Math 10 - Linear Algebra

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This is Math 10 - *Linear Algebra and Applications* at PCC. I took this class during Summer 2019 with Dr. Socrates. We use the book *A Portrait of Linear Algebra* (3^{rd} edition) by *Jude Socrates* (yes, we are really grateful to have the author of the book as our professor for the class). Please use this notes with great caution and let me know if you find anything mathematically wrong/concerning.

Contents

1 Sets, Axioms, Theorem & Proofs

Mathematics is a language, and logic is its grammar.

Set Theory and Basic Logic:

A *set* is an unordered collection of objects. There are two ways to describe a set:

1. Roster Method
2. Set Builder Notation

Example 1.1

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\} = \textit{ natural number}$$
$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} = \textit{ integers}$$
$$\mathbb{Q} = \left\{\frac{a}{b} | a, b \in \mathbb{Z}, \;\; b \neq 0\right\} = \textit{ rationals}$$
$$\mathbb{R} = \textit{ All real numbers}$$

Logical statement $\to$ fact $\to$ either true or false. *Axioms* is logical statements that we will accept as true without questions.

The Field Axioms for the Set of Real Numbers

1. The Closure Property (Add/Mult)

$$\forall x, y \in \mathbb{R} : x + y \in \mathbb{R} \text{ as well}$$
$$x \cdot y \in \mathbb{R} \text{ as well}$$

2. Commutative Properties

$$\forall x, y \in \mathbb{R} : x + y = y + x$$
$$\text{and } \; x \cdot y = y \cdot x$$

3. Associative Properties

$$\forall x, y, z \in \mathbb{R} : x + (y + z) = (x + y) + z$$
$$\text{and } \; x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

4. Distribution Property

$$\forall x, y, z \in \mathbb{R} : x \cdot (y + z) = (x \cdot y) + (x \cdot z)$$

5. Existence of Identity Elements

$$\exists 0 \in \mathbb{R} : \forall x \in \mathbb{R} : 0 + x = x = x + 0$$
$$\exists 1 \in \mathbb{R}, \;\; 0 \neq 1 : \forall x \in \mathbb{R} : 1 \cdot x = x = x \cdot 1$$

6. Existence of Inverse

$$\forall x \in \mathbb{R},\ \ \exists -x \in \mathbb{R} : x + (-x) = 0 = (-x) + x$$
$$\forall x \in \mathbb{R},\ \ x \neq 0 : \exists \frac{1}{x} \in \mathbb{R} : x\left(\frac{1}{x}\right) = 1 = \left(\frac{1}{x}\right)x$$

Note: We can define $x + (-y) = x - y$ and $x\left(\frac{1}{y}\right) = \frac{x}{y},\ \ x \neq 0$

Theorem: a true logical statement that requires proofs. It's usually in the form of if p then q or $p \implies q$.

- $q \implies p$: the converse of $p \implies q$
- $\neg p \implies \neg q$: the inverse of $p \implies q$
- $\neg q \implies \neg p$: the contrapositive of $p \implies q$

Logical Equivalence: $p \implies q$ and $q \implies p$ are both true then

$$p \iff q$$

$p \implies q$ is logically equivalent to its contrapositive:

$$p \implies q \iff \neg q \implies \neg p$$

Note

$\overline{p}$ and $\overline{q}$ also denotes negation

THEOREM 1.1 De Morgan's Laws

- Not (p and q) is logically equivalent to (not p) or (not q)
- Not (p or q) is logically equivalent to (not p) and (not q)

Proofs:

Definition 1.1 *A proof for a theorem is a sequence of true logical statements which convincingly and completely explains why a theorem is true.*

Tips to write proofs: Identify what's given (hypothesis) and what you want to show (the conclusion)

Example 1.2

Given: $a \in \mathbb{R}$

Show: $0 \cdot a = 0 = a \cdot 0$

Proof.

- By closure: $0 \cdot a \in \mathbb{R}$
- By commutativity :$0 \cdot a = a \cdot 0$
- By Add. Identity Prop, with $x = 0 : 0 + 0 = 0$
- By substitution : $(0 + 0) \cdot a = 0 \cdot a$

- By the Dist. Prop: $(0 \cdot a) + (0 \cdot a) = 0 \cdot a$

Since $0 \cdot a \in \mathbb{R}$, it has an additive inverse which is $-(0 \cdot a)$

- By substitution: $-(0 \cdot a) + (0 \cdot a + 0 \cdot a) = -(0 \cdot a) + (0 \cdot a)$
- By associativity: $(-(0 \cdot a) + 0 \cdot a) + 0 \cdot a = -(0 \cdot a) + (0 \cdot a)$
- By Inv. Prop: $0 + 0 \cdot a = 0$
- By identity: $0 \cdot a = 0$

■

Example 1.3

$$\forall a, b \in \mathbb{R} : a \cdot b = 0 \iff a = 0 \;\; or \;\; b = 0$$

Proof. ($\implies$)

$$\text{Given:} \;\; a \cdot b = 0$$

$$\text{Show:} \;\; a = 0 \;\; \text{or} \;\; b = 0$$

Case 1 $a = 0$

Since $a = 0$ or $b = 0$, this conclusion is true

Case 2 $a \neq 0$. We know: $a \cdot b = 0$

- By Mult. Inv Prop: $\exists \frac{1}{a} \in \mathbb{R}$
- By subst: $\frac{1}{a}(a \cdot b) = \frac{1}{a} \cdot 0$
- By associativity: $\left(\frac{1}{a} \cdot a\right) \cdot b = \frac{1}{a} \cdot 0$
- By Mult. Prop of 0 : $\left(\frac{1}{a} \cdot a\right) \cdot b = 0$
- By Mult Inv Prop: $1 \cdot b = 0$
- By identity Prop: $b = 0$

($\impliedby$)

$$\text{Given:} \;\; a = 0 \;\; \text{or} \;\; b = 0$$

$$\text{Show:} \;\; a \cdot b = 0$$

Case 1 (a = 0) We get $a \cdot b = 0 \cdot b = 0$ by the Mult.Prop of 0

Case 2 (b = 0) We get $a \cdot b = a \cdot 0 = 0$ by the same reasoning ■

Example 1.4 *Prove by using contrapositive*

$$\forall a, b \in \mathbb{Z} : \;\; \textit{if } a \cdot b \textit{ is even, then either } a \textit{ is even or } b \textit{ is even}$$

The contrapositive of the above statement would be if a is odd and b is odd, then $a \cdot b$ *is odd*

Proof.

$$a = 2c + 1 \quad \text{and}$$
$$b = 2d + 1 \quad \text{where} \quad c, d \in \mathbb{Z}$$

$$\begin{aligned} a \cdot b &= (2c+1)(2d+1) \\ &= 4cd + 2c + 2d + 1 \\ &= 2(2cd + c + d) + 1 \end{aligned}$$

By closure, $c \cdot d$ and $2cd$ are integers, so $2cd + c + d \in \mathbb{Z}$. Thus, ab is odd ■

Corollary *If $a \in \mathbb{Z}$ and if a^2 is even, then a is even*

Example 1.5 *Prove $\sqrt{2}$ is irrational*

$$\textit{Given:} \quad \sqrt{2} \in \mathbb{R}$$
$$\textit{Show:} \quad \sqrt{2} \quad \textit{is irrational}$$

Assume $\sqrt{2}$ is rational, so $\sqrt{2} = \frac{a}{b}$, where $a, b \in \mathbb{Z}, b \neq 0$, a and b have no common factor except ± 1.

- *By substitution:* $2 = \frac{a^2}{b^2}$
- *By substitution:* $a^2 = 2b^2$

Thus, a^2 is even, and by corollary above, a is even. So $a = 2c, c \in \mathbb{Z}$

$$2b^2 = a^2 = (2c)^2 = 4c^2$$
$$b^2 = 2c^2$$

b^2 is even, so again b is even. So a & b have a common factor of 2 which contradicts our initial assumption. Thus, $\sqrt{2}$ must be irrational

2 Euclidean Spaces and Subspaces

2.1 Euclidean Spaces

Example 2.1

$$\vec{v} =< 7, -2, \pi, 0, 4 >\in \mathbb{R}^5$$

Definition 2.1 (Vector) *An ordered n-tuple or vector is an ordered list of n real numbers*

$$\vec{v} =< v_1, v_2, \ldots, v_n >$$

Definition 2.2 ($\mathbb{R}^n$) *The set of all possible n-vectors is called Euclidean n-space, denoted by the symbol* $\mathbb{R}^n$

$$\mathbb{R}^n = \{\vec{v} =< v_1, v_2, \ldots, v_n | v_1, v_2, \ldots, v_n \in \mathbb{R}\}$$

Definition 2.3 (Zero Vector) *Each $\mathbb{R}^n$ has a special element called the zero vector, all of whose components are 0.*

$$\vec{0}_n =< 0, 0, \dots, 0 >$$

Example 2.2

$$\vec{0}_7 =< 0, 0, 0, 0, 0, 0, 0 >$$

Definition 2.4 (Vector Arithmetic) *If $\vec{u} =< u_1, u_2, \dots, u_n >$ and $\vec{v} =< v_1, v_2, \dots, v_n >$ are vectors in $\mathbb{R}^n$, we define the vector sum*

$$\vec{u} + \vec{v} =< u_1 + v_1, u_2 + v_2, \dots, u_n + v_n >$$

and if $r \in \mathbb{R}$, we define the scalar product

$$r \cdot \vec{v} =< rv_1, rv_2, \dots, rv_n >$$

THEOREM 2.1 **The Multiplicative Property of the Scalar 0**

Given that $\vec{v} \in \mathbb{R}^n$ and $\vec{v} =< v_1, v_2, \dots, v_n >$, then

$$0 \cdot \vec{v} = \vec{0}_n$$

Proof. We have:

$$\begin{aligned} 0 \cdot \vec{v} &=< 0 \cdot v_1, 0 \cdot v_2, \dots, 0 \cdot v_n > \quad \text{(Def of scalar mult)} \\ &=< 0, 0, \dots, 0 > \quad \text{(by Mult.Prop of 0)} \\ &= \vec{0}_n \end{aligned}$$

■

Translating Vectors in $\mathbb{R}^2$:

THEOREM 2.2

Let $\vec{u} =< u_1, u_2 >\in \mathbb{R}^2$, and $P(a_1, b_1)$ is a point on the Cartesian plane. If $\vec{u}$ is translated to P, then head of $\vec{u}$ will be located at $Q(a_2, b_2)$ where

$$a_2 = a_1 + u_1, \text{ and } b_2 = b_1 + u_2$$

Conversely, if $P(a_1, b_1)$ and $Q(a_2, b_2)$ are two points on the Cartesian plane, then the vector $\vec{v} \in \mathbb{R}^2$ from P to Q is:

$$\vec{u} = \overrightarrow{PQ} =< a_2 - a_1, b_2 - b_1 >$$

Axioms for Parallel Vectors:

We say that 2 vectors $\vec{u}$ and $\vec{v} \in \mathbb{R}^n$ are parallel to each other if there exists either $a \in \mathbb{R}$ or $b \in \mathbb{R} \ni$:

$$\vec{u} = a \cdot \vec{v} \text{ or } \vec{v} = b \cdot \vec{u}$$

Consequently, this means that $\vec{0}_n$ is parallel to all vectors $\vec{v} \in \mathbb{R}^n$, since $\vec{0}_n = 0 \cdot \vec{v}$

Example 2.3 *Prove:*

$$\forall \vec{u}, \vec{v} \in \mathbb{R}^n, \quad \textit{and} \quad r \in \mathbb{R}:$$
$$r\left(\vec{u}+\vec{v}\right) = r\vec{u} + r\vec{v}$$

Proof. Let $\vec{u} =< u_1, u_2, \ldots, u_n >$ and $\vec{v} =< v_1, v_2, \ldots, v_n >$

$$\begin{aligned} r\left(\vec{u}+\vec{v}\right) &=< r(u_1+v_1), r(u_2+v_2), \ldots, r(u_n+v_n) > \\ &=< ru_1 + rv_1, ru_2 + rv_2, \ldots, ru_n + rv_n > \quad \text{(By Distributive Property)} \end{aligned}$$

Now, RHS: $r\vec{u} + r\vec{v} =< ru_1 + rv_1, ru_2 + rv_2, \ldots, ru_n + rv_n >$ ∎

The Length of a Vector:

Definition 2.5 *Let $\vec{v} =< v_1, v_2 >\in \mathbb{R}^2$ and $\vec{w} =< w_1, w_2, w_3 >\in \mathbb{R}^3$. We define the length/norm/magnitude of them as:*

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2} \quad \textit{and} \quad \|\vec{w}\| = \sqrt{w_1^2 + w_2^2 + w_3^2}$$

We say that $\vec{v}$ is a unit vector if $\|\vec{v}\| = 1$ and similarly for $\|\vec{w}\| = 1$

THEOREM 2.3

For any scalar $k \in \mathbb{R}$ and vector $\vec{v} \in \mathbb{R}^2$ or $\mathbb{R}^3$:

$$\|k\vec{v}\| = |k|\|\vec{v}\|$$

Furthermore, $\|\vec{v}\| \geq 0$ and $\|\vec{v}\| = 0$ iff $\vec{v} = \vec{0}_2$ or $\vec{0}_3$. Consequently, if $\vec{v}$ is a non-zero vector, then

$$\vec{u}_1 = \frac{1}{\|\vec{v}\|} \cdot \vec{v} \quad \text{and} \quad \vec{u}_2 = -\frac{1}{\|\vec{v}\|} \cdot \vec{v}$$

are units vectors parallel to $\vec{v}$

2.2 The Span of a Set of Vectors

Definition 2.6 *Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a set of vectors form some $\mathbb{R}^n$. We define* Span(S) *as all linear combination of the vector in S*

$$\textit{Span(S)} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k | c_1, c_2, \ldots, c_k \in \mathbb{R}\}$$

Example 2.4

$$S = \left\{\vec{0}_n\right\}$$
$$\textit{Span(S)} = \left\{c \cdot \vec{0}_n | c \in \mathbb{R}\right\} = \left\{\vec{0}_n\right\}$$

Example 2.5

$$S = \{\vec{e}_1, \dots, \vec{e}_n\} \subset \mathbb{R}^n$$
$$< v_1, \dots, v_n >= v_1\vec{e}_1 + v_2\vec{e}_2 + \dots + v_n\vec{e}_n$$
$$\textit{Span}\left(\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}\right) = \mathbb{R}^n$$

Example 2.6

$$S = \{< 3, -2 >\} \subset \mathbb{R}^2$$
$$\textit{Span(S)} = \{c\cdot < 3, -2 > | c \in \mathbb{R}\}$$

c	$c < 3, -2 >$
-2	$< -6, 4 >$
0	$< 0, 0 >$
1	$< 3, -2 >$
$\frac{5}{2}$	$< \frac{15}{2}, -5 >$

Example 2.7

$$S = \{< 3, -2 >, < -9, 6 >\}$$
$$\textit{Span(S)} = \{c < 3, -2 > + d < -9, 6 >\}, \quad c, d \in \mathbb{R}$$

c	d	$c < 3, -2 > + d < -9, 6 >$
1	0	$< 3, -2 >$
0	1	$< -9, 6 >$
0	0	$< 0, 0 >$
a	b	$k < 3, -2 >$

Example 2.8

$$S = \{< 3, -2 >, < -5, 6 >\}$$
$$\textit{Span(S)} = \{c < 3, -2 > + d < -5, 6 > | c, d \in \mathbb{R}\}$$

c	d	Span(S)
1	0	$< 3, -2 >$
0	1	$< -5, 6 >$
0	0	$< 0, 0 >$
2	-1	$< 11, -10 >$

Guess: we can make <u>any</u> vector we want in $\mathbb{R}^2$

$$< x, y >= c < 3, -2 > + d < -5, 6 >$$

Given x, y, we can always solve for c and d

$$\begin{cases} x = 3c - 5d & \implies \quad d = \frac{2x+3y}{8} \\ y = -2c + 6d & \implies \quad c = \frac{6x+5y}{8} \end{cases}$$

THEOREM 2.4

If $\vec{u}, \vec{v} \in \mathbb{R}^2$ are non-parallel vectors, then:

$$\text{Span}\left(\{\vec{u}, \vec{v}\}\right) = \mathbb{R}^2$$

In other words, any vectors $\vec{w} \in \mathbb{R}^2$ can be expressed as a linear combination:

$$\vec{w} = r\vec{u} + s\vec{v}$$

for some scalar r and s.

Example 2.9

$$S = \{< 3, -2, 4 >, < 5, 1, -6 >\}$$

c	d	Span(S)
0	0	$< 0, 0, 0 >$
1	0	$< 3, -2, 4 >$
0	1	$< 5, 1, -6 >$
4	3	$< 27, -5, -2 >$

Goal: describe all vectors $< x, y, z >$ *in span S.*

$$< x, y, z >= c < 3, -2, 4 > +d < 5, 1, -6 >$$

1. If there is a soln, is there more than one?

$$c < 3, -2, 4 > +d < 5, 1, -6 >= c' < 3, -2, 4 > +d' < 5, 1, -6 >$$

$$< 3, -2, 4 >= \frac{d' - d}{c - c'} < 5, 1, -6 >$$

which is not possible since the vectors in S are not parallel

2. So if a soln to c and d exists, it must be unique.

$$\begin{cases} x = 3c + 5d \\ y = -2c + d & \implies \quad c = \frac{6y+z}{-8} = \frac{x-5y}{13} \\ z = 4c - 6d \end{cases}$$

Since c is unique, $\frac{x-5y}{13} = \frac{6y+z}{-8}$

$$8x + 38y + 13z = 0$$

which is the Cartesian equation for the plane span S.

Translation of a span:

$$Q = \{\vec{q} + \vec{v} | \vec{v} \in \text{Span(S)}\}$$

for some fixed non-zero vector $\vec{q} \in \mathbb{R}^n$

Example 2.10 *Find an eqn for the line passing through* $P(4, -2, 3)$ *and* $Q(7, 1, -5)$

$$\overrightarrow{PQ} =< 3, 3, -8 > \quad \textit{(direction vector for L)}$$
$$\vec{q} = \overrightarrow{OP} =< 4, -2, 3 >$$

So,

$$< x, y, z >=< 4, -2, 3 > +t < 3, 3, -8 > , \quad t \in \mathbb{R}$$

Example 2.11 *Find a Cartesian eqn for the plane through P,Q(from last example) and* $R(1, 0, -2)$.

$$\overrightarrow{PQ} =< 3, 3, -8 >$$
$$\overrightarrow{PR} =< -3, 2, -5 >$$

This confirms that PQR forms a triangle, not a straight line. We then can express the vector equation of the plane as

$$< x, y, z >=< 4, -2, 3 > +c < 3, 3, -8 > +d < -3, 2, -5 >$$

1. *If a soln for c and d exists, again it must be unique*
2. $$\begin{cases} x = 4 + 3c - 3d \\ y = -2 + 3c + 2d \\ z = 3 - 8c - 5d \end{cases}$$

After some algebra, we obtain:

$$x + 39y + 15z = -29$$

Different way to get the above equation:

$$\begin{cases} x = 4 + 3t \\ y = -2 + 3t \\ z = 3 - 8t \end{cases} \implies \quad t = \tfrac{x-4}{3} = \tfrac{y+2}{3} = \tfrac{z-3}{-8}$$

Definition 2.7 *A line L in Cartesian space passing through the point* (x_0, y_0, z_0) *and with non-zero direction vector* $\vec{d} =< a, b, c >$ *can be specified using a vector equation in the form:*

$$< x, y, z >=< x_0, y_0, z_0 > +t < a, b, c > , \quad t \in \mathbb{R}$$

If none of the components of d are 0, we can obtain symmetric equations for L, of the form:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

2.3 The Dot Product and Orthogonality

Definition 2.8 *If* $\vec{u} =< u_1, u_2, \ldots, u_n >$ *and* $\vec{v} =< v_1, v_2, \ldots, v_n >$ *are vectors from* $\mathbb{R}^n$*, we define their product:*

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

Example 2.12 *If* $\vec{u} =< 4, -3, -6, 5, -2 >$ *and* $\vec{v} =< 3, -5, 4, -7, -1 >$ *then:*

$$\vec{u} \cdot \vec{v} = -30$$

Length of a Vector:

Definition 2.9 *We define the length of a vector in* $\mathbb{R}^n$ *. . . (1.1). It follows directly from the definition of the dot product that*

$$\|\vec{v}\|^2 = \vec{v} \cdot \vec{v} \ \ or \ \ \|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$$

and

$$\|k\vec{v}\| = |k| \|\vec{v}\|$$
$$\|\vec{v}\| = 0 \ \ iff \ \ \vec{v} = \vec{0}_n$$

Example 2.13 $\|\vec{u}\| = 3$ *,* $\|\vec{v}\| = 7$ *and* $\vec{u} \cdot \vec{v} = 16$*. Find* $\|7\vec{u} - 2\vec{v}\|$

$$\begin{aligned}
\|7\vec{u} - 2\vec{v}\|^2 &= (7\vec{u} - 2\vec{v}) \cdot (7\vec{u} - 2\vec{v}) \\
&= 7\vec{u} \cdot (7\vec{u} - 2\vec{v}) - 2\vec{v} \cdot (7\vec{u} - 2\vec{v}) \\
&= 49\vec{u} \cdot \vec{u} - 28\vec{u} \cdot \vec{v} + 4\vec{v} \cdot \vec{v} \\
&= 49(9) - 28(16) + 4(49) \\
&= 189
\end{aligned}$$

The Law of Cosines:

$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos\theta$$
$$(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos\theta$$
$$\|\vec{u}\|^2 - 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos\theta$$
$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

The Cauchy - Schwartz Inequality

Proof. Case 2 : Neither $\vec{u}$ nor $\vec{v}$ is $\vec{0}_n \implies \|\vec{u}\| > 0$ and $\|\vec{v}\| > 0$

$$\text{Create:} \ \ \vec{w} = a\vec{u} + b\vec{v}, \ \ \text{where} \ \ a, b \in \mathbb{R}$$

Since a,b could be 0, $\vec{w}$ could be $\vec{0}_n$, so $\|\vec{w}\| \geq 0$ and thus $\|\vec{w}\|^2 \geq 0$. Therefore,

$$\begin{aligned} 0 \leq \vec{w} \cdot \vec{w} &= (a\vec{u} + b\vec{v}) \cdot (a\vec{u} + b\vec{v}) \\ &= a^2\|\vec{u}\|^2 + 2ab\vec{u} \cdot \vec{v} + b^2\|\vec{v}\|^2 \end{aligned}$$

Let $a = \|\vec{v}\| \neq 0$

$$0 \leq \|\vec{v}\|^4\|\vec{u}\|^2 + 2\|\vec{v}\|^2 b\vec{u} \cdot \vec{v} + b^2\|\vec{v}\|^2$$
$$0 \leq \|\vec{v}\|^2\|\vec{u}\|^2 + 2b\vec{u} \cdot \vec{v} + b^2$$

If we let $b = -\vec{u} \cdot \vec{v}$, we get:

$$0 \leq \|\vec{v}\|^2\|\vec{u}\|^2 + (-2)(\vec{u} \cdot \vec{v})(\vec{u} \cdot \vec{v}) + (-(\vec{u} \cdot \vec{v}))^2$$
$$\to (\vec{u} \cdot \vec{v})^2 \leq \|\vec{u}\|^2\|\vec{v}\|^2 \qquad \blacksquare$$

2.4 System of Linear Equation

Example 2.14 *Decide if $\vec{b} =< 10, -9, -5, -7 >$ is a member of the span of the following five vectors from $\mathbb{R}^4$*

$$c_1 < 3, -4, 1, -6 >, \quad c_2 < 2, -3, 2, -5 >, \quad c_3 < 1, 1, -9, 5 >,$$
$$c_4 < 1, -2, 2, -4 >, \quad c_5 < 9, -7, -8, -3 >$$

If so, express $\vec{b}$ as a linear combination of these 5 vectors in simplest way possible.

$$3c_1 + 2c_2 + c_3 + c_4 + 9c_5 = 10$$
$$-4c_1 - 3c_2 + c_3 - 2c_4 - 7c_5 = -9$$
$$c_1 + 2c_2 - 9c_3 + 2c_4 - 8c_5 = -5$$
$$-6c_1 - 5c_2 + 5c_3 - 4c_4 - 3c_5 = -7$$

This is a system of 4 linear equation in 5 unknowns. Matrix form:

$$\left[\begin{array}{ccccc|c} 3 & 2 & 1 & 1 & 9 & 10 \\ -4 & -3 & 1 & -2 & -7 & -9 \\ 1 & 2 & -9 & 2 & -8 & -5 \\ -6 & -5 & 5 & -4 & -3 & -7 \end{array}\right]$$

4 rows, 6 columns, 4×6 augmented matrix Now, let's us apply some row operations on this matrix and see how we can turn it into rref form.

Change the current display of row operations to a more friendly state (left-side)

$$R_1 \leftrightarrow R_3:$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & -9 & 2 & -8 & -5 \\ -4 & -3 & 1 & -2 & -7 & -9 \\ 3 & 2 & 1 & 1 & 9 & 10 \\ -6 & -5 & 5 & -4 & -3 & -7 \end{array}\right]$$

$$R_2 \to R_2 + 4R_1, \quad R_3 \to R_3 - 3R_1, \quad R_4 \to R_4 + 6R_1:$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & -9 & 2 & -8 & -5 \\ 0 & 5 & -35 & 6 & -39 & -29 \\ 0 & -4 & 28 & 5 & 33 & 25 \\ 0 & 7 & -49 & 8 & -51 & -37 \end{array}\right]$$

$$R_2 \to R_2 + R_3, \quad R_3 \to R_3 + 4R_2, \quad R_4 \to R_4 - 7R_2:$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & -9 & 2 & -8 & -5 \\ 0 & 1 & -7 & 1 & -6 & -4 \\ 0 & 0 & 0 & -1 & 9 & 9 \\ 0 & 0 & 0 & 1 & -9 & -9 \end{array}\right]$$

$$R_3 \leftrightarrow R_4, \quad R_4 \to R_4 + R_3:$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & -9 & 2 & -8 & -5 \\ 0 & 1 & -7 & 1 & -6 & -4 \\ 0 & 0 & 0 & 1 & -9 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 \to R_1 - 2R_3, \quad R_2 \to R_2 - R_3:$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & -9 & 0 & 10 & 13 \\ 0 & 1 & -7 & 0 & 3 & 5 \\ 0 & 0 & 0 & 1 & -9 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 \to R_1 - 2R_2:$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 5 & 0 & 4 & 3 \\ 0 & 1 & -7 & 0 & 3 & 5 \\ 0 & 0 & 0 & 1 & -9 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Leading : c_1, c_2, c_4 *, free:* c_3, c_5*. Let* $c_3 = a, \quad c_5 = b$

$$\vec{x} = < 3 - 5a - 4b, 5 + 7a - 3b, a, -a + 9b, b >$$

Simplest form: $\vec{x} = < 3, 5, 0, -9, 0 >$

Definition 2.10 (The Identity Matrices) *The* $n \times n$ *identity matrix, denoted* I_n*, is the matrix which* *contains $\vec{e}_1$ in column 1, $\vec{e}_2$ in column 2, ..., $\vec{e}_n$ in column n:*

$$I_n = [\vec{e}_1 \quad \vec{e}_2 \quad \dots \vec{e}_n] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

2.5 Linear System and Linear Independence

Definition 2.11 *A linear system is called consistent if it has at least one solution and vice versa.*

THEOREM 2.5

Let $\vec{b} \in \mathbb{R}^m$ and let $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be a set of vectors from $\mathbb{R}^m$. Then $\vec{b} \in$ Span(s) iff the system of eqn corresponding to the augmented matrix

$$A = \left[\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n | \vec{b}\right]$$

is consistent

Definition 2.12 (Homogeneous System) *A homogeneous system of m eqns in n unknowns is a system of linear eqn where the right side of eqn consists entirely of 0. In other words, the augmented matrix has the form*

$$\left[A|\vec{0}_m\right]$$

where A is an $m \times n$ matrix. If the right side $\vec{b}$ is not $\vec{0}_m$, we call it non-homogeneous.

Let's solve the following homogeneous system (rref):

$$\textit{Underdetermined} \left.\begin{array}{ccccc|c} 1 & 0 & 5 & 0 & 4 & 0 \\ 0 & 1 & -7 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right.$$

Soln: leading: x_1, x_2, x_4 and free: x_3, x_5

$$\vec{x} =< -5x_3 - 4x_5, 7x_3 - 3x_5, x_3, 9x_5, x_5 >$$

$$\text{Simplest Soln:} \quad \vec{x} =< 0, 0, 0, 0, 0 >$$

Go deeper:

$$\begin{aligned} \vec{x} &=< -5x_3, 7x_3, x_3, 0, 0 > + < -4x_5, -3x_5, 0, 9x_5, x_5 > \\ &= x_3 < -5, 7, 1, 0, 0 > + x_5 < -4, -3, 0, 9, 1 > \end{aligned}$$

$\implies$ A linear comb of 2 vectors wit coeff x_3, x_5 (the free vars!)

THEOREM 2.6

An underdetermined homogeneous system always has an infinite number of solns. In other words, homogeneous system with more variables then eqns has an infinite number of solns

Example 2.15

$$\begin{bmatrix} 7 & -1 & -2 & 6 \\ -2 & 5 & 3 & -4 \\ 8 & 3 & -5 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 3 \\ 5 \end{bmatrix}$$

$$\begin{aligned} A : 3 \times 4 \,, \vec{x} : 4 \times 1 \implies \quad & A\vec{x} : 3 \times 1 \\ & = 4 \begin{bmatrix} 7 \\ -2 \\ 8 \end{bmatrix} - 2 \begin{bmatrix} -1 \\ 5 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} + 5 \begin{bmatrix} 6 \\ -4 \\ 1 \end{bmatrix} \\ & = \begin{bmatrix} 54 \\ -29 \\ 16 \end{bmatrix} \end{aligned}$$

THEOREM 2.7

Properties of Matrix Multiplication

$\forall m \times n$ matrices A, $\forall \vec{x}, \vec{y} \in \mathbb{R}^n$, and $\forall k \in \mathbb{R}$, matrix multiplication enjoys:

- The additivity Prop: $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$
- The homogeneity Prop: $A(k\vec{x}) = k(A\vec{x})$

Proof. $\forall \vec{x}, \vec{y} \in \mathbb{R}^n$, $A : m \times n$,

$$A = [\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n] \,, \quad A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$$

$$\begin{aligned} \vec{x} &= < x_1, \ldots, x_n > \\ \vec{y} &= < y_1, \ldots, y_n > \\ \vec{x} + \vec{y} &= < x_1 + y_1, \ldots, x_n + y_n > \\ A(\vec{x} + \vec{y}) &= [\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n] \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_n + y_n \end{bmatrix} \\ &= (x_1 + y_1)\vec{c}_1 + \ldots + (x_n + y_n)\vec{c}_n \end{aligned}$$

On the other hand,

$$\begin{aligned} A\vec{x} &= x_1\vec{c}_1 + x_2\vec{c}_2 + \ldots + x_n\vec{c}_n \\ A\vec{y} &= y_1\vec{c}_1 + y_2\vec{c}_2 + \ldots + y_n\vec{c}_n \\ A\vec{x} + A\vec{y} &= (x_1 + y_1)\vec{c}_1 + (x_2 + y_2)\vec{c}_2 + \ldots + (x_n + y_n)\vec{c}_n \end{aligned}$$

■

The Matrix Product Form of Linear System:

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n = \vec{b}$$

We formed the augmented matrix $\left[\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n|\vec{b}\right]$ and looked at its rref. Another alternative way:

$$\begin{bmatrix}\vec{v}_1 & \vec{v}_2 \ldots \vec{v}_n\end{bmatrix}\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix} = \vec{b}_n$$

Matrix Equation : $A\vec{x} = \vec{b}$

THEOREM 2.8

Suppose that $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a set of vectors from $\mathbb{R}^m$, and $\vec{b} \in \mathbb{R}^m$. Let's form the $m \times n$ matrix:

$$A = \begin{bmatrix}\vec{v}_1 & \vec{v}_2 \ldots \vec{v}_n\end{bmatrix}$$

Then, $\vec{b} \in$ Span(S) iff the matrix eqn

$$A\vec{x} = \vec{b}$$

is consistent

Linear Dependence and Independence:

Example 2.16

$$\begin{aligned}S &= \{< 1, 0, 0, 0 >, < 0, 1, 0, 0 >, < 0, 0, 0, 1 >, < 0, 0, 0, 1 >\}\\&= \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\} \subseteq \mathbb{R}^4\end{aligned}$$

Test eqn:

$$x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3 + x_4\vec{e}_4 = \vec{0}_4$$
$$< x_1, x_2, x_3, x_4 >=< 0, 0, 0, 0 >$$

Only trivial soln! So S is independent. Follow up:

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

which is already in rref and there's no free vars and thus is independent. On the contrary, a matrix in its rref like this

$$\begin{bmatrix}1&0&5&0&4\\0&1&-7&0&3\\0&0&0&1&-9\\0&0&0&0&0\end{bmatrix}$$

is certainly dependent because of the two free vars. Observe that:

$$\vec{v}_3 = 5\vec{v}_1 - 7\vec{v}_2$$
$$\vec{v}_5 = 4\vec{v}_1 + 3\vec{v}_2 - 9\vec{v}_4$$

Example 2.17

$$S = \{\vec{v}\} \subseteq \mathbb{R}^m$$

Test eqn: $x\vec{v} = \vec{0}_m$

Case 1 x has to be 0 $\implies$ S is independent and $\vec{v}$ can be a nonzero vector.

Case 2 $\vec{v} = \vec{0}_m$, $S = \left\{\vec{0}_m\right\}$ is dependent.

Consider:

$$S = \{\vec{u}, \vec{v}\} \subseteq \mathbb{R}^m$$

Test eqn: $x_1\vec{u} + x_2\vec{v} = \vec{0}_m$. Non-trivial soln?

$$x_1 \neq 0 \quad \implies \quad \vec{u} = \frac{-x_2}{x_1}\vec{v}, \;\; \vec{u} \parallel \vec{v}$$

$$x_2 \neq 0 \quad \implies \quad \vec{v} = \frac{-x_1}{x_2}\vec{u}$$

So, $\vec{u} - k\vec{v} = \vec{0}_m$ or $\vec{v} - k\vec{u} = \vec{0}_m$
Consider another set of vectors:

$$S = \{\vec{u}, \vec{v}, \vec{w}\}$$

Test: $x_1\vec{u} + x_2\vec{v} + x_3\vec{w} = \vec{0}_m$

$$x_1 \neq 0 : \;\; \vec{u} = \frac{-x_2}{x_1}\vec{v} - \frac{x_3}{x_1}\vec{w}$$

If there is a non-trivial soln, then one vector is a linear combination of the other 2. Conversely, if one vector is a linear combination of the other 2, say: $\vec{v} = a\vec{u} + b\vec{w}$, then there is a non-trivial soln:

$$a\vec{u} - \vec{v} + b\vec{w} = \vec{0}_m$$

2.6 Independent Sets versus Spanning Sets

THEOREM 2.9

Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a linearly independent set of vectors from $\mathbb{R}^m$, and suppose $\vec{v}_{n+1}$ is not a member of Span(S). Then, the extended set:

$$S' = S \cup \{\vec{v}_{n+1}\} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n, \vec{v}_{n+1}\}$$

is still linearly independent.

Proof.

$$\text{Given: } \{\vec{v}_1, \ldots, \vec{v}_n\} \text{ is indep and } \vec{v}_{n+1} \notin \text{Span(S)}$$
$$\text{Show: } S' = \{\vec{v}_1, \ldots, \vec{v}_n, \vec{v}_{n+1}\} \text{ is still independent}$$

Test eqn for S':

$$c_1\vec{v}_1 + \ldots + c_n\vec{v}_n + c_{n+1}\vec{v}_{n+1} = \vec{0}_m$$

Show all $c_i = 0$

Case 1 ($c_{n+1} = 0$)

$$c_1\vec{v}_1 + \ldots + c_n\vec{v}_n + \vec{0}_m = \vec{0}_m$$

Since S is indep, all $c_1 \ldots c_n = 0$

Case 2 ($c_{n+1} \neq 0$)

$$\vec{v}_{n+1} = \frac{-c_1\vec{v}_1}{c_{n+1}} - \ldots - \frac{-c_n\vec{v}_n}{c_{n+1}}$$

This eqn says $\vec{v}_{n+1} \in$ Span(S), which contradicts the given. So $c_{n+1} \neq 0$ is impossible, and only case 1 is possible. Therefore, S' is independent. ∎

2.7 Subspaces of Euclidean Spaces; Basis and Dimension

Definition 2.13 *A subspace W of $\mathbb{R}^n$ is a non-empty subset of vectors of $\mathbb{R}^n$ such that if $\vec{u}$, $\vec{v} \in W$, and $r \in \mathbb{R}$, then:*

$$\vec{u} + \vec{v} \in W, \quad r \cdot \vec{w} \in W$$

We say W is under vector addition and scalar multiplication.

$$W \trianglelefteq \mathbb{R}^n$$

to indicate that W is a subspace of $\mathbb{R}^n$. We call $\mathbb{R}^n$ the ambient space of W.

THEOREM 2.10 If $S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ is a non-empty set of vectors from $\mathbb{R}^n$, then W = Span(S) is a subspace of $\mathbb{R}^n$

Proof. Recall:

$$\text{Span}(\vec{u}_1, \ldots, \vec{u}_k) = \{x_1\vec{u}_1 + \ldots + x_k\vec{u}_k | x_1, \ldots, x_k \in \mathbb{R}\}$$

W is closed under addition, so

$$\vec{a} = x_1\vec{u}_1 + \ldots + x_k\vec{u}_k$$
$$\vec{b} = y_1\vec{u}_1 + \ldots + y_k\vec{u}_k$$
$$\vec{a} + \vec{b} = (x_1 + y_1)\vec{u}_1 + \ldots + (x_k + y_k)\vec{u}_k \quad \in W$$

Closed under scalar multiplication:

$$r \cdot \vec{a} = (rx_1)\vec{u}_1 + \ldots + (rx_k)\vec{u}_k \quad \in W$$ ∎

Basis for a Subspace:

Definition 2.14 *A basis for a non-zero subspace* $W \trianglelefteq \mathbb{R}^n$ *is a non-empty set of vectors* $B = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ *which spans W and is also linearly independent.*

Example 2.18

$$y = -\frac{3}{5}x \trianglelefteq \mathbb{R}^2$$

$B = \{< 5, -3 >\}$ *or* $B = \{< 10, -6 >\}$

Example 2.19

$$4x - 6y + 7z = 0$$

$B = \{< 3, 2, 0 >, < 7, 0, -4 >\}$

Note Basis: Spans W and Independent.

THEOREM 2.11 **Existence of a Basis Theorem**

If W is any non-zero subspace of $\mathbb{R}^n$, then there exists a basis $B = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ for W. In other words,

$$W = \text{Span}(B) = \text{Span}\left(\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}\right)$$

where B is a linearly independent set that spans W. Furthermore, we must have $k \leq n$.

Proof. We use Extension Theorem to make a basis for W.

1. W is not the zero subspace, let's pick any $\vec{w}_1 \in W, \vec{w}_1 \neq \vec{0}_n$. Let $B_1 = \{\vec{w}_1\}$. Since $\vec{w}_1 \neq \vec{0}_n$, B_1 is indep. Span(B_1) = W? Yes, B is a basis for W. Done! ✓(No: we need another vector)
2. Pick $\vec{w}_2 \notin$ Span(B_1). By Ext:

   $$B_2 = \{\vec{w}_1, \vec{w}_2\} \quad \text{is indep}$$

   Test: Span(B_2) = W? Yes, B_2 is a basis. Done! ✓
3. If no, pick $\vec{w}_3 \notin$ Span(B_2) . . . ■

THEOREM 2.12 **The Independent Sets from Spaning Sets Theorem**

Suppose we have a set of n vectors $S = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ from some Euclidean space $\mathbb{R}^k$, and we form Span(S). Suppose now we randomly choose a set of m vectors from Span(S) to form a new set:

$$L = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m\}$$

We can now conclude that if L is independent, then $m \leq n$

2.8 The Fundamental Matrix Spaces

THEOREM 2.13 **The Four Fundamental Matrix Spaces (Definition)**

Let A be an $m \times n$ matrix. The rowspace of A is the Span of the rows of A.

- The columnspace of A is the Span of the columns of A.
- The nullspace of A is the set of all solution to $A\vec{x} = \vec{0}_m$

$$\text{rowspace(A)} = \text{Span}\left(\{\vec{r}_1, \vec{r}_2, \dots, \vec{r}_m\}\right)$$
$$\text{colspace(A)} = \text{Span}\left(\{\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n\}\right)$$
$$\text{nullspace(A)} = \left\{\vec{x} \in \mathbb{R}^n | A\vec{x} = \vec{0}_m\right\},$$

where $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_m$ are the rows of A (considered as vectors from $\mathbb{R}^n$), and $\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n$ are the column of A (considered as vectors from $\mathbb{R}^m$).
Let us define the transpose matrix operation, where $A^\top$ is the $n \times m$ matrix obtained from A by writing row 1 of A as column 1 of $A^\top$, writing row 2 of A as column 2 of $A^\top$ and so on. Same goes for column. The fourth fundamental matrix space is:

$$\text{nullspace}(A^\top) = \left\{\vec{y} \in \mathbb{R}^m | A^\top \vec{y} = \vec{0}_n\right\}$$

Under these definitions, the subspaces and the corresponding ambient spaces are:

$$\text{rowspace(A)} = \text{colspace}(A^\top) \trianglelefteq \mathbb{R}^n, \quad \text{colspace(A)} = \text{rowspace}\left(A^\top\right) \trianglelefteq \mathbb{R}^m,$$
$$\text{nullspace(A)} \trianglelefteq \mathbb{R}^n, \quad \text{and nullspace}\left(A^\top\right) \trianglelefteq \mathbb{R}^m$$

Note that this includes both definition and theorem

Let A be an $m \times n$ matrix. Show that nullspace(A) $\trianglelefteq \mathbb{R}^n = \left\{\vec{x} \in \mathbb{R}^n | A\vec{x} = \vec{0}_m\right\}$

Proof.

- Non-empty set ✓
- Closed under addition
- Closed under scalar multiplication

Let $x_1, x_2 \in$ nullspace(A). We then need to show $\vec{x}_1 + \vec{x}_2 \in$ nullspace(A)

$$A(\vec{x}_1 + \vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = \vec{0}_m + \vec{0}_m = \vec{0}_m$$

And,

$$k\vec{x}_1 \in \text{ nullspace(A)}, \quad A(k\vec{x}_1) = k(A\vec{x}_1) = k\vec{0}_m = \vec{0}_m$$ ■

Example 2.20 *A: 4×7. Find a basis for the 4 fundamental subspaces.*

$$R = \begin{bmatrix} 1 & -4 & 0 & 3 & 0 & 5 & 6 \\ 0 & 0 & 1 & -2 & 0 & 7 & -3 \\ 0 & 0 & 0 & 0 & 1 & -8 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1. *Basis for rowspace(A): $\trianglelefteq \mathbb{R}^7$*

$$= \{ \textit{row } 1, 2, \textit{ and } 3 \textit{ of rref} \}$$
$$= \{ <1, -4, 0, 3, 0, 5, 6>, <0, 0, 1, -2, 0, 7, -3>, <0, 0, 0, 0, 1, -8, 4> \}$$

   dim(rowspace(A)) = 3

2. *Colspace basis: $\trianglelefteq \mathbb{R}^4$*

$$= \{ \textit{col } 1, 3, 5 \textit{ of original matrix} \}$$
$$= \{ <7, -3, -1, 2>, <2, 4, 24, -3>, <-3, 2, 4, 4> \}$$
$$\vec{c}_6 = 5\vec{c}_1 + 7\vec{c}_3 - 8\vec{c}_5$$

$$\begin{aligned} \textit{dim(cols(A))} &= 3 = \textit{dim(rowsp(A))} \\ &= \textit{No. of leading 1s} \\ &= \textit{rank of } A \end{aligned}$$

3. *Basis for nullspace(A). Solve for $A\vec{x} = \vec{0}_4$*

$$\vec{x} = < 4x_2 - 3x_4 - 5x_6 - 6x_7, x_2, 2x_4 - 7x_6 + 3x_7, x_4, 8x_6 - 4x_7, x_6, x_7 >$$

$$\begin{aligned} \vec{x} = x_2 < 4, 1, 0, 0, 0, 0, 0 > + x_4 < -3, 0, 2, 1, 0, 0, 0 > + x_6 < -5, 0, -7, 0, 8, 1, 0 > \\ + x_7 < -6, 0, 3, 0, -4, 0, 1 > \end{aligned}$$

   Basis: $\{ <4, 1, 0, 0, 0, 0, 0>, <-3, 0, 2, 1, 0, 0, 0>, <-5, 0, -7, 0, 8, 1, 0>, <-6, 0, 3, 0, -4, 0, 1> \}$.
   Dim(nullsp(A)) = 4.

4. *Nullsp (A^{+}) basis $\trianglelefteq \mathbb{R}^4$:*

$$y_3 \textit{ is free}, \quad \{ <-2, -5, 1, 0> \}$$

   Dim(nullsp (A^{+})) = 1

2.9 Orthogonal Complement

Prove: $W \cap W^{\perp} = \left\{ \vec{0}_n \right\}$

Proof. Let $\vec{w} \in W \cap W^{\perp}$. We need to show that $\vec{w} = \vec{0}_n$. So $\vec{w} \in W$ and $\vec{w} \in W^{\perp}$. Thus, $\vec{w}$ acts like one of the "v" in definition of $W^{\perp}$. Hence:

$$\vec{w} \cdot \vec{w} = 0$$
$$\|\vec{w}\|^2 = 0 \implies \|\vec{w}\| = 0$$

So, $\vec{w} = \vec{0}_n$! ■

Example 2.21 *Consider the following rref:*

$$\begin{bmatrix} 1 & 0 & 3 & -3 & -4 \\ 0 & 1 & 4 & -7 & -6 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

dim(W) = 2 from rref. Basis for $W^{\perp}$: $\{< -3, -4, 1, 0, 0 >, < 3, 7, 0, 1, 0 >, < 4, 6, 0, 0, 1 >\}$

Example 2.22 *From Ex 2.21: dim(W) = 2 and a basis for W is:*

$$B_1 = \{< 1, 0, 3, -3, -4 >, < 0, 1, 4, -7, -6 >\}$$
$$B_2 = \{< 1, -2, -5, 11, 8 >, < 5, -3, 3, 6, -2 >\}$$
$$B_3 = \{< 1, 0, 3, -3, 4 >, < -9, 6, -3, -15, 0 >\}$$
$$B_4 = \{< -9, 6, -3, -15, 0 >, < 3, -2, 1, 5, 0 >\} \implies \quad \textit{they are parallel}$$
$$B_5 = \{< 1, -2, -5, 11, 8 >, < 0, 1, 4, -7, -6 >, < 3, -2, 1, 5, 0 >\}$$

B_5 *is not a basis because the number of vectors exceeds the number of dimension.*

3 Linear Transformation on Euclidean Spaces

3.1 Mapping Spaces: Introduction to Linear Transformation

Example 3.1 *Construct* $T : \mathbb{R}^3 \to \mathbb{R}^2$

$$T\,(< x, y, z >) =< 2x - 5z, x + 3y + 7z >$$

a) Compute $T\,(< 4, -2, 3 >)$

$$T\,(< 4, -2, 3 >) =< -7, 19 >$$

b) Compute $T < 0, 0, 0 >=< 0, 0 >$

$$T\left(\vec{0}_3\right) = \vec{0}_2$$

c) Additive?

$$\vec{u} =< u_1, u_2, u_3 > , \quad \vec{v} =< v_1, v_2, v_3 >$$
$$\vec{u} + \vec{v} =< u_1 + v_1, u_2 + v_2, u_3 + v_3 >$$
$$T(\vec{u} + \vec{v}) =< 2(u_1 + v_1) - 5(u_3 + v_3) + < 2v_1 - 5v_3, v_1 + 3v_2 + 7v_3 >$$

V.s.

$$T(\vec{u} + T(\vec{v}) =< 2u_1 - 5u_3, u_1 + 3u_2 + 7u_3 > + < 2v_1 - 5v_3, v_1 + 3v_2 + 7v_3 >$$

d) Homogeneity?

$$k\vec{u} =< ku_1, ku_2, ku_3 >$$

$$\begin{aligned} T(k\vec{u}) &=< 2ku_1 - 5ku_2, ku_1 + 3ku_2 + 7ku_3 \\ &= k < 2u_1 - 5u_2, u_1 + 3u_2 + 7u_3 > \\ &= kT(\vec{u}) \end{aligned}$$

e)Add "4" to $< 2x - 5z + 4 >, \ldots >$. Additive? NO! Homog? NO!

f) Rewrite $\begin{bmatrix} 2x - 5z \\ x + 3y + 7z \end{bmatrix}$ *as a matrix product.*

$$\begin{bmatrix} 2x - 5z \\ x + 3y + 7z \end{bmatrix} = \begin{bmatrix} 2 & 0 & -5 \\ 1 & 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

g) Compute $T(\vec{e}_1) = T < 1, 0, 0 >= \begin{bmatrix} 2 \\ 1 \end{bmatrix}, T(\vec{e}_2) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, T(\vec{e}_3) = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$

Proof of Thm (pg. 160)

Proof. ($\implies$)

Given: T is a linear transformation

Show: we can find A: $m \times n \ni$

$$T(\vec{x}) = A\vec{x}\forall\vec{x} \in \mathbb{R}^n$$

How do we make A from T?

$$\begin{aligned} T\left(< x_1, x_2, \ldots, x_n >\right) &= T(x_1\vec{e}_1 + x_2\vec{e}_2 + \ldots + x_n\vec{e}_n \\ &= T(x_1\vec{e}_1) + T(x_2\vec{e}_2 + \ldots + T(x_n\vec{e}_n) \text{ By Additivity} \\ &= x_1T(\vec{e}_1) + x_2T(\vec{e}_2) + \ldots + x_nT(\vec{e}_n) \text{ By Homogeneous} \\ &= \begin{bmatrix} T(\vec{e}_1)\ T(\vec{e}_2) \ldots T(\vec{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \end{aligned}$$

■

Example 3.2 $S_5 : \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{aligned} S_5(\vec{e}_1) &= 5\vec{e}_1 \\ S_5(\vec{e}_2) &= 5\vec{e}_2 \\ S_5(\vec{e}_3) &= 5\vec{e}_3 \\ [S_5] &= \begin{bmatrix} 5\vec{e}_1 & 5\vec{e}_2 & 5\vec{e}_3 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \\ &= 5I_3 \end{aligned}$$

Example 3.3

1. *Multiply row 2 of I_2 by $\frac{2}{3}$:*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & \frac{2}{3} \end{bmatrix}$$

$\to$ *vertical effect*

2. *Multiply row 1 by $-\frac{3}{2}$*

$$\begin{bmatrix} -\frac{3}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

$\to$ *horizontal effect*

3. $R_1 \to R_1 - \frac{1}{2}R_2$ *:*

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

$\to$ *Horizontal shearing operator*

3.2 Rotation, Projections, and Reflections

Example 3.4 $\theta = \sin^{-1}\left(\frac{3}{5}\right)$

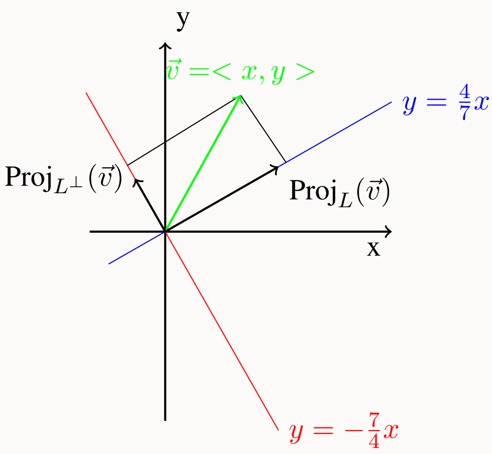
$$\begin{aligned} \sin\theta &= \frac{3}{5} \\ \cos\theta &= \frac{4}{5} \end{aligned}$$

$$\begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$$

Where does $< \frac{7}{3} >$ *go?*

$$\begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{19}{5} \\ \frac{33}{5} \end{bmatrix}$$

Example 3.5 *Proj + Ref in* $\mathbb{R}^2$*. Let L be* $y = \frac{4}{7}x$*, so* $L^\perp$ *is* $y = -\frac{7}{4}x$



Let $< x, y >$ *be any vector in* $\mathbb{R}^2$

$$\begin{cases} Proj_L(\vec{v}) = a < 7, 4 > \\ Proj_{L^\perp}(\vec{v}) = b < -7, 4 > \end{cases} \quad =< x, y >$$

$$x = 7a - 4b$$

$$y = 4a + 7b$$

$$a = \frac{7x + 4y}{65}, \quad b = \frac{-4x + 7y}{65}$$

$$\begin{aligned} Proj_L(\vec{v}) &= \frac{7x + 4y}{65} < 7, 4 > \\ &= \frac{1}{65} < 49x + 28y, 28x + 16y > \\ &= \begin{bmatrix} \frac{49}{65} & \frac{28}{65} \\ \frac{28}{65} & \frac{16}{65} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ Proj_{L^\perp}(\vec{v}) &= \frac{-4x + 7y}{65} < -4, 7 > \\ &= \frac{1}{65} < 16x - 28y, -28x + 49y > \\ &= \begin{bmatrix} \frac{16}{65} & -\frac{28}{65} \\ -\frac{28}{65} & \frac{49}{65} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ [Refl_L] &= 1^{st} - 2^{nd} \quad (\textit{matrices}) \\ &= \frac{1}{65} \begin{bmatrix} 33 & 56 \\ 56 & -33 \end{bmatrix} \end{aligned}$$

Example 3.6

$$\pi : 7x - 4y + 2z = 0$$

We want $\vec{v} =< x, y, z >= proj_\pi(\vec{v}) + proj_L(\vec{v})$. *Since* $proj_L(\vec{r}) \in L : proj_L(\vec{v}) = k < 7, -4, 2 >$. *But we also want* $proj_\pi(\vec{v}) \circ < 7, -4, 2 >= 0$

$$[< x, y, z > -k < 7, -4, 2 >] \circ < 7, -4, 2 >= 0$$

So,

$$7x - 4y + 2z - 69k = 0$$
$$k = \frac{7x - 4y + 2z}{69}$$

Then,

$$\begin{aligned} proj_L(\vec{v}) &= \frac{7x - 4y + 2z}{69} < 7, -4, 2 > \\ &= \frac{1}{69} < 49x - 28y + 14z, -28x + 16y - 8z, 14x - 8y + 4z > \\ &= \frac{1}{69} \begin{bmatrix} 49 & -28 & 14 \\ -28 & 16 & -8 \\ 14 & -8 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \end{aligned}$$

$\implies$ *It's a symmetric matrix!*

$$A = A^\top$$
$$proj_\pi(\vec{v}) = \frac{1}{69} \begin{bmatrix} 20 & 28 & -14 \\ 28 & 53 & 8 \\ -14 & 8 & 65 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

which is obtained by taking $< x, y, z > - < 49x - 28y + 14z, -28x + 16y - 8z, 14x - 8y + 4z >$ $[Proj_\pi]$ - $[Proj_L]$ *:*

$$refl_\pi(\vec{v}) = \frac{1}{69} \begin{bmatrix} -29 & 56 & -28 \\ 56 & 37 & 16 \\ -28 & 16 & 61 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

3.3 Operations on Linear Transformation and Matrices

Example 3.7 $T_1, T_2 : \mathbb{R}^2 \to \mathbb{R}^3$

$$T_1 (< x, y >) =< 5x - 2y, 3x + y, -4y >$$
$$T_2 (< x, y >) =< 7x, 6x - 3y, x + 5y >$$

a) $(T_1 + T_2) (< 3, -2 >)$

$$\begin{aligned} &= T_1 (< 3, -2 >) + T_2 (< 3, -2 >) \\ &=< 15 + 4, 9 - 2, 8 > + < 21, 18 + 6, 3 - 10 > \\ &=< 40, 31, 1 > \end{aligned}$$

b) Find $[T_1]$ *and* $[T_2]$

$$[T_1] = \begin{bmatrix} 5 & -2 \\ 3 & 1 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[T_2] = \begin{bmatrix} 7 & 0 \\ 6 & -3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

c) Find $(T_1 + T_2)\,(< x, y >))$

$$=< 12x - 2y, 9x - 2y, x + y >$$

d)

$$[T_1 + T_2] = \begin{bmatrix} 12 & -2 \\ 9 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

e)

$$\begin{aligned} -3T_2 < x, y > &= -3 < 7x, 6x - 3y, x + 5y > \\ &=< -21x, -18x + 9y, -3x - 15y > \end{aligned}$$

f) $[-3T_2]$

$$\begin{bmatrix} -21 & 0 \\ -18 & 0 \\ -3 & -15 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Proof. Additivity of $I_1 + I_2$. Show: $(T_1 + T_2)(\vec{u} + \vec{v})$

$$\begin{aligned} (T_1 + T_2)(\vec{u} + \vec{v}) &= (T_1 + T_2)(\vec{u}) + (T_1 + T_2)(\vec{v}) \\ &= T_1(\vec{u}) + T_1(\vec{v}) + T_2(\vec{u}) + T_2(\vec{v}) \\ &= (T_1 + T_2)(\vec{u}) + (T_1 + T_2)(\vec{v}) \end{aligned}$$

■

Example 3.8 $T_1 : \mathbb{R}^4 \to \mathbb{R}^2$

$$T\,(< x_1, x_2, x_3, x_4 >) =< 4x_1 - x_3, 3x_1 + 5x_2 - x_4 >$$

Keep $T_2 :< 7x, 6x - 3y, x + 5y >$

a)

$$[T_1] = \begin{bmatrix} 4 & 0 & -1 & 0 \\ 3 & 5 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

b) Find $(T_2 \circ T_1)(< -6, -3, 5, 4 >)$

$$\begin{aligned} T_2 \circ T_1 &= T_2\left(T_1(< 6, -3, 5, 4 >)\right) \\ &= T_2\left(< 24 - 5, 18 - 15 - 4 >\right) \\ &= T_2\left(< 19, -1 >\right) \\ &=< 133, 117, 14 > \end{aligned}$$

c) Find $(T_2 \circ T_1)(< x_1, x_2, x_3, x_4 >)$

$$\begin{aligned} T_2 \circ T_1 &= T_2\left(< 4x_1 - x_3, 3x_1 + 5x_2 - x_4 >\right) \\ &=< 7(4x_1 - x_3), 6(4x_1 - x_3) - 3(3x_1 + 5x_2 - x_4), 4x_1 - x_3 + 5(3x_1 + 5x_2 - x_4) > \end{aligned}$$

$$[T_2 \circ T_1] = \begin{bmatrix} 28 & 0 & -7 & 0 \\ 15 & -15 & -6 & 3 \\ 19 & 25 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Numbers of rows in T_1 = *numbers of columns in* T_2. *We can form* $[T_2][T_1]$

$$\begin{bmatrix} 7 & 0 \\ 6 & -3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 4 & 0 & -1 & 0 \\ 3 & 5 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 28 + 0 & 0 + 0 & -7 + 0 & 0 + 0 \\ 24 - 9 & 0 - 15 & -6 + 0 & 0 + 3 \\ 4 + 15 & 0 + 25 & -1 + 0 & 0 - 5 \end{bmatrix}$$

$$\begin{bmatrix} 28 & 0 & -7 & 0 \\ 15 & -15 & -6 & 3 \\ 19 & 25 & -1 & -5 \end{bmatrix} : 3 \times 4$$

Example 3.9

$$\begin{bmatrix} 7 & -2 & 4 & 3 & 0 \\ 6 & 8 & 1 & -6 & 2 \\ 4 & -4 & 9 & 1 & 7 \end{bmatrix} \begin{bmatrix} 4 & 7 \\ 8 & -7 \\ -6 & 3 \\ 3 & 9 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 28 - 16 - 24 + 9 & 49 + 14 + 12 + 27 \\ 29 + 64 - 6 - 18 - 9 & 42 - 56 + 3 - 54 + 2 \\ 16 - 32 - 54 + 13 + 14 & 28 + 28 + 27 + 9 - 7 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 102 \\ 60 & -63 \\ -53 & 85 \end{bmatrix}$$

By the last example in 3.2:

$$\begin{aligned} [proj_\pi] &= I_3 - [proj_L] \\ [refl_L] &= [proj_\pi] - [proj_L] \\ &= I_3 - 2[proj_L] \end{aligned}$$

3.4 Properties of Operations on Linear Transformations and Matrices

Idea: adding two matrices/scalar mult can be done by partitioning A, B into rows or columns

$$A = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_m \end{bmatrix}, \quad B = \begin{bmatrix} \vec{s}_1 \\ \vec{s}_2 \\ \vdots \\ \vec{s}_m \end{bmatrix} \implies A + B = \begin{bmatrix} \vec{r}_1 + \vec{s}_1 \\ \vdots \\ \vdots \\ \vec{r}_m + \vec{s}_m \end{bmatrix}$$

or if $A = \begin{bmatrix} \vec{c}_1 \dots \vec{c}_n \end{bmatrix}, B = \begin{bmatrix} \vec{d}_1 \dots \vec{d}_n \end{bmatrix} \implies A + B = \begin{bmatrix} \vec{c}_1 + \vec{d}_1 \dots \vec{c}_n + \vec{d}_n \end{bmatrix}$

Proof. Show: $r(A+B) = rA + rB$

$$\begin{aligned} \text{LHS} &= \begin{bmatrix} \vec{c}_1 + \vec{d}_1 \dots \vec{c}_n + \vec{d}_n \end{bmatrix} = \begin{bmatrix} r(\vec{c}_1 + \vec{d}_1) \dots \end{bmatrix} \\ \text{RHS} &= [r\vec{c}_1 \; r\vec{c}_2 \dots r\vec{c}_n] + \begin{bmatrix} r\vec{d}_1 \; r\vec{d}_2 \dots r\vec{d}_n \end{bmatrix} \\ &= \begin{bmatrix} r\vec{c}_1 + r\vec{d}_1 \dots r\vec{c}_n + r\vec{d}_n \end{bmatrix} \end{aligned}$$

■

Proof. Associative Prop of Matrix Mult
Idea: start with $C = \vec{x} : q \times 1$. Show: $A(B\vec{x}) = AB(\vec{x})$
Partition: $B = \begin{bmatrix} \vec{b}_1 \; \vec{b}_2 \dots \vec{b}_q \end{bmatrix}$

$$\begin{aligned} B\vec{x} &= x_1\vec{b}_1 + \dots + x_q\vec{b}_q \\ A(B\vec{x}) &= A(x_1\vec{b}_1) + \dots + A(x_q\vec{b}_q) \\ &= x_1(A\vec{b}_1) + \dots + x_q(A\vec{b}_q) \\ &= \begin{bmatrix} A\vec{b}_1 \dots A\vec{b}_q \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_q \end{bmatrix} \\ &= (AB)\vec{x} \end{aligned}$$

Now, let $C = [\vec{c}_1 \dots \vec{c}_n]$. Show: $(AB)C = A(BC)$

$$\begin{aligned} (AB)\left[\vec{c}_1 \dots \vec{c}_n\right] &= [(AB)\vec{c}_1 \dots (AB)\vec{c}_n] \\ &= [A(B\vec{c}_1 \dots A(B\vec{c}_n)] \\ &= A\left[B\vec{c}_1 \dots B\vec{c}_n\right] \\ &= A(BC) \end{aligned}$$

■

[T] is UNIQUE. Meaning: if $T(\vec{v}) = A\vec{v}$ for any $\vec{v} \in \mathbb{R}^n$, then $A = B$. Show:

$$[T_2 \circ T_1] = [T_2][T_1]$$

Proof. Let $\vec{v}$ be any vector from $\mathbb{R}^n$

$$\begin{aligned}(T_2 \circ T_1)(\vec{v}) &= T_2(T_1(\vec{v})) \quad \text{(def)}\\ &= B(A\vec{v}), \quad \text{where} A = [T_1], \quad B = [T_2]\\ &= (BA)\vec{v} \qquad \text{by Associative Prop}\end{aligned}$$

By the Uniqueness of the matrix of $T_2 \circ T_1$:

$$[T_2 \circ T_1] = BA = [T_2][T_1]$$

■

Example 3.10 *Magic!*

$$A = \begin{bmatrix} 6 & -7 \\ -6 & 9 \end{bmatrix}$$

Compute (A) where $p(x) = x^2 - 15x + 12$

$$p(A) = A^2 - 15A + 12I_2$$

$$A^2 = \begin{bmatrix} 78 & -105 \\ -90 & 123 \end{bmatrix}$$

$$\begin{aligned}\begin{bmatrix} 78 & -105 \\ -90 & 123 \end{bmatrix} - 15 \begin{bmatrix} 6 & -7 \\ -6 & 9 \end{bmatrix} + 12 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\\ = \begin{bmatrix} 78 - 90 + 12 & -105 + 105 \\ -90 + 90 & 123 - 135 + 12 \end{bmatrix}\\ = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\end{aligned}$$

Note This is a demonstration of the Cayley - Hamilton Theorem which states that $p(A) = 0_{n\times n}$ where p(x) is the characteristics polynomial of A.

Uniqueness of Representation for any Basis:

$$B = \{\vec{v}_1, \dots, \vec{v}_n\} \in \mathbb{R}^n$$

Any $\vec{v} \in \mathbb{R}^n$ can be written as $\vec{v} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$ for exactly one set of coordinates: $c_1, \ldots, c_n$. IF there were two ways:

$$\vec{v} = d_1\vec{v}_1 + \ldots + d_n\vec{v}_n$$

So,

$$\begin{aligned}c_1\vec{v}_1 + \ldots + c_n\vec{v}_n &= d_1\vec{v}_1 + \ldots + d_n\vec{v}_n\\ (c_1 - d_1)\vec{v}_1 + \ldots + (c_n - d_n)\vec{v}_n &= \vec{0}_n\\ c_1 - d_1 = \ldots = c_n - d_n &= 0\end{aligned}$$

Thus, there would be one and only one set of such coordinates.

Example 3.11 $T : \mathbb{R}^2 \to \mathbb{R}^4$

$$T\left(< 5, -3 >\right) =< 7, 4, 2, -8 >$$
$$T\left(< -3, 2 >\right) =< 0, 6, -3, 1 >$$

Find $T\left(< 6, -7 >\right)$

$$< 6, -7 >= a < 5, -3 > +b < -3, 2 >$$
$$\begin{cases} 5a - 3b = 6 \\ -3a + 2b = -7 \end{cases} \implies a = -9, \quad b = -17$$

$$\begin{aligned} T\left(< 6, -7 >\right) &= T\left(-9 < 5, -3 > -17 < -3, 2 >\right) \\ &= -9T\left(< 5, -3 >\right) - 17T\left(< -3, 2 >\right) \\ &= -9 < 7, 4, 2, -8 > -17 < 0, 6, -3, -1 > \\ &=< -63, -120, 15, 89 > \end{aligned}$$

3.5 Kernel and Range

$$\text{Given: T is 1-1}$$
$$\text{Show: Ker(T)} = \left\{\vec{0}_n\right\}$$

Proof. ($\implies$) Let $\vec{v} \in$ ker(T). Then,

$$T(\vec{v}) = \vec{0}_m$$

but $T(\vec{0}_n = \vec{0}_m$, $\vec{v} = \vec{0}_n$ since T is 1-1!
($\impliedby$)

$$\text{Given: ker(T)} = \left\{\vec{0}_n\right\}$$
$$\text{Show: T is 1-1}$$

Let $T(\vec{v}_1) = T(\vec{v}_2)$, then

$$T(\vec{v}_1 - \vec{v}_2) = \vec{0}_m$$
$$\vec{v}_1 - \vec{v}_2 \ \in \text{ker(T)} = \left\{\vec{0}_n\right\}$$
$$\vec{v}_1 - \vec{v}_2 = \vec{0}_n$$
$$\vec{v}_1 = \vec{v}_2$$

■

Example 3.12

$$T_1 : \begin{bmatrix} 1 & -3 & 4 \\ 2 & -6 & 9 \\ 5 & -15 & 4 \\ -3 & 9 & -7 \end{bmatrix}$$

$$R_1 : \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

T_1 : *nullity = 1, rank = 2, and NOT 1-1*

$$T_2 : \begin{bmatrix} 1 & -2 & 4 \\ 2 & -6 & 9 \\ 5 & -15 & 4 \\ -3 & 9 & -7 \end{bmatrix}$$

$$R_2 : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

T_2 : *is 1-1, nullity = 0, rank = 3, nullsp*($\{[T]\}$) $= \left\{\vec{0}_3\right\}$.

Note
- $\mathbb{R}^3 \to \mathbb{R}^5$: cannot be ONTO but could be 1-1
- $\mathbb{R}^5 \to \mathbb{R}^3$: cannot be 1-1 but could be onto if full rank (=3)

3.6 Invertible Operator and Matrices

To be invertible means to be both one-to-one and onto.

Proof. ($\Longrightarrow$)

Given: f is invertible set

Show: g exists with all these properties $(f : A \to B,\ \ g : B \to A)$

1. Create g using f. Start with $b \in B$, who is g(b)? Some $a \in A$. Not just argument "a" but some "a" so that $f(a) = b$

   $\to$ This "a" exists by ONTO property

   $\to$ This "a" is unique by 1-1 property

2. Since "a" is unique $\forall b \in B$, g is unique

3. $(f \circ g)(b) = f(g(b)) = f(a) = b$✓
   $(g \circ f)(a) = g(f(a)) = g(b) = a$✓
4. g is also 1-1.

$$\text{Suppose: } g(b_1) = g(b_2)$$
$$\text{Show: } b_1 = b_2$$

   Let $g(b_1) = a_1$ where $f(a_1) = b_1 \in A$; $g(b_2) = a_2$, where $f(a_2) = b_2 \in A$. So $a_1 = a_2$

$$f(a_1) = f(a_2)$$
$$b_1 = b_2 \checkmark$$

5. g is also onto.

$$\text{Let } a \in A$$
$$\text{Show: we can find } b \in B \text{ where } g(b) = a$$

   We can find $f(a) \in B$. Let $f(a) = b$, so $g(b) = a$. Let $g^{-1} = h$. So show that $h(a) = f(a) \forall a \in A$. We know: $h(a) = b$ where $g(b) = a$. But b is the unique element of B where $f(a) = b$. Thus, $h(a) = f(a)$, also $h = f$

($\Longleftarrow$)

$$\text{Given: g exists with 3 amazing properties}$$
$$\text{Show: f is invertible}$$

- f is 1-1 since

$$f(a_1) = f(a_2) \in B$$
$$g(f(a_1)) = g(f(a_2))$$
$$a_1 = a_2$$

  since $(g \circ f)(a) = a$
- f is onto: let $b \in B$. Find $a \in A$ so that $f(a) = b$. Find $g(b) \in A$, so let $g(b) = a$

$$f(a) = f(g(b)) = b \checkmark$$
$$(f \circ g)(b) = b$$

■

Example 3.13

$$A = \begin{bmatrix} -3 & -5 \\ 5 & 7 \end{bmatrix}$$

$ad - bc = -21 + 25 = 4$

$$A^{-1} = \frac{1}{4} \begin{bmatrix} 7 & 5 \\ -5 & -3 \end{bmatrix}$$

Example 3.14 $T:\mathbb{R}^2 \to \mathbb{R}^2$

$$T(<x,y>) = <3x+7y, 2x-6y>$$

$$[T]\begin{bmatrix}3 & 7\\ 2 & -6\end{bmatrix} \implies ad-bc=-32$$

$$[T^{-1}] = -\frac{1}{32}\begin{bmatrix}-6 & -7\\ -2 & 3\end{bmatrix}$$

$$T^{-1}(<x,y>) = <\frac{6}{32}x+\frac{7}{32}y, \frac{1}{16}x-\frac{3}{32}y>$$

3.7 Finding the Inverse of a Matrix

Proofs of the Thm - 2 for 1 - for the Matrix Inverse

find the theorem in the book

Proof. ($\implies$)

$$\text{Given: } A, B \text{ are inverse of each other}$$

$$\text{Show: } AB = I_n \text{ or } BA = I_n$$

We know:

$$AB = AA^{-1} = I_n$$

$$BA = A^{-1}A = I_n \checkmark$$

($\impliedby$)

$$\text{Given: } AB = I_n \text{ or } BA = I_n$$

$$\text{Show: A and B are both invertible and } A^{-1} = B,\ B^{-1} = A$$

Case 1 ($AB = I_n$)

$$A : T_1 : \mathbb{R}^n \to \mathbb{R}^n$$

$$B : T_2 : \mathbb{R}^n \to \mathbb{R}^n$$

$AB \leftrightarrow T_1 \circ T_2 \leftrightarrow$ identity : both 1-1 and onto. $\implies$ T_2 is 1-1 and T_1 is onto.

$$\text{Nullity}(T_2) = 0$$

$$\text{Rank}(T_2) = n$$

$$0 + n = n \checkmark$$

$\implies T_2$ is onto and thus invertible. We have:

$$
\begin{aligned}
AB &= I_n \\
(AB)B^{-1} &= I_n B^{-1} \\
A(BB^{-1}) &= I_n B^{-1} \\
AI_n &= B^{-1} I_n \\
A &= B^{-1}
\end{aligned}
$$

A & B are inverse of each other

Case 2 ($BA = I_n$) This is also proven similarly to case 1. ■

3.8 Conditions for Invertibility

All the conditions can be found on page 248
Theorem - The Really Big Theorem on Invertibility

Proof. Use 2-for-1 Thm:

$$\text{Guess: } (AB)^{-1} = B^{-1}A^{-1}$$

$$
\begin{aligned}
(AB)(AB)^{-1} &= (AB)(B^{-1}A^{-1}) \\
&= \left(A(BB^{-1})\right) A^{-1} \\
&= (AI_n)A^{-1} \\
&= AA^{-1} \\
&= I_n \checkmark
\end{aligned}
$$

We can now prove: If A is the product of the elementary matrices, then A is also invertible.

$$A = F_1 F_2 \dots F_j$$

where each F_j is elementary and each F_j is also invertible. A is invertible, so colspace(A) $= \mathbb{R}^n$ ($\implies$)

$$\text{Given: A is invertible}$$

$$A \leftrightarrow T_1, \text{ T is invertible}$$

1-1 and onto, range(T) $= \mathbb{R}^n$ (colspace) ✓
($\impliedby$) Given: colspace $= \mathbb{R}^n$. So, range $= \mathbb{R}^n \to$ T is onto and also 1-1. Thus, it's invertible. ■

4 Permutation Theory and Determinants

4.1 Permutation and the Determinant Concept

To decide if a term is positive or negative, we will count the inversion in a permutation – every time a number on the LEFT is bigger than the number on the right.

Permutation of the columns	Inversion	Count	Sign
1,2,3	none	0: even	$+a_{1,1}a_{2,2}a_{3,3}$
1,3,2	$3 > 2$	1: odd	$-a_{1,1}a_{2,3}a_{3,2}$
2,1,3	$2 > 1$	1: odd	$-a_{1,2}a_{2,1}a_{3,3}$
2,3,1	$2 > 1, 3 > 1$	2: even	$+a_{1,2}a_{2,3}a_{3,1}$
3,1,2	$3 > 1, 3 > 2$	2: even	$+a_{1,3}a_{2,1}a_{3,2}$
3,2,1	$3 > 2, 3 > 1$ $2 > 1$	3: odd	$-a_{1,3}a_{2,2}a_{3,1}$

Table 1: The six terms of a 3×3 determinant

Example 4.1 *Find the determinant of A*

- - -

$$A = \begin{bmatrix} 7 & -4 & 6 \\ 2 & 3 & -8 \\ -5 & 1 & 9 \end{bmatrix}$$ 7 -42

+ + +

Refer to the notebook to change the layout later through okular

$$Det(A) = 189 - 160 + 12 - (-90 - 56 - 72) = 259$$

Example 4.2

$$\sigma = (5, 2, 8, 3, 6, 7, 4, 1)$$

$$5 > 2, 3, 4, 1 : 4$$
$$2 > 1 : 1$$
$$8 > 3, 6, 7, 4, 1 : 5$$
$$3 > 1 : 1$$
$$6 > 4, 1 : 2$$
$$7 > 4, 1 : 2$$
$$4 > 1 : 1$$

Number of inversion $= 16$ *which is even. So, Sgn*$(\sigma) = +1$. *In addition,*

$$\sigma^{-1} = (8, 2, 4, 7, 1, 5, 6, 3)$$

change to physical notebook layout

Prove:

$$\text{sgn}(\sigma') = -\text{sgn}(\sigma)$$

Proof.

Case 1 Components are adjacent, so the number of inversions increases or decreases by 1.

Case 2 Components are NOT adjacent. Strategy here is to switch adjacent components which eventually results in 2k-1 steps (odd). ■

Need more clarifications from the book

Example 4.3 $1 \to 5$

$$\begin{array}{ccccccc} 3 & 1 & 5 & 4 & 2 & \to & \textit{odd} \\ 1 & 3 & 5 & 4 & 2 & \to & \textit{even} \end{array}$$

4.2 A Note About Calculating Determinant

We can do column operations to find the determinant and remember to record (-) sign whenever exchanging rows.

4.3 Properties of Determinant

A matrix is invertible $\iff R = I_n$

Proof.

Case 1 ($R = I_n$) So A is invertible.

$$\text{det(R)} = \det(E_t) \cdot \det(E_{t-1}) \dots \det(E_1) \cdot \det(A) = 1$$

Since $\det(E_t), \dots, \det(E_1) \neq 0$, det(A) $\neq 0$

Case 2 ($R \neq I_n$) R has a row of 0's $\implies$ A is not invertible.

$$0 = \det(E_t) \dots \det(E_1) \cdot \det(A)$$

Using the same argument as case 1, det(A) = 0 ■

THEOREM 4.1 Let A and B be $n \times n$ matrices. Then,

$$\det(A \cdot B) = \det(A) \cdot \det(B)$$

Proof.

Case 1 (A is invertible)

$$\det(A) \neq 0$$

and $A = E_1 E_2 \dots E_r$ where E_i is elementary matrix

$$\begin{aligned}\det(A) &= \det(E_1)\det(E_2)\dots\det(E_r)\\ \det(AB) &= \det\left[E_1 E_2 \dots E_r\right] B\\ &= \det(E_1)\det(E_2)\dots\det(E_r)\det(B)\\ &= \det(A)\det(B)\end{aligned}$$

Case 2 (A is not invertible)

$$\det(A) = 0$$

AB is not invertible $\implies$ det(AB) = 0 ■

Cofactor Expansion

Example 4.4

+

$$\begin{bmatrix} -1 & 2 & 6 & -5 \\ 0 & 8 & 21 & -12 \\ 0 & 4 & 26 & -22 \\ 0 & -13 & -7 & 19 \end{bmatrix}$$

$|A| = (-1)(3410) = -3410$

refer to the notebook to show how to find the determinant using cofactor Change layout

Example 4.5

$$A = \begin{bmatrix} 8 & -2 & 3 & -7 \\ -3 & 0 & 4 & 8 \\ 6 & 2 & -1 & -5 \\ 5 & -9 & -2 & 9 \end{bmatrix}$$

Make all entries in col2 except the last one into zero by the following operations: $R_1 \to R_1 + R_3, R_4 \to R_4 + 5R_3, R_3 \to R_3 - 2R_4$ *and do a cofactor expansion along col2.*

add cofactor sign below

$$\begin{bmatrix} 14 & 0 & 2 & -12 \\ -3 & 0 & 4 & 8 \\ -64 & 0 & 13 & 27 \\ -55 & 1 & -7 & -16 \end{bmatrix}$$

+

$$|A| = +1 \begin{vmatrix} 14 & 2 & -12 \\ -3 & 4 & 8 \\ -64 & 13 & 27 \end{vmatrix}$$

C3 -> C3 - 2C2

$$\begin{vmatrix} 62 & 16 & -16 \\ -8 & 1 & 0 \\ -217 & -51 & 1 \end{vmatrix}$$

$$|A| = 1 \cdot 1 \cdot \begin{vmatrix} 62 & -16 \\ -217 & 1 \end{vmatrix} = -3410 \checkmark$$

add column operators above – refer to notebook

5 Eigentheory and Diagonalization

5.1 The Eigentheory of Square Matrices

Solve: $A\vec{v} = \lambda\vec{v}$, for λ and $\vec{v} \neq \vec{0}_n$

$$A\vec{v} = (\lambda I_n)\vec{v}$$

$$\vec{0}_n = (\lambda I_n)\vec{v} - A\vec{v} = (\lambda I_n - A)\vec{v}$$

So we are looking for $\lambda \in \mathbb{R}$, $\vec{v} \neq \vec{0}_n$ so that $(\lambda I_n - A)\vec{v} = \vec{0}_n$. So, $\vec{v}$ is a non-zero vector in ker $(\lambda I_n - A)$. Thus, $\lambda I_n - A$ is definitely not invertible.

$$\det(\lambda I_n - A) = 0$$

Example 5.1

$$\begin{bmatrix} 285 & 504 \\ -160 & -283 \end{bmatrix}$$

$$\lambda I_2 - A = \begin{bmatrix} \lambda - 285 & -504 \\ 160 & \lambda + 283 \end{bmatrix}$$

$$\begin{aligned} p(\lambda) &= (\lambda - 285)(\lambda + 283) + 160(504) \\ &= \lambda^2 - 285\lambda + 283\lambda - 285(283) + 160(504) \\ &= \lambda^2 - 2\lambda - 15 \\ &= (\lambda - 5)(\lambda + 3) \\ \lambda &= -3, 5 \end{aligned}$$

Find the eigenvectors (ker $(\lambda I_n - A)$ *or ker* $(A - \lambda I_n)$

$$A + 3I_2 = \begin{bmatrix} 288 & 504 \\ -160 & -280 \end{bmatrix} = \begin{bmatrix} 288 & 504 \\ 1 & \frac{7}{4} \end{bmatrix} = \begin{bmatrix} 1 & \frac{7}{4} \\ 0 & 0 \end{bmatrix}$$

$\implies$ *ker:* $< -7, 4 >$

$$\begin{bmatrix} 285 & 504 \\ -160 & -283 \end{bmatrix} \begin{bmatrix} -7 \\ 4 \end{bmatrix} \begin{bmatrix} 21 \\ -12 \end{bmatrix}$$

remember to change like the notebook layout

parallel by a fa

$\lambda = 5:$

$$A - 5I_2 = \begin{bmatrix} 280 & 504 \\ -160 & -288 \end{bmatrix} = \begin{bmatrix} 1 & \frac{9}{5} \\ 0 & 0 \end{bmatrix}$$

$\implies$ *ker:* $< -9, 5 >$

$$\begin{bmatrix} 285 & 504 \\ -160 & -283 \end{bmatrix} \begin{bmatrix} -9 \\ 5 \end{bmatrix} \begin{bmatrix} -45 \\ 25 \end{bmatrix}$$

- *Eigen(A, -3) has basis* $\{< -7, 4 >\}$
- *Eigen(A, 5) has basis* $\{< -9, 5 >\}$

Example 5.2

$$\begin{bmatrix} 5 & 14 & -6 \\ 0 & -2 & 3 \\ 0 & 0 & 5 \end{bmatrix} \to \textit{upper triangular}$$

$$p(\lambda) = (\lambda - 5)^2(\lambda + 2)$$

$$\lambda = -2, 5$$

- $\lambda = -2:$

$$A + 2I_3 = \begin{bmatrix} 7 & 14 & -6 \\ 0 & 0 & 3 \\ 0 & 0 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigen(A,-2) has basis $\{< -2, 1, 0 >\}$

- $\lambda = 5:$

$$A - 5I_3 = \begin{bmatrix} 0 & -14 & -6 \\ 0 & -7 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 & -\frac{3}{7} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigen(A,5) has basis $\{< 1, 0, 0 >, < 0, 3, 7 >\}$

Eigen $(A, \lambda) = \{\vec{v} \in \mathbb{R}^n | A\vec{v} = \lambda\vec{v}\} =$ eigenspace

5.2 Computational Techniques

Prove that A is invertible iff $\lambda = 0$ is not an eigen-value.

Proof. Use: $p(\lambda) =$ det $(\lambda I_n - A)$

$$p(0) = \det(-A)$$
$$= (-1)^n \det(A)$$

Thus, $p(0) = 0 \iff$ det $(A) = 0$. Thus, $p(0) \neq 0 \iff$ det $(A) \neq 0$ (iff A is invertible) ■

Example 5.3 *Det* $(\lambda I_3 - A)$ *:*

$$\begin{vmatrix} \lambda + 25 & -11 & -11 \\ 132 & \lambda - 63 & -66 \\ -66 & 33 & \lambda + 36 \end{vmatrix} = \lambda^3 - 2\lambda^2 - 39\lambda - 72$$
$$= (\lambda - 8)(\lambda^2 - 16\lambda + 9)$$
$$= (\lambda - 8)(\lambda + 3)^2$$
$$\lambda = 8, -3$$

- *Eigen(A,8):*

$$A - 8I_3 = \begin{bmatrix} -33 & 11 & 11 \\ -132 & 55 & 66 \\ 66 & -33 & -44 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis: $\{< -1, -6, 3 >\}$

- *Eigen(A,-3):*

$$A + 3I_3 = \begin{bmatrix} -22 & 11 & 11 \\ -132 & 66 & 66 \\ 66 & -33 & -33 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -\frac{1}{2} & \frac{-1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis: $\{< 1, 2, 0 >, < 1, 0, 2 >\}$

5.3 Diagonalization of Square Matrices

Warm-up:

$$CD : \begin{bmatrix} -5 & 7 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} -15 & -14 \\ 9 & -8 \end{bmatrix}$$
$$= \begin{bmatrix} 3 \begin{bmatrix} -5 \\ 3 \end{bmatrix} - 2 \begin{bmatrix} 7 \\ 4 \end{bmatrix} \end{bmatrix}$$

Now: $AC = CD$

$$\begin{bmatrix} A\vec{c}_1 & A\vec{c}_2 \dots A\vec{c}_n \end{bmatrix} = \begin{bmatrix} d_1\vec{c}_1 \dots d_n\vec{c}_n \end{bmatrix}$$

$$\boxed{AC = CD \rightarrow \begin{cases} C^{-1}AC = D \\ A = CDC^{-1} \end{cases}}$$

Why is this useful?

$$A = CDC^{-1}$$
$$A^2 = (CDC^{-1})(CDC^{-1})$$
$$= CD^2C^{-1}$$

Thus,

$$\boxed{A^k = CD^kC^{-1}}$$

In math 55: system of diff eqns y_1, y_2, y_3 is function of t

$$[A]\, y's = f(t)$$

Diagonalize A, if possible. Let's find e^{At}

$$\begin{aligned}
e^x &= \sum_{k=0}^{\infty} \frac{A^k}{k!} \\
&= I_n + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots \\
e^A &= \sum_{k=0}^{\infty} \frac{CD^kC^{-1}}{k!} \\
&= C\left[\sum_{k=0}^{\infty} \frac{D^k}{k!}\right]C^{-1} \\
&= C\begin{bmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n}\end{bmatrix}C^{-1} \\
e^{At} &= C\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t}\end{bmatrix}C^{-1}
\end{aligned}$$

<u>Proof of Indep of Eigenvectors</u>

Proof. Induction on k

$$k = 1: \quad S = \{\vec{v}_1\}, \text{ indep } \iff \vec{v}_1 \neq \vec{0}_n$$

Good, since eigenvectors $\neq \vec{0}_n$

$$\begin{gathered}
\text{Assume: } S = \{\vec{v}_1, \dots, \vec{v}_i\} \text{ is indep} \leftrightarrow \lambda_1, \dots, \lambda_i \text{ distinct} \\
\text{Show: } S' = \{\vec{v}_1, \dots, \vec{v}_i, \vec{v}_{i+1}\} \text{ is still indep } \leftrightarrow \lambda_1, \dots, \lambda_i, \lambda_{i+1} \text{ distinct} \\
\text{Test: } c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_i\vec{v}_i + c_{i+1}\vec{v}_{i+1} = \vec{0}_n \\
\text{Show: } c_1, \dots, c_i, c_{i+1} \text{ all zero}
\end{gathered}$$

We will do 2 things to the test equation:

$$\begin{gathered}
A\left[c_1\vec{v}_1 + \dots + c_{i+1}\vec{v}_{i+1}\right] = A\vec{0}_n \\
c_1A\vec{v}_1 + \dots + c_iA\vec{v}_i + c_{i+1}A\vec{v}_{i+1} = \vec{0}_n \\
c_1\lambda_1\vec{v}_1 + \dots + c_1\lambda_i\vec{v}_i + c_{i+1}A\vec{v}_{i+1} = \vec{0}_n \qquad (2)
\end{gathered}$$

Secondly,

$$\begin{gathered}
\lambda_{i+1}\left(c_1\vec{v}_1 + \dots c_{i+1}\vec{v}_{i+1}\right) = \lambda_{i+1}\vec{0}_n \\
c_1\lambda_{i+1}\vec{v}_1 + \dots + c_{i+1}\lambda_{i+1}\vec{v}_{i+1} = \vec{0}_n \qquad (3)
\end{gathered}$$

(3) - (2) gives us:

$$c_1(\lambda_{i+1} - \lambda_1)\vec{v}_1 + \ldots + c_i(\lambda_{i+1} - \lambda_i)\vec{v}_i = \vec{0}_n$$

But we assume that $\{\vec{v}_1, \ldots, \vec{v}_i\}$ is indep and $\lambda_{i+1} - \lambda_1 \neq 0 \ldots \lambda_{i+1} - \lambda_i \neq 0$. So, $c_1 = c_2 = \ldots = c_i = 0$. But if $c_{i+1}\vec{v}_{i+1} = \vec{0}_n$ and eigenvector cannot be 0, then $c_{i+1} = 0$ ■

Example 5.4 $A : 10 \times 10$

$$p(\lambda) = (\lambda + 5)^2(\lambda + 2)^3(\lambda - 1)(\lambda - 3)^4$$
$$1 \leq dim(Eig(A, -5)) \leq 2$$
$$1 \leq dim(Eig(A, -2)) \leq 3$$
$$dim(Eig(A, 1)) = 1$$
$$1 \leq dim(Eig(A, 3)) = 4$$

Suppose those dimensions actually equals to the upper-bound limit. Show A is diagonalizable. Basis for:

- $\lambda = -5 : \{\vec{v}_1, \vec{v}_2\}$
- $\lambda = -2 : \{\vec{v}_3, \vec{v}_4, \vec{v}_5\}$
- $\lambda = 1 : \{\vec{v}_6\}$
- $\lambda = 3 : \{\vec{v}_7, \vec{v}_8, \vec{v}_9, \vec{v}_{10}\}$

For $C = [\vec{v}_1 \ldots \vec{v}_{10}]$ *to be invertible, the set* $S = \{\vec{v}_1, \ldots, \vec{v}_{10}\}$ *must be indep. Induction on the eigenspace:*

- $k = 1$ *:* $\{\vec{v}_1, \vec{v}_2\}$ *indep? Yes, by definition of basis.*
- $k = 2$ *: Show* $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$ *still indep. Test:*

$$c_1\vec{v}_1 + \ldots + c_5\vec{v}_5 = \vec{0}_{10} \quad (1)$$
$$c_1A\vec{v}_1 + \ldots + c_5A\vec{v}_5 = \vec{0}_{10}$$
$$-5c_1\vec{v}_1 - 5c_2\vec{v}_2 - 2c_3\vec{v}_3 - 2c_4\vec{v}_4 - 2c_5\vec{v}_5 = \vec{0}_{10} \quad (2)$$

$-2 \cdot (1) + (2)$ *:*

$$-3c_1\vec{v}_1 - 3c_2\vec{v}_2 = \vec{0}_{10}$$
$$c_3\vec{v}_3 + c_4\vec{v}_4 + c_5\vec{v}_5 = \vec{0}_{10}$$
$$c_3 = c_4 = c_5 = 0$$

since $\{\vec{v}_3, \vec{v}_4, \vec{v}_5\}$ *is basis. Prove similarly for other cases.*

6 Inner Product Spaces

6.1 Orthonormal Sets and the Gram - Schmidt Algorithm

$$< f(x)|g(x) > : \quad \text{inner product}$$

<u>55</u>: $\int_a^b f(x)g(x)dx$

1. $< f(x)|g(x) >=< g(x)|f(x) >$
2. $< f(x)|kg(x) >= k < f(x)|g(x) >$
3. $< f|g + h >=< f|g > + < f|h >$
4. $< f|f >\geq 0$
5. $< f(x)|f(x) >= 0 \iff f(x) = z(x) = 0 \forall [a, b]$
6. $< f|g >^2 \leq < f|f > \cdot < g|g >$ which allows us to define

$$\cos\theta = \frac{< f|g >}{||f|| ||g||}$$

Definition 6.1 *S is orthonormal if* $\vec{v}_i \circ \vec{v}_j = 0$ *if* $i \neq j$ *and* $||\vec{v}_i|| = 1$

Example 6.1 *Any* $\mathbb{R}^n : S = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$

$$\text{Given:} \quad c_1\vec{v}_1 + \ldots + c_k\vec{v}_k = \vec{0}_n$$
$$\text{Show:} \quad c_1 = \ldots = c_k = 0$$

Proof.

$$\vec{v}_1 \circ (c_1\vec{v}_1 + \ldots + c_k\vec{v}_k) = \vec{v}_1 \circ \vec{0}_n$$
$$c_1(\vec{v}_1 \circ \vec{v}_1) + \ldots + c_k(\vec{v}_k \circ \vec{v}_1) = 0$$

where $\vec{v}_1 \circ \vec{v}_1 = 1$ and $\vec{v}_2 \circ \vec{v}_1 = \ldots = \vec{v}_k \circ \vec{v}_1 = 0$, so $c_1 = 0$. Continuing thusly, $c_2 = c_3 = \ldots = c_k = 0$ ■

<u>Gram - Schmidt Algorithm:</u>

Input: Any basis $\{\vec{w}_1, \ldots, \vec{w}_n\}$ for $\mathbb{R}^n$

Output: An orthogonal set $\{\vec{v}_1, \ldots, \vec{v}_n\}$ with the property: Span($\vec{w}_1$ = Span($\vec{v}_1$)

1. $\vec{v}_1 = \vec{w}_1$

2. Span($\vec{w}_1, \vec{w}_2$) = Span($\vec{v}_1, \vec{v}_2$). To find $\vec{v}_2$: decompose $\vec{w}_2$ as $\vec{w}_2 = \vec{u} + \vec{v}_2$

$$\vec{u} \parallel \vec{v}_1, \quad \vec{u} = \text{proj}_{\text{span}(\vec{v}_1)} \vec{w}_2$$
$$\vec{u} = k\vec{v}_1$$
$$\vec{v}_2 = \vec{w}_2 - k\vec{v}_1, \quad \text{force orthogonal to } \vec{v}_1$$
$$\vec{v}_1 \circ \vec{v}_2 = \vec{v}_1 \circ \vec{w}_2 - k\vec{v}_1 \circ \vec{v}_1 = 0$$
$$k = \frac{\vec{w}_2 \circ \vec{v}_1}{\vec{v}_1 \circ \vec{v}_1}$$
$$\implies \quad \vec{v}_2 = \vec{w}_2 - \frac{\vec{w}_2 \circ \vec{v}_1}{\vec{v}_1 \circ \vec{v}_1}\vec{v}_1$$

   So, Span($\vec{v}_1, \vec{v}_2$) = Span($\vec{w}_1, \vec{w}_2$)

3. $$\vec{v}_3 = \vec{w}_3 - \frac{\vec{w}_3 \circ \vec{v}_1}{\vec{v}_1 \circ \vec{v}_1}\vec{v}_1 - \frac{\vec{w}_3 \circ \vec{v}_2}{\vec{v}_2 \circ \vec{v}_2}\vec{v}_2$$

4. $$\vec{v}_4 = \vec{w}_4 - \frac{\vec{w}_4 \circ \vec{v}_1}{\vec{v}_1 \circ \vec{v}_1}\vec{v}_1 - \frac{\vec{w}_4 \circ \vec{v}_2}{\vec{v}_2 \circ \vec{v}_2}\vec{v}_2 - \frac{\vec{w}_4 \circ \vec{v}_3}{\vec{v}_3 \circ \vec{v}_3}\vec{v}_3$$

Example 6.2 *Input:* $\{\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4\} = \{< -1, 1, 0, 1 >, < 0, -1, -1, 2 >, < 1, 1, 2, 0 >, < 1, 1, 0, -1 >\}$

1. $\vec{v}_1 =< -1, 1, 0, 1 >$

2.

$$\begin{aligned} \vec{v}_2 &= \vec{w}_2 - \frac{\vec{w}_2 \circ \vec{v}_1}{\vec{v}_1 \circ \vec{v}_1}\vec{v}_1 \\ &=< 0, -1, -1, 2 > - \frac{< 0, -1, -, 1, 2 > \circ < -1, 1, 0, 1 >}{< -, 1, 1, 0, 1 > \circ < -1, 1, 0, 1 >} < -1, 1, 0, 1 > \\ &= \frac{1}{3} < 1, -4, -3, 5 > \end{aligned}$$

   Check: $\vec{v}_2 \circ \vec{v}_1 = -1 - 4 + 0 + 5 = 0$ ✓. *Choose:* $\vec{v}_2 =< 1, -4, -3, 5 >$

3.

$$\begin{aligned} \vec{v}_3 &=< 1, 1, 2, 0 > -0 - \frac{1 - 4 - 6 + 0}{1^2 + 16 + 9 + 25} < 1, -4, -3, 5 > \\ &= \frac{1}{17} < 20, 5, 25, 15 > \end{aligned}$$

   Choose: $\vec{v}_3 =< 4, 1, 5, 3 >$

4.

$$\begin{aligned} \vec{v}_4 &=< 1, 1, 0, -1 > - \frac{-1}{3} < -1, 1, 0, 1 > - \frac{1 - 4 - 5}{51} < 1, -4, -3, 5 > - \frac{4 + 1 - 3}{51} < 4, 1, 5, 3 > \\ &=< 34, 34, -34, 0 > \end{aligned}$$

   Choose: $\vec{v}_4 =< 1, 1, -1, 0 >$

Output: $\left\{\frac{\vec{v}_1}{\sqrt{3}}, \frac{\vec{v}_2}{\sqrt{51}}, \frac{\vec{v}_3}{\sqrt{51}}, \frac{\vec{v}_4}{\sqrt{3}}\right\}$

6.2 Orthogonal Complement and Decompositions

Let's use the last example from 7.1.

$$\text{Create } \ W = \text{Span}(\vec{w})_1, \vec{w}_2), \ \ \dim(W) = 2$$
$$= \text{Span}(\vec{v}_1, \vec{v}_2) \ \text{ by Gram - Schmidt}$$

Let $\vec{v} =< 5, -7, 9, 4 >\in \mathbb{R}^4$. Find $< \vec{v} >_S$

$$c_1 = \vec{v} \circ \vec{v}_1 = -\frac{8}{\sqrt{3}}$$
$$c_2 = \frac{26}{\sqrt{51}}$$
$$c_3 = \frac{70}{\sqrt{51}}$$
$$c_4 = \frac{7}{\sqrt{3}}$$

$\implies \ < \frac{-8}{\sqrt{3}}, \frac{26}{\sqrt{51}}, \frac{70}{\sqrt{51}}, -\frac{11}{\sqrt{3}} >$
Goal: decompose $\vec{v} \in \mathbb{R}^n$

$$\vec{v} = \vec{w}_1 + \vec{w}_2$$
$$\vec{w}_1 \in W, \ \ \vec{w}_2 \in W^\perp$$

$\vec{v}_i \circ \vec{v}_j = 0$ for all $i \neq j$ and so $\vec{v}_3$ and $\vec{v}_4 \in W^\perp$. Also, $\vec{v}_1, \vec{v}_2 \in W$, so

$$c_1\vec{v}_1 + c_2\vec{v}_2 \in W$$
$$c_3\vec{v}_3 + c_4\vec{v}_4 \in W^\perp$$

Thus, $\vec{v} = \vec{w}_1 + \vec{w}_2$ where

$$\vec{w}_1 = c_1\vec{v}_1 + c_2\vec{v}_2 \in W$$
$$\vec{w}_2 = c_3\vec{v}_3 + c_4\vec{v}_4 \in W^\perp$$
$$\vec{w}_1 =< \frac{54}{17}, \frac{-80}{17}, \frac{-26}{17}, \frac{266}{51} >$$
$$\vec{w}_2 =< \frac{31}{17}, \frac{-39}{17}, \frac{179}{17}, \frac{70}{17} >$$
$$\vec{w}_1 \circ \vec{w}_2 = 0$$

Proof. We know $\{\vec{u}_1, \dots, \vec{u}_n\}$ is indep. (orthonormal set). So any subset is still indep. Thus, $\{\vec{u}_{k+1}, \dots, \vec{u}_n\}$ indep. Spanning?
Let $\vec{v} \in \mathbb{R}^n$, $\vec{v} \circ \vec{u}_1 = (c_1\vec{u}_1 + \dots + c_n\vec{u}_n)\vec{u}_1$ (uniquely). When is $\vec{v} \in W^\perp$?

$$\vec{v} \circ \vec{u}_1 = \dots = \vec{v} \circ \vec{u}_k = 0$$
$$\vec{v} \circ \vec{u}_1 = c_1\vec{u}_1 \circ \vec{u}_1 = c_1 = \dots = c_k = 0$$

$\implies \ \ \vec{v} = c_{k+1}\vec{u}_{k+1} + \dots + c_n\vec{u}_n$. So, $\{\vec{u}_{k+1} \dots \vec{u}_n\}$ span $W^\perp$ ■

Bonus: $\dim(W^\perp) = n - k$, but $\dim(W) = k$. Note that $W \cap W^\perp = \left\{\vec{0}_n\right\}$

THEOREM 6.1

The decomposition $\vec{v} = \vec{w}_1 + \vec{w}_2, \ \ \vec{w}_1 \in W, \ \ \vec{w}_2 \in W^\perp$ is unique. This means:

$$\vec{v} = \vec{z}_1 + \vec{z}_2, \ \ \vec{z}_1 \in W, \ \ \vec{z}_2 \in W^\perp$$
$$\vec{w}_1 + \vec{w}_2 = \vec{z}_1 + \vec{z}_2$$
$$\vec{w}_1 - \vec{z}_1 = -\vec{w}_2 + \vec{z}_2 \quad \in W \cap W^\perp \qquad \text{(closure)}$$

6.3 Orthonormal Bases and Projection Operators

Example 6.3 $W = $ *Span*$(\vec{w}_1, \vec{w}_2$ *) as before*

$$U \cdot U^\top = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{51}} \\ \frac{1}{\sqrt{3}} & -\frac{4}{\sqrt{51}} \\ 0 & -\frac{3}{\sqrt{51}} \\ \frac{1}{\sqrt{3}} & \frac{5}{\sqrt{51}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{51}} & -\frac{4}{\sqrt{51}} & -\frac{5}{\sqrt{51}} & \frac{5}{\sqrt{51}} \end{bmatrix}$$
$$= \frac{1}{17} \begin{bmatrix} 6 & -7 & -1 & -4 \\ -7 & 11 & 4 & -1 \\ -1 & 4 & 3 & -5 \\ -4 & -1 & -5 & 14 \end{bmatrix}$$

Proj$_w(< 5, -7, 9, 4 >) = \frac{1}{17} < 54, -80, -26, -266 >$

6.4 Orthogonal Matrices

Suppose $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ is the output of Gram - Schmidt, where input was a basis for $\mathbb{R}^n$. We assemble $Q = [\vec{u}_1 \ \vec{u}_2 \ \ldots \vec{u}_n]_{n \times n}$. Show $Q^\top Q = I_n$

$$\begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \\ \vdots \\ \vec{u}_n \end{bmatrix} \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \ldots & \vec{u}_n \end{bmatrix}$$
$$= \begin{bmatrix} \vec{u}_1 \circ \vec{u}_1 & \vec{u}_1 \circ \vec{u}_2 & \ldots & \vec{u}_1 \circ \vec{u}_n \\ \vec{u}_2 \circ \vec{u}_1 & \vec{u}_2 \circ \vec{u}_2 & \ldots & \vec{u}_2 \circ \vec{u}_n \\ \vec{u}_n \circ \vec{u}_1 & \vec{u}_n \circ \vec{u}_2 & \ldots & \vec{u}_n \circ \vec{u}_n \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ 0 & 0 & \ldots & 1 \end{bmatrix} : \ n \times n$$

By 2 for 1 Thm, $Q^\top = Q^{-1}$ already, and so $Q \cdot Q^\top = I_n$

THEOREM 6.2

$$Q = \begin{bmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_n \end{bmatrix} \quad \vec{r}_1 \dots \vec{r}_n \ : \text{orthonormal vectors}$$

$$\begin{aligned} QQ^\top &= \begin{bmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_n \end{bmatrix} \begin{bmatrix} \vec{r}_1 & \dots & \vec{r}_n \end{bmatrix} \\ &= I_n \end{aligned}$$

$$\begin{aligned} QQ^\top &= I_n \\ \det(Q \cdot Q^\top) &= \det(I_n) = 1 \\ \det(Q) \cdot \det(Q^\top) &= 1 \end{aligned}$$

Since det(Q) = det($Q^\top$)

$$\begin{aligned} [\det(Q)]^2 &= 1 \\ \det(Q) &= 1 \text{ or } -1 \end{aligned}$$

Show: If $\vec{u}_1, \dots, \vec{u}_k$ is an orthonormal basis for W, $u = [\vec{u}_1 \dots \vec{u}_k]$, then $[proj_w] = u \cdot u^\top$

Proof. If $\vec{v} \in \mathbb{R}^n$: $\vec{w}_1 = proj_w(\vec{v})$

$$\vec{w}_1 = (\vec{v} \circ \vec{u}_1)\vec{u}_1 + (\vec{v} \circ \vec{u}_2)\vec{u}_2 + \dots + (\vec{v} \circ \vec{u}_k)\vec{u}_k$$

Compute: $(uu^\top)\vec{v} = u(u^\top \vec{v})$

$$\begin{aligned} \begin{bmatrix} \vec{u}_1 & \dots & \vec{u}_k \end{bmatrix} \begin{bmatrix} \vec{u}_1 \\ \vdots \\ \vec{u}_k \end{bmatrix} \vec{v} &= \begin{bmatrix} \vec{u}_1 & \dots & \vec{u}_k \end{bmatrix} \begin{bmatrix} \vec{u}_1 \circ \vec{v} \\ \vdots \\ \vec{u}_k \circ \vec{v} \end{bmatrix} \\ &= (\vec{u}_1 \circ \vec{v})\vec{u}_1 + \dots + (\vec{u}_k \circ \vec{v})\vec{u}_k \end{aligned}$$

■

6.5 Orthogonal Diagonalization of Symmetric Matrices

$$A = A^\top$$

Magical Properties:

- All the eigenvalues are real numbers (Hard! Chapter 8)

- If $\vec{v}_1, \vec{v}_2$ are eigen-vectors from distinct eigenspaces $\lambda_1, \lambda_2 (\lambda_1 \neq \lambda_2$, then $\vec{v}_1 \perp \vec{v}_2$

Proof. Instead of $\vec{v}_1 \circ \vec{v}_2$, think of $\vec{v}_1 \circ (A[\vec{v}_2])$

$$\begin{aligned}
\vec{v})_1 \circ (\lambda_2 \vec{v}_2) &= (A[\vec{v}_2])^\top \cdot [\vec{v}_1] \\
&= \left([\vec{v}_2]^\top \cdot A^\top\right) \cdot [\vec{v}_1] \\
&= \left([\vec{v}_2]^\top A\right) [\vec{v}_1] \\
&= [\vec{v}_2](A[\vec{v}_1]) \\
&= (\lambda_1 \vec{v}_1) \circ \vec{v}_2 \\
\lambda_2(\vec{v}_1 \circ \vec{v}_2) = \lambda_1(\vec{v}_1 \circ \vec{v}_2) \\
(\lambda_2 - \lambda_1)(\vec{v}_1 \circ \vec{v}_2) = 0
\end{aligned}$$

where $\lambda_2 - \lambda_1 \neq 0$ and $\vec{v}_1 \circ \vec{v}_2 = 0$ ∎

THEOREM 6.3 **Spectral Theorem for Symmetric Matrices**

All symmetric matrices are diagonalizable. Furthermore, we can choose C to be an orthogonal matrix Q ($QQ^\top = I_n$) such that $Q^\top AQ = D$, a diagonalizable matrix.

Example 6.4 *$< 1, -1, 2 >$ is already $\perp$ to $< 1, 1, 0 >, < -2, 0, 1 >$. So apply Gram - Schmidt gives us:*

$$\begin{aligned}
\vec{v}_1 &=< 1, 1, 0 > \\
\vec{v}_2 &=< -1, 1, 1 >
\end{aligned}$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix}$$

7 General Vector Spaces

7.1 Axioms for a Vector Space

Redefine abstract vector space - operation, $\oplus$ is the usual vector

1. r $\odot < x, y > = < -rx, -ry >$
2. r $\odot (< a, b > \oplus < c, d >)$

3.

$$\begin{aligned} r\odot < a+c, b+d > &=< -r(a+c), -r(b+d) > \\ &=< -ra-rc, -rb-rd > \\ &=< -ra, -rb > \oplus < -rc, -rd > \\ &= r\odot < a, b > \oplus r < c, d > \checkmark \end{aligned}$$

4.

$$\begin{aligned} r \odot (s\odot < x, y >) &= r\odot < -sx, -sy > \\ &=< rsx, rsy > \\ &= (rs)\odot < x, y > \\ &= k\odot < x, y > \\ &=< -kx, -ky > \end{aligned}$$

No!

Example 7.1

$$\mathbb{R}^+ = \{\vec{x} | x \in \mathbb{R}, \ \textit{and } x > 0\}$$
$$\vec{x} \oplus \vec{y} = \vec{xy}$$
$$r \odot \vec{x} = \vec{x^r} = e^{r \ln \vec{(x)}}$$

- $\vec{z} \oplus \vec{y} = \overrightarrow{zy} = \vec{y}$ *only if* $\vec{z} = \vec{0}$
- $\vec{v} \oplus -\vec{v} = 1$
- $$\begin{aligned} (r+s) \odot \vec{x} &= \overrightarrow{x^{r+s}} \\ &= \overrightarrow{x^r x^s} \\ &= (r \odot \vec{x}) \oplus (s \odot \vec{x}) \end{aligned}$$

7.2 Linearity Properties for a Finite Set of Vectors

Example 7.2

$$S = \left\{e^{-5x}, e^{-2x}, e^{4x}, e^{6x}\right\} \subseteq C^\infty(I)$$

Indep/Dep?

$$\textit{Idea:} \ \lim_{x\to\infty} e^{3x} = \infty$$
$$\lim_{x\to-\infty} e^{3x} = 0$$

Test eq'n

$$c_1e^{-5x} + c_2e^{-2x} + c_3e^{4x} + c_4e^{6x} = z(x)$$

where $z(x) =$ *fxn which outputs 0 for all* $x \in \mathbb{R}$. *Divide both sides by* e^{-5x}

$$c_1 + c_2e^{3x} + c_3e^{9x} + c_4e^{11x} = z(x)$$

As $x \to -\infty, c_1 = 0$. *Now, we have:*

$$c_2e^{-2x} + c_3e^{4x} + c_4e^{6x} = z(x)$$
$$c_2 + c_3e^{6x} + c_4e^{8x} = z(x)$$
$$x \to -\infty, \quad c_2 = 0$$

Keep going: $c_3 = c_4 = 0$, *so S is independent.*

7.3 Linearity Properties for Infinite Sets of Vectors

Example 7.3 $\|\mathbb{Z}\| = \|\mathbb{N}\| = \aleph_0$. *Both* $\mathbb{N}$ *and* $\mathbb{Z}$ *are countable.*

Suppose $\mathbb{R}$ is countable, we can write ALL real number exactly one on a list. Idea here is no matter ho you list them we will have at least one missing real number.

$$|\mathbb{R}| \neq |\mathbb{N}|, \quad |\mathbb{R}| = \underbrace{"C"}_{\text{continuum}}$$

$|\mathbb{N}| = \aleph_0 = |\mathbb{Z}| = |\mathbb{Q}|$ and $|\mathbb{R}| = C$

Example 7.4 $\{x^n | n \in \mathbb{N}\} = \{x^0, x^1, x^2, \dots, x^n, \dots\}$

$$S_1 = \left\{e^{kx} | k \in \mathbb{N}\right\} = \{e^0, e^x, e^{2x}, \dots\}$$
$$S_2 = \left\{e^{kx} | k \in \mathbb{Z}\right\} = \{\dots, e^{-2x}, e^{-x}, 1, e^x, \dots\}$$
$$S_3 = \left\{e^{kx} | k \in \mathbb{Q}\right\}$$
$$S_4 = \left\{e^{kx} | k \in \mathbb{R}\right\}$$

$S_1 \subset S_2 \subset S_3 \subset S_4$ *and* $|S_1| = |S_2| = |S_3| = \aleph_0$ *and* $|S_4| = C$

Example 7.5 *From* S_3 *construct* $\left\{e^{\frac{3x}{4}}, e^{-\frac{5x}{7}}, e^{-2x}, e^{\frac{x}{2}}\right\}$
Indices: $\frac{3}{4}, -\frac{5}{7}, -2, \frac{1}{2}$

$$7e^{-2x} + \pi e^{-\frac{5x}{7}} - \frac{4}{3}e^{\frac{x}{2}} + 6e^{\frac{3x}{4}}$$

is a linear combination of vectors from S_3

Example 7.6

$$S_4 = \left\{e^{kx} | k \in \mathbb{R}\right\}$$

Is S_4 indep/dep?

$$c_1 e^{k_1 x} + c_2 e^{k_2 x} + \ldots + c_n e^{k_n x} = z(x)$$
$$k_1 < k_2 < \ldots < k_n \in \mathbb{R}$$

Divide both sides by $e^{k_1 x}$

$$c_1 + c_2 e^{(k_2 - k_1)x} + \ldots + c_n e^{(k_n - k_1)x} = z(x)$$

As $x \to \infty$, $e^{(k_2-k_1)x} \to 0$. So $c_1 = 0$. Repeating this logic, we get $c_1 = c_2 = \ldots = c_n = 0$. S is indep.

7.4 Subspaces, Basis and Dimension

Example 7.7

$$S = \{x^n | n \in \mathbb{N}\}$$

Spans P and is lin.indep. So S is a basis for P!

$$S = \left\{ \underbrace{1, x_1, \ldots, x^{n-1}, x^n}_{n+1} \right\}$$

is a basis for P^n

$$P^3 : \quad S = \left\{1, x, x^2, x^3\right\}$$

Proof. Given: V is any non-zero vector space, try to make a basis for V

1. Pick any $\vec{v}_1 \in V, \;\; \vec{v}_1 \neq \vec{0}_v$.

$$\text{Make} \quad S_1 = \{\vec{v}_1\} : \quad S_1 \text{ is indep}$$
$$\text{Span}(S_1) = V? \quad \text{Yes: done!}$$

   If NO,
2. Pick $\vec{v}_2 \in V, \quad \notin \text{Span}(S_1)$. Make $S_2 = \{\vec{v}_1, \vec{v}_2\}$. In order to stop this process going forever, it requires trans-finite induction / Zorn's Lemma to stop. ∎

Example 7.8 *Consider: $W \subseteq P^3$, defined by*

$$W = \left\{p(x) \in P^3 | p(-2) = 3p(1) \text{ and } p'(-1) = p(2)\right\}$$

a) Is $z(x) \in W$?

$$\begin{cases} z(-2) = 0 \\ z(1) = 0 \end{cases} \quad \checkmark$$

$z'(x) = z(x)$ so $z'(1) = 0 = z(2)$ ✓

b) Closure under +:

$$p_1(-2) = 3p_1(1) \qquad p_2(-2) = 3p_2(1)$$
$$p_1'(-1) = p_1(2) \qquad p_2'(-1) = p_2(2)$$
$$(p_1 + p_2)(-2) = 3(p_1 + p_2)(1) ✓$$
$$(p_1 + p_2)'(-1) = (p_1 + p_2)(2) ✓$$

c) Closure under $\cdot$:

$$kp_1(-2) = 3kp_1(1)$$
$$(kp_1)'(-1) = (kp_1)(2)$$

d) Now that we know that $W \trianglelefteq \mathbb{R}^3$, find a basis for W.

$$p(-2) = c_0 - 2c_1 + 4c_2 - 8c_3$$
$$3p(1) = 3c_0 + 3c_1 + 3c_2 + 3c_3$$
$$p'(1) = c_1 - 2c_2 + 3c_3$$
$$p(2) = c_0 + 2c_1 + 4c_2 + 8c_3$$

Our coefficient must satisfy

$$c_0 - 2c_1 + 4c_2 - 8c_3 = 3c_0 + 3c_1 + 3c_2 + 3c_3$$
$$c_1 - 2c_2 + 3c_3 = c_0 + 2c_1 + 4c_2 + 8c_3$$
$$2c_0 + 5c_1 - c_2 + 11c_3 = 0$$
$$c_0 + c_1 + 6c_2 + 5c_3 = 0$$

After some algebras, we obtain

$$\begin{aligned} c_0 &= -\frac{31}{3}c_2 - \frac{14}{3}c_3 \\ c_1 &= \frac{13}{3}c_2 - \frac{1}{3}c_3 \\ p(x) &= -\frac{31}{3}c_2 - \frac{14}{3}c_3 + \left(\frac{13}{3}c_2 - \frac{1}{3}c_3\right)x + c_2x^2 + c_3x^3 \\ &= \frac{c_2}{3}\left(-31 + 13x + 3x^2\right) + \frac{c_3}{3}\left(-14 - x + 3x^3\right) \end{aligned}$$

If we let

$$\begin{cases} q_1(x) = -31 + 13x + 3x^2 \\ q_2(x) = -14 - x + 3x^3 \end{cases} \in W$$

then any $p(x) \in W$ is a lin.comb of q_1 and q_2 and $B = \{q_1(x), q_2(x)\}$ is also lin.indep. B is also a basis for W!

7.5 Linear Transformation on General Vector Spaces

Example 7.9

$$a = 0$$
$$E_0(\cos x) = 1$$
$$E_0(\sqrt{x}) = 0$$
$$E_0(\frac{1}{2}e^x) = \frac{1}{2}$$
$$E_0(f(x) + g(x)) = E_0(f) + E_0(g)$$
$$E_0(fk) = kf(0) = kE_0(f)$$

Let $\vec{a} =< -2, 1, 3 >$

$$E_{\vec{a}}(q_2) =< -14 + 2 - 24, -14 - 1 + 3, -14 - 3 + 81 >=< -36, -12, 64 >$$

Example 7.10

$$D : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$$
$$D(e^{5x}) = 5e^{5x}$$
$$D(7x^3) = 21x^2$$

Example 7.11 *Int:* $C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$

$$Int(f) = \int_0^1 f(x)dx$$
$$Int(e^{5x}) = \int_0^1 e^{5x}dx = \frac{1}{5}(e^5 - 1)$$
$$\int_0^1 (f + g)(x)dx = \int_0^1 f(x)dx + \int_0^1 g(x)dx$$
$$\int_0^1 kf(x)dx = k\int_0^1 f(x)dx$$
$$Ant(f) = \int_0^x f(t)dt$$

$$\begin{aligned} Ant(e^{5x}) &= \int_0^x e^{5t}dt \\ &= \frac{1}{5}e^{5t}\Big|_0^x \\ &= \frac{1}{5}(e^{5x} - 1) \end{aligned}$$

7.6 Isomorphisms and Their Applications

Find a fxn $y = f(x)$ that satisfies:

$$5y'' - 3y' + 4y = \underbrace{7x^2e^{4x} - 2xe^{4x} + 6e^{4x}}_{\text{a linear combination}}$$

Example 7.12 *What's the smallest vector space V such that V contains the fxn $f(x) = x^2e^{4x} \in V$ such that the derivative of all fxn in V is also in V? ($D : V \to V$)*

$$f'(x) = \underbrace{2xe^{4x}}_{\text{force this to be in } V} + \underbrace{4x^2e^{4x}}_{\in V}$$

$$g(x) = xe^{4x} \quad \textit{must be in } V$$

$$g'(x) = \underbrace{e^{4x}}_{\text{force this to be in } V} + \underbrace{4xe^{4x}}_{\in V}$$

$$h(x) = e^{4x} \in V$$

$$h'(x) = 4e^{4x} \in V$$

The smallest V containing all these fxn is $V =$ Span($f(x), g(x), h(x)$)

Kernel & Range:

If $\vec{v}, \vec{w} \in \ker(T)$

$$T(\vec{v}) = \vec{0}_w, \;\; T(\vec{w}) = \vec{0}_w$$

$$\implies \quad T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w}) = \vec{0}_w + \vec{0}_w$$

Example 7.13 $D : P^3 \to P^2$ *(or P^3)*

$$\ker(D) = \quad \textit{constant fxns}$$

7.7 Coordinate Vectors and Matrices for Linear Transformation

Example 7.14 $V = P^2$, *standard basis* $\{1, x, x^2\} = B$

$$p(x) = 5x^2 - 3x + 8$$

$$< p(x) >_B = < 8, -3, 5 >$$

Example 7.15 *Let* $B' = \{4x^2 - 1, 2x + 3, x - 1\}$

- *Prove that B' is also a basis for P^2.*

$$\underbrace{c_1(4x^2 - 1)}_{c_1 = 0} + c_2(2x + 3) + c_3(x - 1) = z(x)$$

$$c_2 = c_3 = 0$$

Indep and Basis: Yes ✓

- $< p(x) >'_B$?

$$\left[\begin{array}{ccc|c} 4 & 0 & 0 & 5 \\ 0 & 2 & 3 & -3 \\ -1 & 3 & -1 & 8 \end{array}\right]$$

$$\xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{5}{4} \\ 1 & 0 & 0 & \frac{5}{4} \\ 0 & 0 & 1 & -\frac{11}{2} \end{array}\right]$$

$< p(x) >'_B = < \frac{5}{4}, \frac{5}{4}, -\frac{11}{2} >$

Example 7.16 *Take the differential equation from 7.6,*

$$[D]_B = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 4 & 0 \\ 0 & 1 & 4 \end{bmatrix}$$

$B = \left\{x^2e^{4x}, xe^{4x}, e^{4x}\right\}$ *basis for V. Find* $\frac{d}{dx}\left(5x^2e^{4x} - 3xe^{4x} + 8e^{4x}\right)$
Do it with $[D]_B$

$$\begin{bmatrix} 4 & 0 & 0 \\ 2 & 4 & 0 \\ 0 & 1 & 4 \end{bmatrix} \underbrace{\cdot}_{2.\ multiply} \underbrace{\begin{bmatrix} 5 \\ -3 \\ 8 \end{bmatrix}}_{1.Encode} \underbrace{\begin{bmatrix} 20 \\ -2 \\ 29 \end{bmatrix}}_{3.Decode}$$

We observed the $[D]_B$ *is invertible.*

$$[D]_B^{-1} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ -\frac{1}{8} & \frac{1}{4} & 0 \\ \frac{1}{32} & -\frac{1}{16} & \frac{1}{4} \end{bmatrix}$$

Find

$$\int x^2e^{4x}dx = \frac{1}{4}x^2e^{4x} - \frac{1}{8}xe^{4x} + \frac{1}{32}e^{4x} + C$$

Find $\int \left(5x^2e^{4x} + 3xe^{4x} - 8e^{4x}\right) dx$

$$\begin{bmatrix} \frac{1}{4} & 0 & 0 \\ -\frac{1}{8} & \frac{1}{4} & 0 \\ \frac{1}{32} & -\frac{1}{16} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -8 \end{bmatrix} = \begin{bmatrix} \frac{5}{4} \\ \frac{1}{8} \\ -\frac{65}{32} \end{bmatrix}$$

$$\frac{5}{4}x^2e^{4x} + \frac{1}{8}xe^{4x} - \frac{65}{32}e^{4x} + C$$

Use $[D]$ *to find a fxn* $y = f(x) \ni$

$$\underbrace{5y'' - 2y' + 3y}_{\text{interpret this as a linear comb of several transformation}} = 7x^2e^{4x} - 8xe^{4x} + 6e^{4x}$$

$$5y'' - 2y' + 3y = 5[D]_B^2 - 2[D]_B + 3I_3$$

$$= \begin{bmatrix} 75 & 0 & 0 \\ 76 & 75 & 0 \\ 10 & 38 & 75 \end{bmatrix}$$

Find $f(x)$ *so that* $f(x) \xrightarrow{\text{transform into}} 7x^2 \ldots + 6e^{4x}$ $(A\vec{x} = \vec{b} \implies \vec{x} = A^{-1}\vec{b})$

$$\text{Inverse:} \begin{bmatrix} \frac{1}{75} & 0 & 0 \\ -\frac{76}{5625} & \frac{1}{75} & 0 \\ \frac{2138}{421875} & -\frac{38}{5625} & \frac{1}{75} \end{bmatrix} \begin{bmatrix} 7 \\ -8 \\ 6 \end{bmatrix} = \begin{bmatrix} \frac{7}{75} \\ -\frac{1132}{75^2} \\ \frac{71516}{75^3} \end{bmatrix}$$

$$f(x) = \frac{7}{75}x^2e^{4x} - \frac{1132}{75^2}xe^{4x} + \frac{71516}{75^3}e^{4x}$$

Consider: $T : P^3 \to P^2$ where

$$T\left(p(x)\right) = 4p'(x) + p''(x)(2x-3) + p(-2)(-x^2+5x+2)$$

a) Warm-up: find $T\left(7x^3 - 5x^2 + 4x - 6\right)$

$$\begin{aligned} p'(x) &= 21x^2 - 10x + 4 \\ p''(x) &= 42x - 10 \\ p(-2) &= -90 \\ T\left(p(x)\right) &= 4(21x^2 - 10x + 4) + (42x - 10)(2x - 3) - 90(-x^2 + 5x + 2) \\ &= 258x^2 - 636x - 134 \end{aligned}$$

b) Briefly explain why $T\left(p(x)\right)$ belongs in $P^2 \forall p(x) \in P^3$
If deg $p(x) \leq 3$, $p'(x)$ has deg ≤ 2 and $p''(x)$ has deg ≤ 1. So deg $4p'(x) \leq 2$, deg $p''(x)(2x-3) \leq 2$ and constant $x\left(-x^2+5x+2\right)\big)$ has deg 0 or 2.
c) Show that T is additive

$$\begin{aligned} T(p+q) &= 4(p+q)' + (p+q)''(2x-3) + (p+q)(-2)(-x^2+5x+2) \\ &= T(p) + T(q) \end{aligned}$$

d) $T(kp) = kT(p)$ ✓
e) Let

$$\begin{aligned} B &= \left\{1, x, x^2, x^3\right\} \\ B' &= \left\{1, x, x^2\right\} \end{aligned}$$

Find $[T]_{B,B'}$

$$\begin{aligned} T(1) &= -x^2 + 5x + 2 \\ T(x) &= 2x^2 - 10x \\ T(x^2) &= -4x^2 + 32x + 2 \\ T(x^3) &= 32x^2 - 58x - 16 \\ [T]_{B,B'} &= \begin{bmatrix} 2 & 0 & 2 & -16 \\ 5 & -10 & 32 & -58 \\ -1 & 2 & -4 & 32 \end{bmatrix} \end{aligned}$$

Nullity(T) + Rank(T) = n = dim(V) = 4
f) The rref of $[T]_{B,B'}$ is

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{33}{2} \\ 0 & 1 & 0 & \frac{99}{4} \\ 0 & 0 & 1 & \frac{17}{2} \end{bmatrix}$$

Find a basis for

- Ker(T): $< 66, -99, -34, 4 > \leftrightarrow \{66 - 99x - 34x^2 + 4x^3\}$
- Range(T): $\vec{c}_1, \vec{c}_2, \vec{c}_3 \leftrightarrow \{2 + 5x - x^2, -10x + 2x^2, 2 + 32x - 4x^2\}$

Nullity(T) + Rank(T) = 1 + 3 = 4 = dim(V)

Example 7.17 *Consider* $T : P^2 \to P^3$

$$[T]_{B,B'} = \begin{bmatrix} 5 & 2 & 9 \\ 2 & -4 & -6 \\ -3 & 3 & 6 \\ 1 & 1 & 3 \end{bmatrix}$$

where

$$B = \{x^2 + 1, -x + 3, 4\} \in P^2$$
$$B' = \{1 - x, 1 + x^2, x^3, 1\} \in P^3$$

a)Compute $T(3x^2 - 5x + 2)$

$$3(x^2 + 1) + 5(-x + 3) - 4 \cdot 4 = 3x^2 - 5x + 2$$

$$\begin{bmatrix} 5 & 2 & 9 \\ 2 & -4 & -6 \\ -3 & 3 & 3 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ -4 \end{bmatrix} = \begin{bmatrix} -11 \\ 10 \\ -6 \\ -4 \end{bmatrix} \leftrightarrow -11(1 - x) + 10(1 + x^2) - 6x^3 - 4$$
$$= -6x^3 + 10x^2 - 11x - 9$$

b) rref:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Kernel(T): $< 1, -2, 1 > \leftrightarrow \{-(x^2 + 1) - 2(-x + 3) + 4\} \leftrightarrow \{-x^2 + 2x - 3\}$
Range: $\vec{c}_1, \vec{c}_2$

$$\{5(1 - x) + 2(1 + x^2) - 3(x^3) + 1, 2(1 - x) - 4(1 + x^2) + 3(x^3) + 1\}$$
$$\{-3x^3 + 2x^2 - 5x + 8, 3x^3 - 4x^2 - 2x - 1\}$$

Remember