximated uniformly by trigonometric polynomials.

# §28 Lec 28: Jun 2, 2021

## §28.1 Fourier Series (Cont'd)

Recall that for $n \in \mathbb{Z}$ we define the character $e_n : \mathbb{R} \to \mathbb{C}$

$$e_n(x) = e^{2\pi inx}$$

For a continuous, 1-periodic function $f : \mathbb{R} \to \mathbb{C}$, we define its $n^{\text{th}}$ Fourier coefficient via

$$\hat{f}(n) = \int_0^1 f(x)\overline{e_n(x)}\,dx = \int_0^1 f(x)e^{-2\pi inx}\,dx \qquad \forall n \in \mathbb{Z}$$

and the partial Fourier series

$$\left[S_N(f)\right](x) = \sum_{|n|\le N} \hat{f}(n)e_n(x) \qquad \forall N \ge 0$$

We observed $S_N(f) = f * D_N$ where $D_N$ denotes the Dirichlet kernel

$$D_N = \sum_{|n|\le N} e_n \qquad \forall N \ge 0$$

Using

$$D_N = \frac{e_{N+1} - e_{-N}}{e_1 - 1} \tag{1}$$

We obtained the explicit formula

$$D_N(x) = \frac{\sin\left(2\pi\left(N+\frac{1}{2}\right)x\right)}{\sin(\pi x)}$$

and computed

$$\int_0^1 |D_N(x)|\,dx \xrightarrow[N\to\infty]{} \infty$$

In particular, $\{D_N\}_{N\ge1}$ do not form an approximation to the identity. Instead, we define the Fejer Kernel

$$F_N = \frac{D_0 + \ldots + D_{N-1}}{N} \qquad \forall N \ge 1$$

So

$$\sigma(f) = f * F_N = \frac{1}{N}\sum_{n=0}^{N-1} f * D_n = \frac{1}{N}\sum_{n=0}^{N-1} S_n(f)$$

**Claim 28.1.** $\{F_N\}_{N\ge1}$ form an approximation to the identity and thus $\sigma(f) \xrightarrow[n\to\infty]{u} f$ for any continuous, 1-periodic $f : \mathbb{R} \to \mathbb{C}$.

*Proof.* First, we have

$$\int_0^1 e_n(x)\,dx = \int_0^1 \cos(2\pi nx)\,dx + i\int_0^1 \sin(2\pi ni)\,dx = \begin{cases} 1, & n = 0 \\ 0, & n \ne 0 \end{cases}$$

we get

$$\int_0^1 D_N(x)\,dx = \sum_{|n|\le N} \int_0^1 e_n(x)\,dx = 1 \qquad \forall N \ge 0$$

and so

$$\int_0^1 F_N(x)\,dx = \frac{1}{N} \sum_{n=0}^{N-1} \int_0^1 D_n(x)\,dx = 1 \qquad \forall N \ge 1$$

Net, we compute an explicit formula for $F_N$

$$\begin{aligned} NF_N &= D_0 + \ldots + D_{N-1} \\ &\overset{(1)}{=} \frac{e_1 - e_0}{e_1 - 1} + \frac{e_2 - e_{-1}}{e_1 - 1} + \ldots + \frac{e_N - e_{-N+1}}{e_1 - 1} \\ &= \frac{(e_1 + e_2 + \ldots + e_N) - (e_0 + e_{-1} + \ldots + e_{-N+1})}{e_1 - 1} \\ &= \frac{(e_1 - 1)(e_1 + e_2 + \ldots + e_N) - (e_1 - 1)(e_0 + e_{-1} + \ldots + e_{-N+1})}{(e_1 - 1)^2} \end{aligned}$$

Notice that

$$\begin{aligned} (e_1 - 1)(e_1 + \ldots + e_N) &= e_2 + \ldots + e_{N+1} - e_1 - \ldots - e_N = e_{N+1} - e_1 \\ (e_1 - 1)(e_0 + \ldots + e_{-N+1}) &= e_1 + \ldots + e_{-N+2} - e_0 - \ldots - e_{-N+1} = e_1 - e_{-N+1} \end{aligned}$$

So

$$\begin{aligned} NF_N(x) &= \frac{e_{N+1}(x) + e_{-N+1}(x) - 2e_1(x)}{\left(e^{2\pi i x} - 1\right)^2} \\ &= \frac{e_1(x)\left(e^{2\pi i N x} + e^{-2\pi i N x} - 2\right)}{e_1(x)\left(e^{\pi i x} - e^{-\pi i x}\right)^2} \\ &= \frac{2\left(\cos(2\pi N x) - 1\right)}{\left[2i\sin(\pi x)\right]^2} \\ &= \left[\frac{\sin(\pi N x)}{\sin(\pi x)}\right]^2 \end{aligned}$$

which implies

$$F_N(x) = \frac{1}{N}\left[\frac{\sin(\pi N x)}{\sin(\pi x)}\right]^2 \ge 0 \qquad \forall N \ge 1$$

Thus,

$$\int_0^1 |F_N(x)|\,dx = \int_0^1 F_N(x)\,dx = 1 \qquad \forall N \ge 1$$

Lastly, we have to verify that $\forall 0 < \delta < 1$

$$\int_\delta^{1-\delta} |F_N(x)|\,dx \underset{N\to\infty}{\longrightarrow} 0$$

Fix $\delta > 0$. Then

$$\delta \le x \le 1 - \delta \implies \pi\delta \le \pi x \le \pi - \pi\delta$$

$\implies \exists c_\delta > 0$ s.t.
$$|\sin(\pi x)|^2 \geq c_\delta \qquad \forall x \in [\delta, 1-\delta]$$

So
$$\begin{aligned}\int_\delta^{1-\delta} |F_N(x)|\, dx &= \frac{1}{N}\int_\delta^{1-\delta} \left|\frac{\sin(\pi N x)}{\sin(\pi x)}\right|^2 dx \\ &\leq \frac{1}{N}\int_\delta^{1-\delta} \frac{1}{c_\delta}\, dx \\ &= \frac{1}{N}\frac{1-2\delta}{c_\delta} \underset{N\to\infty}{\longrightarrow} 0\end{aligned}$$

This proves that $\{F_N\}_{N\geq 1}$ form an approximation to the identity. □

## §28.2 Topology Addendum

**Lemma 28.1**

Let $(X,d)$ be a metric space. A set $A \subseteq X$ is dense in $X$ if and only if $A \cap W \neq \emptyset$ for every non-empty open set $W \subseteq X$.

*Proof.* " $\implies$ " Let $A \subseteq X$ be such that $\overline{A} = X$. Assume, towards a contradiction that $\exists \emptyset \neq W = \mathring{W} \subseteq X$ s.t.
$$\begin{aligned}A \cap W = \emptyset &\implies W \subseteq {}^cA \\ &\implies W = \mathring{W} \subseteq \widehat{{}^cA}^{\circ} = {}^c(\overline{A}) = {}^cX = \emptyset\end{aligned}$$

which is a contradiction as $W \neq \emptyset$.
" $\impliedby$ " Assume, towards a contradiction, that
$$\overline{A} \neq X \implies \left.\begin{matrix}{}^c(\overline{A}) \neq \emptyset \\ {}^c(\overline{A}) = \widehat{{}^cA}^{\circ}\end{matrix}\right\} \implies \widehat{{}^cA}^{\circ} \neq \emptyset$$

which implies
$$\exists x \in {}^cA \text{ and } \exists r > 0 \text{ s.t. } B_r(x) \subseteq {}^cA$$

So $\underbrace{B_r(x)}_{\neq\emptyset \text{ open}} \cap A \neq \emptyset$ – contradiction! □

**Theorem 28.2**

Let $(X,d)$ be a complete metric space. Then $X$ has the property of Baire, that is, for every sequence $\{A_n\}_{n\geq 1}$ of open dense sets we have
$$\overline{\bigcap_{n\geq 1} A_n} = X$$

*Proof.* Using the lemma, it suffices to show

$$\bigcap_{n\geq 1} A_n \cap W \neq \emptyset \qquad \forall \emptyset \neq W = \mathring{W} \subseteq X$$

Fix $\emptyset \neq W = \mathring{W} \subseteq X$.

$$\overline{A_1} = x \implies A_1 \cap W \neq \emptyset \implies \exists x_1 \in \underbrace{A_1 \cap W}_{\text{open}} \implies \exists 0 < r_1 < 1 \text{ s.t.}$$

$$K_{r_1}(x_1) = \{y \in X : d(y, x_1) \leq r_1\} \subseteq A_1 \cap W$$

$$\overline{A_2} = X \implies A_2 \cap B_{r_1}(x_1) \neq \emptyset \implies \exists x_2 \in \underbrace{A_2 \cap B_{r_1}(x_1)}_{\text{open}} \implies \exists 0 < r_2 < \frac{1}{2} \text{ s.t.}$$

$$K_{r_2}(x_2) \subseteq A_1 \cap B_{r_1}(x_1)$$

Proceeding inductively, we find a sequence $\{x_n\}_{n\geq 1} \subseteq X$ and $\{r_n\}_{n\geq 1}$ s.t.

$$\begin{cases} 0 < r_n < \frac{1}{n} & \forall n \geq 1 \\ K_{r_{n+1}}(x_{n+1}) \subseteq A_{n+1} \cap B_{r_n}(x_n) \subseteq K_{r_n}(x_n) & \forall n \geq 1 \end{cases}$$

Note that $\{K_{r_n}(x_n)\}_{n\geq 1}$ is a sequence of nested closed sets whose diameters decrease to zero. As $(X, d)$ is complete, we find

$$\bigcap_{n\geq 1} K_{r_n}(x_n) = \{x\}$$

for some $x \in X$. In addition,

$$\{x\} = \bigcap_{n\geq 1} K_{r_n}(x_n) \subseteq A_1 \cap W \cap \bigcap_{n\geq 2} A_n \cap B_{r_{n-1}}(x_{n-1}) \subseteq \left(\bigcap_{n\geq 1} A_n\right) \cap W$$

which implies $\left(\bigcap_{n\geq 1} A_n\right) \cap W \neq \emptyset$. □

**Lemma 28.3**

Let $(X, d)$ be a metric space. Then the following are equivalent:

1. For every $\{A_n\}_{n\geq 1}$ of open dense sets we have $\overline{\bigcap_{n\geq 1} A_n} = X$.
2. For every $\{F_n\}_{n\geq 1}$ of closed sets with empty interiors, we have

$$\widehat{\bigcup_{n\geq 1} F_n}^{\circ} = \emptyset$$

*Proof.* Left as exercise. □

# §29 Lec 29: Jun 4, 2021

## §29.1 Topology Addendum (Cont'd)

**Lemma 29.1**

Let $(X,d)$ be a metric space that has the Baire property. If $\emptyset \neq W = \mathring{W} \subseteq X$, then $W$ has the Baire property.

*Proof.* Fix $\emptyset \neq W = \mathring{W} \subseteq X$. Let $\{D_n\}_{n\geq 1}$ be open dense sets in $W$.
$D_n$ open in $W \implies \exists G_n$ open in $X$ s.t. $D_n = G_n \cap W$ open in $X$ as $G_n$ and $W$ are open.
$D_n$ dense in $W \implies \overline{D_n} \cap W = W \implies W \subseteq \overline{D_n} \implies \overline{W} \subseteq \overline{D_n}$.
Define $A_n = D_n \cup {}^c(\overline{W})$ open in $X$.

$$\overline{A_n} = \overline{D_n \cup {}^c(\overline{W})} = \overline{D_n} \cup \overline{{}^c(\overline{W})} = \overline{D_n} \cup {}^c\left(\mathring{\overline{W}}\right) \supseteq \overline{W} \cup {}^c(\overline{W}) = X$$

Thus $\{A_n\}_n$ are dense open sets in $X$ and as $X$ has the Baire property,

$$\overline{\bigcap_{n\geq 1} A_n} = X$$

Then,

$$X = \overline{\bigcap_{n\geq 1} A_n} = \overline{\bigcap_{n\geq 1} \left[D_n \cup {}^c(\overline{W})\right]} = \overline{\left(\bigcap_{n\geq 1} D_n\right) \cup {}^c(\overline{W})} = \overline{\bigcap_{n\geq 1} D_n} \cup {}^c\left(\mathring{\overline{W}}\right)$$

which implies

$$W = \left[\overline{\bigcap_{n\geq 1} D_n} \cup {}^c\left(\mathring{\overline{W}}\right)\right] \cap W$$

$$= \left[\overline{\bigcap_{n\geq 1} D_n} \cap W\right] \cup \left[{}^c\left(\mathring{\overline{W}}\right) \cap W\right]$$

$$\left.\mathring{\overline{W}} \supseteq \mathring{W} = W \implies {}^c\left(\mathring{\overline{W}}\right) \subseteq {}^cW \implies {}^c\left(\mathring{\overline{W}}\right) \cap W = \emptyset\right\}$$

$\implies \overline{\bigcap_{n\geq 1} D_n} \cap W = W$ i.e. $\bigcap_{n\geq 1} D_n$ is dense in $W$. □

**Theorem 29.2**

Let $(X,d)$ be a metric space with the Baire property. Let $f_n : X \to \mathbb{R}$ be continuous function that converges pointwise to a function $f : X \to \mathbb{R}$. Then the set

$$C = \{x \in X : \ f \text{ is continuous at } x\} \text{ is dense in } X$$

*Proof.* We can observe that it suffices to prove the theorem under the additional hypothesis

$$|f_n(x)| \leq 1 \qquad \forall x \in X \quad \forall n \geq 1$$

Indeed, if $\{f_n\}_{n\geq 1}$ is as in the theorem, then we consider

$\phi : \mathbb{R} \to (-1,1), \quad \phi(x) = \dfrac{x}{1+|x|}$ continuous, bijective, with the inverse $\phi^{-1}(y) = \dfrac{y}{1-|y|}$

So $\phi \circ f_n : X \to (-1,1)$ is continuous and $|\phi \circ f_n(x)| \leq 1$ for all $n \geq 1$ and $x \in X$. Also, $f_n \underset{n\to\infty}{\longrightarrow} f$ pointwise $\implies \phi \circ f_n \underset{n\to\infty}{\longrightarrow} \phi \circ f$ pointwise. If the theorem holds with the additional uniform boundedness hypothesis, we get

$$\left.\begin{aligned}&\{x \in X : \ \phi \circ f \text{ is continuous at } x\}\\ &\{x \in X : \ f \text{ is continuous at } x\}\end{aligned}\right\} \text{ is dense in } X$$

So without the loss of generality, we assume

$$|f_n(x)| \leq 1 \qquad \forall n \geq 1 \quad \forall x \in X \tag{1}$$

Then,

$$\begin{aligned} C &= \{x \in X : \ f \text{ is continuous at } x\} \\ &= \{x \in X : \ \omega(f,x) = 0\} \\ &= \bigcap_{n\geq 1} \underbrace{\left\{x \in X : \ \omega(f,x) < \frac{1}{n}\right\}}_{=:G_n \text{ open in } X} = \bigcap_{n\geq 1} G_n \end{aligned}$$

As $X$ has the Baire property, to prove $\overline{C} = X$ it suffices to show $\overline{G_n} = X \ \forall n \geq 1$. Fix $N \geq 1$. We will show that $G_N = \left\{x \in X : \omega(f,x) < \frac{1}{N}\right\}$ is dense in $X$. By a lemma from last lecture, it suffices to show

$$G_N \cap W \neq \emptyset \qquad \forall \emptyset \neq W = \mathring{W} \subseteq X$$

Fix $\emptyset \neq W = \mathring{W} \subseteq X$. For $n \geq 1$ and $x \in X$, we define

$$u_n(x) = \inf_{m\geq n} f_m(x) \quad \text{and} \quad v_n(x) = \sup_{m\geq n} f_m(x)$$

Then $\{u_n(x)\}_{n\geq 1}$ is increasing and $\{v_n(x)\}_{n\geq 1}$ is decreasing. As $\lim_{n\to\infty} f_n(x) = f(x)$, we have

$$\lim_{n\to\infty} u_n(x) = f(x) = \lim_{n\to\infty} v_n(x) \tag{2}$$

For $n \geq 1$, let

$$\begin{aligned} F_n &= \left\{x \in X : \ v_n(x) - u_n(x) \leq \frac{1}{4N}\right\} \\ &= \left\{x \in X : \ \sup_{m\geq n} f_m(x) - \inf_{l\geq n} f_l(x) < \frac{1}{4N}\right\} \\ &= \left\{x \in X : \ \sup_{m,l\geq n} [f_m(x) - f_l(x)] \leq \frac{1}{4N}\right\} \\ &= \bigcap_{m,l\geq n} \left\{x \in X : \ f_m(x) - f_l(x) \leq \frac{1}{4N}\right\} \\ &\overset{(1)}{=} \bigcap_{m,l\geq n} (f_m - f_l)^{-1}\left(\left[-2, \frac{1}{4N}\right]\right) \end{aligned}$$

$f_m - f_l$ is continuous $\forall m, l \geq n$ and $\left[-2, \frac{1}{4N}\right]$ is closed, so

$$(f_m - f_l)^{-1}\left(\left[-2, \frac{1}{4N}\right]\right) \text{ is closed} \qquad \forall m, l \geq n$$

So $F_n$ is closed in $X$ for all $n \geq 1$. Also,

$$X = \bigcup_{n\geq 1} F_n \qquad \text{by (2)}$$

So

$$\left.\begin{array}{l} W = \left(\bigcup_{n\geq 1} F_n\right) \cap W = \bigcup_{n\geq 1}(F_n \cap W) \\ W = \mathring{W} \neq \emptyset \\ W \text{ has the Baire property} \end{array}\right\} \implies \exists n_1 \in \mathbb{N} \text{ s.t. } \widehat{F_{n_1} \cap W}^{\circ} \neq \emptyset$$

Let $x_0 \in \widehat{F_{n_1} \cap W}^{\circ}$ and let $\delta > 0$ s.t. $B_\delta(x_0) \subseteq F_{n_1} \cap W$. As $f_{n_1}$ is continuous at $x_0$, shrinking $\delta$ if necessary, we may assume

$$\omega(f_{n_1}, B_\delta(x_0)) < \frac{1}{4N}$$

We compute

$$\begin{aligned}
\omega(f, x_0) \leq \omega(f, B_\delta(x_0)) &= \sup_{x \in B_\delta(x_0)} f(x) - \inf_{y \in B_\delta(x_0)} f(y) \\
&= \sup_{x,y \in B_\delta(x_0)} [f(x) - f(y)] \\
&\leq \sup_{x,y \in B_\delta(x_0)} [v_{n_1}(x) - u_{n_1}(y)] \\
&= \sup_{x,y \in B_\delta(x_0)} [v_{n_1}(x) - u_{n_1}(x) + v_{n_1}(y) - u_{n_1}(y) + u_{n_1}(x) - v_{n_1}(y)] \\
(B_\delta(x_0) \subseteq F_{n_1}) &\leq \frac{1}{4N} + \frac{1}{4N} + \sup_{x,y \in B_\delta(x_0)} [u_{n_1}(x) - v_{n_1}(y)] \\
&\leq \frac{1}{2N} + \sup_{x,y \in B_\delta(x_0)} [f_{n_1}(x) - f_{n_1}(y)] \\
&= \frac{1}{2N} + \omega(f_{n_1}; B_\delta(x_0)) \\
&\leq \frac{1}{2N} + \frac{1}{4N} < \frac{1}{N}
\end{aligned}$$

This proves $x_0 \in G_n \cap W \implies G_N \cap W \neq \emptyset$. As $\emptyset \neq W = \mathring{W} \subseteq X$ was arbitrary, we conclude $G_N$ is dense in $X$. □