# Math 55H - Honors Ordinary Differential Equation

Duc Vu
Taught by Pete Riley
Pasadena City College

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This is the last math class in the math sequence at PCC. It is taken during Spring 2020 (Covid-19 period) and thus is online. We use the book *Elementary Differential Equations and Boundary Value Problems* by *Boyce* and *Diprima* ($11^{\text{th}}$ edition). Even though this is an ODE class, we also got to touch a bit upon PDE and Fourier Series (heat conduction problem). Please let me know if you find any mistakes/typos in this notes and I will try to fix them as soon as I can.

## Contents

# 1 Introduction

## 1.1 Classification of ODE

### 1.1.1 Order

**Example 1.1.1**

$$y''' + 2e^t y'' + yy' = t^4$$

*Here we can observe that the highest order of the derivative is 3 which is also the order of the differential equation.*

Generalizing it to $n^{th}$ order ODE, we obtain:

$$F[t, u(t), u'(t), \dots, u^n(t)] = 0$$
$$y^n = f(t, y, y', y'', \dots, y^{n-1})$$

$\Rightarrow$ Simply put, to solve an ODE means to get rid of the derivative. The solution interval of validity is $\alpha < t < \beta$.
$\exists\ \phi \ni$:

$$\phi', \phi'', \dots, \phi^n \quad \text{exist.}$$

and satisfy

$$\phi^n(t) = f[t, \phi(t), \phi'(t), \dots, \phi^{n-1}(t)] \qquad \forall\ t \in (\alpha, \beta)$$

### 1.1.2 Linear & Non-linear

General linear of order n:

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \dots + a_n(t)y = g(t)$$

*Note:* Dependent variables have to be linear

**Example 1.1.2**

$$t^2y'' - 3ty' + 4y = 0\text{: } \textit{linear}$$
$$y''' + 2e^t y'' + yy' = t^4\text{: } \textit{nonlinear}$$
$$y'' - 3y' + y^2 = 0\text{: } \textit{nonlinear}$$
$$y^{(3)} + yy' + \sin y = x^2\text{: } \textit{nonlinear}$$

A notable example of nonlinear differential equation in physics is the differential equation of the motion of a simple pendulum, which can be expressed as

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

For $\theta \approx 0$, the equation can be simplified to

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0 \qquad \text{(linearization)}$$

### 1.1.3 Autonomous & Non-autonomous

**Example 1.1.3**

$$y' = -1 - 2y\text{: } \textit{autonomous}$$
$$y' = t + 2y\text{: } \textit{non-autonomous}$$

From the example above, we can observe that autonomous equation does not depend on t (doesn't contain t) while non-autonomous equation does (contain t)

# 2 First Order Differential Equations

## 2.1 Linear Equations: Method of Integrating Factors

Template for 1st order linear ODE:

$$\frac{dy}{dt} + p(t)y = g(t)$$

p and g are continuous on interval $\alpha < t < \beta$.

**Example 2.1.1**

$$y' + 2y = te^{-2t}, \qquad y(1) = 0 \tag{1}$$

*What would happen if we multiply Eq.(1) by* $e^{2t}$*?*

$$e^{2t}y' + 2e^{2t}y = t$$
$$\left(e^{2t}y\right)' = e^{2t}y' + 2e^{2t}y$$
$$\int \left(e^{2t}y\right)' dt = \int t dt$$
$$ye^{2t} = \frac{1}{2}t^2 + C \tag{1.1}$$
$$y = \frac{1}{2}t^2e^{-2t} + Ce^{-2t} \tag{1.2}$$

*Now, consider the IC:*

$$0 = \frac{1}{2}e^{-2} + Ce^{-2}$$
$$c = -\frac{1}{2}$$

*So,*

$$y = \frac{1}{2}t^2e^{-2t} - \frac{1}{2}e^{-2t} \tag{1.3}$$

In the example above, 1.1 is referred to as *implicit general solution*, 1.2 is called *explicit general solution* and 1.3 is *explicit particular solution*

**Generalize**:

$$y' + p(t)y = g(t) \tag{2}$$

Integrating factor:

$$\mu(t) = \exp \int p(t)dt$$

Multiply Eq.(2) by $\mu(t)$ gives us:

$$\mu(t)y' + \mu(t)p(t)y = \mu(t)g(t)$$

We want the LHS to be result from the product rule which is $\mu(t)p(t)y = \mu'(t)y$. So,

$$\begin{aligned}
\mu'(t) &= \mu(t)p(t) \\
\frac{\mu'(t)}{\mu(t)} &= p(t) \\
\frac{d}{dt}\ln\mu(t) &= p(t) \\
\ln\mu(t) &= \int p(t)dt + K \\
\mu(t) &= \exp \int p(t)dt \qquad \text{(choose k = 0)}
\end{aligned}$$

**Example 2.1.2**

$$y' + 3y = t + e^{-2t}$$

*Let's find the integrating factor*

$$\begin{aligned}
\mu(t) &= \exp \int p(t)dt \\
&= \exp \int 3dt \\
&= e^{3t}
\end{aligned}$$

*Multiply by the integrating factor by both sides gives:*

$$\begin{aligned}
y'e^{3t} + 3ye^{3t} &= te^{3t} + e^t \\
\int \left(ye^{3t}\right)' dt &= \int \left(te^{3t} + e^t\right) dt \\
ye^{3t} &= \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t} + e^t + c \\
y &= \frac{1}{3}t - \frac{1}{9} + e^{-2t} + ce^{-3t}
\end{aligned}$$

*As* $t \to \infty$, $y \to \infty$ *and* $y$ *asymptotically approach the linear function* $y = \frac{1}{3}t - \frac{1}{9}$

**Example 2.1.3**

$$y' = t^2 y + (t-1) \tag{*}$$

*Rearrange the equation so that it fits the template*

$$y' - t^2 y = t - 1$$

*Here* $p(t) = -t^2$, $g(t) = t - 1$. *Then,*

$$\begin{aligned} \mu(t) &= \exp \int -t^2 dt \\ &= e^{-\frac{1}{3}t^3} \end{aligned}$$

*Multiply (*) by* $\mu(t)$*:*

$$\begin{aligned} y' e^{-\frac{1}{3}t^3} - t^2 e^{-\frac{1}{3}t^3} y &= e^{-\frac{1}{3}t^3}(t-1) \\ \int \left( y e^{-\frac{1}{3}t^3} \right)' dt &= \int e^{-\frac{1}{3}t^3}(t-1)dt \\ e^{-\frac{1}{3}t^3} y &= \int e^{-\frac{1}{3}t^3}(t-1)dt \end{aligned}$$

*The integral above has non-elementary solution and thus requires numerical approx.*

## 2.2 Separable Equations

$$\frac{dy}{dx} = f(x, y) \tag{3}$$

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{4}$$

We can derive Eq.(4) from Eq.(3) by setting $M(x, y) = -f(x, y)$ and $N(x, y) = 1$. However, if M is a function of x only and N is a function of y only then Eq.(4) becomes

$$M(x) + N(y)\frac{dy}{dx} = 0$$

called separable. The differential form can be expressed as

$$M(x)dx + N(y)dy = 0$$

**Example 2.2.1**

$$\begin{aligned} y' &= \frac{x^2}{y(1+x^3)} \\ \frac{dy}{dx} &= \frac{x^2}{y(1+x^3)} \\ \int y dy &= \int \frac{x^2}{1+x^3} \\ \frac{1}{2}y^2 &= \frac{1}{3}\ln\left|1+x^3\right| + c_1 \\ 3y^2 - 2\ln\left|1+x^3\right| &= c \end{aligned}$$

*where* $c = 6c_1$. *We can see that the solution is implicit and general*

**Example 2.2.2**

$$y' = \frac{2x}{1+2y} \quad , \quad y(2) = 0$$

*Solve the IVP in explicit form (non-linear)*

$$\int (1+2y)dy = \int 2x dx$$
$$y + y^2 = x^2 + c$$

*Using the IC, we obtain:*

$$0 = 2^2 + c$$
$$c = -4$$
$$\Rightarrow y + y^2 = x^2 - 4$$

*Let's manipulate this equation so that it's in particular explicit form instead of particular implicit.*

$$y^2 + y + \frac{1}{4} = x^2 - 4 + \frac{1}{4}$$
$$\left(y + \frac{1}{2}\right)^2 = x^2 - \frac{15}{4}$$
$$y + \frac{1}{2} = \pm\sqrt{x^2 - \frac{15}{4}}$$
$$y = -\frac{1}{2} \pm \sqrt{x^2 - \frac{15}{4}}$$

*The IC would dictate the* $\pm$ *sign. Since* $y(2) = 0$, *then*

$$y = -\frac{1}{2} + \sqrt{x^2 - \frac{15}{4}}$$

*Let us also try to determine the interval in which the solution is defined. We need* $x^2 - \frac{15}{4} \geq 0 \Rightarrow x \geq \frac{\sqrt{15}}{2}$ *or* $x \leq \frac{-\sqrt{15}}{2}$. *Since* $y(2) = 0$ *is our IC,* $y > \frac{\sqrt{15}}{2}$ *is the interval we want to find*

**Example 2.2.3**

$$y' = 2x\sqrt{y-1} \qquad \text{(non-linear)}$$
$$\int \frac{dy}{\sqrt{y-1}} = \int 2x dx$$
$$2\sqrt{y-1} = x^2 + c$$
$$\sqrt{y-1} = \frac{1}{2}\left(x^2 + c\right)$$
$$y(x) = 1 + \frac{1}{4}\left(x^2 + c\right)^2$$

$\rightarrow$ *Singular solution:* $y(x) \equiv 0$.

***Note***: *There is no singular solution in linear DE*

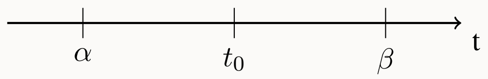


**Figure 1:** Linear case

**THEOREM 2.1** If the function p and g are continuous on an open interval $I : \alpha < t < \beta$ (Fig 1) containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation

$$y' + p(t)y = g(t)$$

for each t in I, and that also satisfies the initial condition

$$y(t_0) = y_0$$

where $y_0$ is an arbitrary prescribed initial value

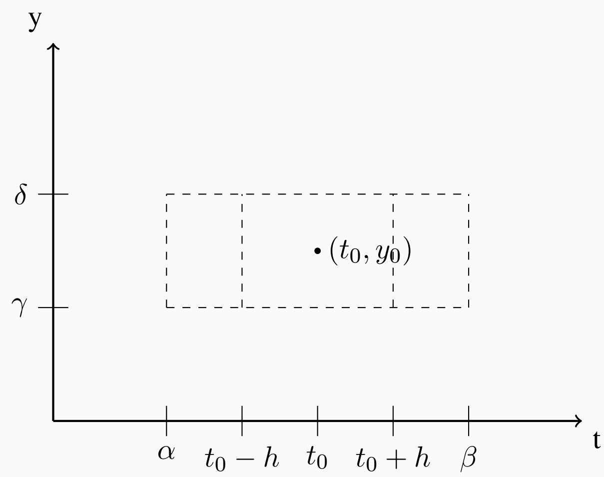


**Figure 2:** Nonlinear case

**THEOREM 2.2** Let the functions $f$ and $\frac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < t < \beta, \gamma < y < \delta$ containing the point $(t_0, y_0)$ (shown in Fig 2). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0$$

## 2.3 Exact Equation

$$\left(2xy^2 + 2y\right) + \left(2x^2y + 2x\right) y' = 0 \tag{*}$$

We can observe:

$$\psi(x,y) = x^2y^2 + 2xy$$
$$\frac{\partial\psi}{\partial x} = 2xy^2 + 2y$$
$$\frac{\partial\psi}{\partial y} = 2x^2y + 2x$$

So, we can rewrite (*) as

$$\frac{\partial}{\partial x}\left(x^2y^2 + 2xy\right) + \frac{\partial}{\partial y}\left(x^2y^2 + 2xy\right)\frac{dy}{dx} = 0$$

But notice, if we assume $y = y(x)$ recalling the chain rule of the LHS is $\frac{d}{dx}\left(x^2y^2 + 2xy\right) = 0$. This means:

$$x^2y^2 + 2xy = C$$

is also a solution to (*). More generally given:

$$M(x,y) + N(x,y)y' = 0 \tag{**}$$

if we can identify a function $\psi = \psi(x,y)$ such that

$$\frac{\partial\psi}{\partial x}(x,y) = M(x,y)$$
$$\frac{\partial\psi}{\partial y}(x,y) = N(x,y)$$

and such that $\psi(x,y) = c$ defines $y = \phi(x)$ implicitly as a differential of x. Then (**) becomes $\frac{d}{dx}\psi[x,\phi(x)] = 0$. Solution of (**) is given as:

$$\psi(x,y) = c$$

(**) is exact $\to M_y(x,y) = N_x(x,y)$. Proof in one direction from Clairaut's Theorem:

$$\frac{\partial\psi}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial\psi}{\partial y} = N(x,y)$$
$$M_y(x,y) = \psi_{xy} \quad \text{and} \quad N_x(x,y) = \psi_{yx}$$

**Note**: Clairaut's Theorem shows that $\psi_{xy} = \psi_{yx}$.

**Example 2.3.1**

$$\frac{dy}{dx} = -\frac{ax - by}{bx - cy}$$

*Rewrite it in differential form:*

$$(bx - cy)dy = -(ax - by)dx$$
$$(ax - by)dx + (bx - cy)dy = 0$$
$$M_y = -b \quad , \quad N_x = b$$
$$M_y \neq N_x$$

$\Rightarrow$ *Not exact!*

**Example 2.3.2**

$$\left(\frac{y}{x} + 6x\right) dx + (\ln x - 2)dy = 0, \quad x > 0$$

*Here,*

$$M_y = N_x = \frac{1}{x}$$

*which is exact. So,*

$$\exists \psi(x, y) \ni:$$

$$\psi_x = M(x, y) = \frac{y}{x} + 6x$$
$$\psi_y = N(x, y) = \ln x - 2$$

*Let's integrate* $\psi_x$ *with respect to x* $x$ *to find* $\psi$

$$\psi = \int \frac{y}{x} + 6x dx$$
$$\psi = y \ln |x| + 3x^2 + h(y)$$

*Then, in order to find* $h(y)$*, we need to use* $\psi_y$

$$\psi_y = \ln x + h'(y) = \ln x - 2$$
$$h'(y) = -2$$
$$h(y) = -2y + c$$

*Therefore,*

$$\psi(x, y) = y \ln x + 3x^2 - 2y + c \qquad \text{(choose } c = 0\text{)}$$
$$y \ln x + 3x^2 - 2y = c$$

**Example 2.3.3**

$$\left(ye^{2xy} + x\right) dx + bxe^{2xy} dy = 0 \tag{*}$$

*Find b so that (*) is exact.*
*Here,* $M(x, y) = ye^{2xy} + x$*, and* $N(x, y) = bxe^{2xy}$*. We need* $M_y = N_x$*,*

$$M_y = 2yxe^{2xy} + e^{2xy}$$
$$N_x = be^{2xy} + 2bxye^{2xy}$$

$\Rightarrow b = 1$
*Solve it using the similar method, we obtain:*

$$e^{2xy} + x^2 = c$$

**Using Integrating Factor**

$$M(x,y)dx + N(x,y)dy = 0$$

maybe exact, but what if it's not exact? Then, we need to utilize integrating factor.

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

Maybe $\exists\mu(x)$ or $\mu(y)$:

**Case 1** If $\frac{M_y - N_x}{N}$ is a function of x only, then $\mu = \mu(x)$ can be found by solving $\frac{d\mu}{dx} = \frac{M_y - N_x}{N} \cdot \mu$

**Case 2** If $\frac{N_x - M_y}{M}$ is a function of y only then $\mu = \mu(y)$ and can be found by solving $\frac{d\mu}{dy} = \frac{N_x - M_y}{M} \cdot \mu$

**Example 2.3.4**

$$ydx + \left(2xy - e^{-2y}\right) dy = 0$$

*which is certainly not exact. Notice:*

$$\frac{N_x - M_y}{M} = \frac{2y-1}{y}$$

*which is a function of y only.* $\exists\mu = \mu(y) \ni$ *:*

$$\frac{d\mu}{dy} = \frac{2y-1}{y} \cdot \mu$$

$$\int \frac{d\mu}{\mu} = \int \left(2 - \frac{1}{y}\right) dy$$

$$\ln|\mu| = 2y - \ln|y| \qquad \textit{(choose c = 0)}$$

$$|\mu| = e^{2y - \ln|y|}$$

$$\mu = \frac{e^{2y}}{y}$$

*Now, we can multiply the function by* $\mu$,

$$\frac{e^{2y}}{y} ydx + \left(\frac{e^{2y}}{y} 2xy - \frac{e^{2y}}{y} e^{2y}\right) dy = 0$$

*which is exact!. Therefore, there must exist* $\psi(x,y) \ni$ *:*

$$\psi_x = M(x,y) = e^{2y}$$

$$\psi_y = N(x,y) = 2xe^{2y} - \frac{1}{y}$$

$$\int \psi_x dx = xe^{2y} + h(y)$$

$$\psi_y = 2xe^{2y} + h'(y)$$

$$h(y) = -\ln|y|$$

$$\psi(x,y) = 2xe^{2y} - \ln|y| = c$$

## 2.4 Homogeneous Equation

$$\frac{dy}{dx} = f(x,y)$$

is homogeneous if f does not depend on x and y separately but depends only on the ration $\frac{y}{x}$ or $\frac{x}{y}$.

$$\implies \frac{dy}{dx} = F(\frac{y}{x})$$

**Example 2.4.1**

$$\frac{dy}{dx} = \frac{x+3y}{x-y}$$

*which is equal to*

$$\frac{dy}{dx} = \frac{1+\frac{3y}{x}}{1-\frac{y}{x}}$$

⇒ *homogeneous!*

**Example 2.4.2**

$$\begin{aligned}\frac{dy}{dx} &= \frac{y^4+2xy^3-3x^2y^2-2x^3y}{2x^2y^2-2x^3y-2x^4} \\ &= \frac{\frac{y^4}{x^4}+\frac{2y^3}{x^3}-\frac{3y^2}{x^2}-\frac{2y}{x}}{\frac{2y^2}{x^2}-\frac{2y}{x}-2} \\ &= F(\frac{y}{x})\end{aligned}$$

**Example 2.4.3**

$$\begin{aligned}\frac{dy}{dx} &= \frac{x^2+3y^2}{2xy} \\ &= \frac{1+3\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)}\end{aligned}$$

*Substituting* $v = \frac{y}{x} \to \quad \frac{dy}{dx} = x\frac{dv}{dx} + v$

$$\begin{aligned}v + x\frac{dv}{dx} &= \frac{1+3v^2}{2v} \\ x\frac{dv}{dx} &= \frac{1+3v^2-2v^2}{2v} \\ \int \frac{dx}{x} &= \int \frac{2v}{1+v^2}dv \\ \ln\left(1+v^2\right) &= \ln|x| + c_1 \\ \ln\left(\frac{1+v^2}{|x|}\right) &= c_1 \\ \ln\left(\frac{x^2+y^2}{|x^3|}\right) &= c_1 \\ \frac{x^2+y^2}{|x^3|} &= c_2 \qquad \text{where } c_2 = e^{c_1} \\ x^2+y^2 &= c_2|x|^3 \\ x^2+y^2-cx^3 &= 0\end{aligned}$$

## 2.5 Bernoulli Equation

$$\frac{dy}{dx} + p(x)y = q(x)y^n \tag{*}$$

Assume $p(x)$, $q(x)$ are continuous on $(a, b)$, $n \in \mathbb{R}$

If $n = 0$ or $n = 1$, then reduce to linear.

Dividing (*) by $y^{1-n}$ :

$$y^{-n}\frac{dy}{dx} + p(x)y^{1-n} = q(x)$$

Now, let $v = y^{1-n}$. This implies that $\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$. (*) then becomes:

$$\frac{1}{1-n}\frac{dv}{dx} + p(x)v = q(x)$$

**Example 2.5.1**

$$\frac{dr}{d\theta} = \frac{r^2 + 2r\theta}{\theta^2}$$

*Let's manipulate this equation to fit the template*

$$\frac{dr}{d\theta} - \frac{2}{\theta}r = \frac{1}{\theta^2}r^2$$

*Dividing it by $r^2$ :*

$$r^{-2}\frac{dr}{d\theta}\frac{-2}{\theta}r^{-1} = \frac{1}{\theta^2}$$

*Substituting $v = r^{1-2} = r^{-1} \to \frac{dv}{d\theta} = -r^{-2}\frac{dr}{d\theta}$*

$$-\frac{dv}{d\theta} - \frac{2}{\theta}v = \frac{1}{\theta^2}$$
$$\frac{dv}{d\theta} + \frac{2}{\theta}v = -\frac{1}{\theta^2}$$

*Using integrating factor:*

$$r(\theta) = \frac{\theta^2}{c - \theta}$$

*Singular solution: $r(\theta) \equiv 0$*

## 2.6 Autonomous ODEs / Population Dynamics

Recall:

$$\frac{dy}{dt} = f(y)$$

is autonomous.

**Exponential Growth**

Rate of change is proportional to the current population.

$$\frac{dy}{dt} = ry$$

$$r = \text{rate of growth} \quad (r > 0)$$

$$r = \text{rate of decay} \quad (r < 0)$$

**Logistic growth**

The growth rate is a function that depends on the current population

$$\frac{dy}{dt} = h(y)y$$

We want: $h(y) \approx r > 0$, where y is small.

$\to h(y)$ decreases as y grow larger.

$\to h(y) < 0$ when sufficiently large.

Simplest model:

$$h(y) = r - ay$$

$$a, r \in \mathbb{R}^+$$

$$\frac{dy}{dt} = (r - ay)y$$

*Note*: *Ansatz* is an educated guess

**Logistic Equation**:

r = intrinsic growth rate $\to \frac{dy}{dt} = r\left(1 - \frac{y}{k}y\right)$. This yields 2 constant solutions. $(k = \frac{r}{a}$ )

$$y = \phi() = 0 \quad \text{and} \quad y = \phi() = k$$

$\implies$ Equilibrium solution

**Case 1**

$$y = k \ : \ \text{sink (asymptotically stable)}$$

**Case 2**

$$y = 0 \ : \ \text{source (unstable solution)}$$

# 3 Second Order Linear Equations

## 3.1 Homogeneous Equations with Constant Coefficients

General form:

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right) \tag{*}$$

$\to$ linear if f is linear in $y$ and $y'$. We have:

$$y'' + p(t)y' + q(t)y = g(t)$$

Or

$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$

If $G(t) \equiv 0$ (forcing term), then equation is homogeneous.
IVP:

$$\text{IC:} \quad y(t_0) = y_0 \quad \text{and} \quad y'(t_0) = y_0'$$

Then,

$$ay'' + by' + cy = 0, \qquad a, b, c \in \mathbb{R}, \quad a \neq 0$$

Consider:

$$\begin{aligned} y'' - y &= 0 \\ y'' &= y \\ \Rightarrow \qquad y_1 = e^t \quad &, \quad y_2 = e^{-t} \end{aligned}$$

Thus,

$$y = c_1 e^t + c_2 e^{-t}$$

which is called the *principle of superposition*.

$$\begin{aligned} ay'' + by' + cy &= 0 \\ y(t) &= e^{rt} \\ y'(t) &= re^{rt} \\ y''(t) &= r^2 e^{rt} \end{aligned} \tag{**}$$

Substitute into (**):

$$\begin{aligned} ar^2 e^{rt} + bre^{rt} + ce^{rt} &= 0 \\ e^{rt}\left(ar^2 + br + c\right) &= 0 \\ ar^2 + br + c &= 0 \qquad \text{(characteristics equation)} \\ r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

**Example 3.1.1**

$$\begin{aligned} y'' + 3y + 2y &= 0 \\ r^2 + 3r - 2 &= 0 \qquad \textit{(characteristics equation)} \\ (r+2)(r+1) &= 0 \\ r_1 = -2, \quad r_2 &= -1 \\ y(t) &= c_1 e^{-t} + c_2 e^{-2t} \end{aligned}$$

**Example 3.1.2**

$$
\begin{aligned}
y'' - 2y' - 2y &= 0 \\
r^2 - 2r - 2 &= 0 \\
(r-1)^2 &= 3 \\
r &= 1 \pm \sqrt{3} \\
y(t) &= c_1 e^{(1-\sqrt{3})t} + c_2 e^{(1+\sqrt{3})t}
\end{aligned}
$$

**Example 3.1.3**

$$y'' + 8y' - 9y = 0, \qquad y(1) = 1, \;\; y'(1) = 0$$

$$
\begin{aligned}
r^2 + 8r + 9 &= 0 \\
r_1 = -9, \;\; r_2 &= 1 \\
y(t) &= c_1 e^t + c_2 e^{-9t} \\
y(t) &= k_1 e^{t-1} + k_2 e^{-9(t-1)}
\end{aligned}
$$

*where* $c_1 = k_1 e^{-1}$, $c_2 = k_2 e^9$. *Using the first IC, we have*

$$
\begin{aligned}
1 &= k_1 e^{t-1} + k_2 e^{-9(t-1)} \\
k_1 + k_2 &= 1
\end{aligned}
$$

*For the 2nd IC,*

$$
\begin{aligned}
0 &= k_1 e^{t-1} - 9k_2 e^{-9(t-1)} \\
0 &= k_1 - 9k_2 \\
k_1 = \frac{9}{10}, &\;\; k_2 = \frac{1}{10} \\
y(t) &= \frac{9}{10} e^{t-1} + \frac{1}{10} e^{-9(t-1)}
\end{aligned}
$$

So, overall we have different cases for r:

**Case 1 (Distinct Root)** Shown in Fig 3

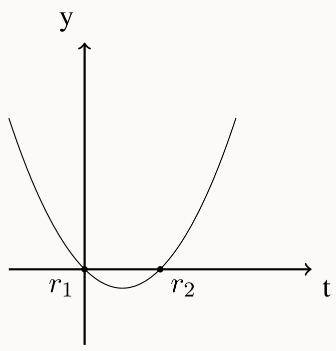


**Figure 3:** $b^2 - 4ac > 0$

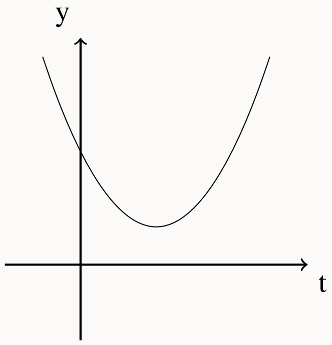


**Figure 4:** $b^2 - 4ac < 0$

**Case 2 (Complex Root)** Shown in Fig 4

**Case 3 (Repeated Root)** Shown in Fig 5

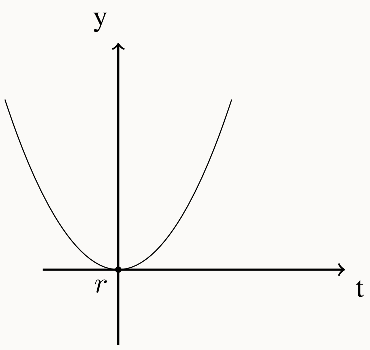


**Figure 5:** $b^2 - 4ac = 0$

## 3.2 Fundamental Solution of Linear Homogeneous Equation

**Differential Operator:**

$$L[\phi] = \phi'' + p\phi + q\phi$$

or

$$L = D^2 + pD + q, \quad \text{D: derivative operator}$$
$$y = \phi(t), \;\; L[y] = y'' + p(t)y' + q(t)y = 0 \tag{*}$$

**Example 3.2.1**

$$t(t-4)y'' + 3ty' + 4y = 2, \quad y(3) = 0$$

*Find the largest interval where we are guaranteed unique solution.*

*Standard form:*

$$y'' + \frac{3}{t-4}y' + \frac{4}{t(t-4)}y = \frac{2}{t(t-4)}$$

$$Dom(p(t)) = \{t | t \neq 4\}$$
$$Dom(q(t)) = \{t | t \neq 0, 4\}$$
$$Dom(g(t)) = \{t | t \neq 0, 4\}$$

$\rightarrow 0 < t < 4$

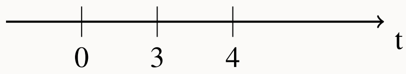


**Figure 6:** Interval of solution

Consider:

$$\text{IC: } \; y(t_0) = y_0, \; y'(t_0) = y'_0$$

$$c_1 y_1(t_0) + c_2 y_2(t_0) = y_0$$
$$c_1 y'_1(t_0) + c_2 y'_2(t_0) = y'_0$$
$$\implies c_1 = \frac{y_0 y'_2(t_0) - y'_0 y_2(t_0)}{y_1(t_0) y'_2(t_0) - y'_1(t_0) y_2(t_0)}$$
$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y'_0 & y'_2(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}}$$
$$c_2 = \frac{\begin{vmatrix} y_0 & y_1(t_0) \\ y'_0 & y'_1(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}}$$

$\rightarrow$ Wronskian determinant:

$$W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}$$

or

$$W = W(y_1, y_2)(t_0)$$

which leads to the following theorem

**THEOREM 3.1**

Suppose that $y_1$ and $y_2$ are two solutions of Eq.(*),

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

and that the Wronskian

$$W = y_1 y_2' - y_1' y_2$$

is not the zero at the point $t_0$ where the initial condition

$$y(t_0) = y_0, \;\; y'(t_0) = y_0'$$

are assigned. Then there is a choice of the constants $c_1$, $c_2$ for which $y = c_1 y_1(t) + c_2 y_2(t)$ satisfies the differential equation (*) and the initial condition above.

**THEOREM 3.2**

**Abel's Theorem**

If $y_1$ and $y_2$ are solutions of the differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

where p and q are continuous on an open interval I, then the Wronskian $W(y_1, y_2)(t)$ is given by

$$W(y_1, y_2)(t) = c \exp\left[-\int p(t)dt\right]$$

where c is a certain constant that depends on $y_1$ and $y_2$ but not on t. Further, $W(y_1, y_2)(t)$ either is zero for all t in I (if $c = 0$ ) or else is never zero in I (if $c \neq 0$ )

**Proof.**

$$y_1'' + p(t)y_1' + q(t)y_1 = 0 \tag{5}$$

$$y_2'' + p(t)y_2' + q(t)y = 0 \tag{6}$$

Multiply Eq.(5) by $-y_2$ and Eq.(6) by $y_1$ and add them, we obtain:

$$\left(y_1 y_2'' - y_1'' y_2\right) + p(t)\left(y_1 y_2' - y_1' y_2\right) = 0 \tag{7}$$

Let $W(t) = y_1 y_2' - y_1' y_2$. Then,

$$\begin{aligned} W'(t) &= [y_1' y_2' + y_1 y_2''] - [y_1' y_2' + y_1'' y_2] \\ &= y_1 y_2'' - y_1'' y_2 \end{aligned}$$

Then, Eq.(7) becomes:

$$\begin{aligned} W' + p(t)W &= 0 \\ \frac{W'}{W} &= -p(t) \\ \ln W &= -\int p(t)dt \\ W &= ce^{-\int p(t)dt} \end{aligned}$$

■

## 3.3 Complex Roots of the Characteristics Equation

Consider:

$$ay'' + by' + cy = 0$$

The characteristics equation is

$$ar^2 + br + c = 0$$

If $b^2 - 4ac < 0$, then

$$\begin{aligned} r_1 &= \lambda + i\mu \\ r_2 &= \lambda - i\mu \end{aligned}$$

So,

$$\begin{aligned} y_1(t) &= e^{(\lambda+i\mu)t} \\ y_2(t) &= e^{(\lambda-i\mu)t} \end{aligned}$$

Euler's Formula:

$$\begin{aligned} e^t &= \sum_{n=0}^{\infty} \frac{t^n}{n!}, \quad -\infty < t < \infty \\ e^{it} &= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \\ e^{it} &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^{2n-1}}{(2n-1)!} \\ e^{it} &= \cos t + i \sin t \\ e^{i\mu t} &= \cos(\mu t) + i \sin(\mu t) \\ e^{(\lambda+i\mu)t} &= e^{\lambda t} \left(\cos(\mu t) + i \sin(\mu t)\right) \end{aligned}$$

Real-valued solution:

$$\begin{aligned} y_1(t) + y_2(t) &= e^{\lambda t} \left(\cos(\mu t) + i \sin(\mu t)\right) + e^{\lambda t} \left(\cos(\mu t) - i \sin(\mu t)\right) \\ &= 2e^{\lambda t} \cos(\mu t) \end{aligned}$$

which is real. Also,

$$y_1(t) - y_2(t) = 2ie^{\lambda t}\sin(\mu t)$$

is real and $2i$ is actually just a number and can be thought as an acceptable real solution. Overall, we have:

$$y(t) = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t) \tag{*}$$

**Example 3.3.1**

$$3u'' - u' + 2u = 0, \quad \textit{IC:}\ \ u(0) = 2,\ \ u'(0) = 0$$

*Characteristics Equation:*

$$3r^2 - r + 2 = 0$$
$$r = \frac{1}{6} \pm \frac{\sqrt{23}}{6}i$$
$$\lambda = \frac{1}{6},\ \ \mu = \frac{\sqrt{23}}{6} u(t) = c_1 e^{\frac{t}{6}}\cos\frac{\sqrt{23}}{6}t + c_2 e^{\frac{t}{6}}\sin\frac{\sqrt{23}}{6}t$$

*Using ICs, we obtain:*

$$u(t) = 2e^{\frac{t}{6}}\cos\frac{\sqrt{23}}{6}t - \frac{2}{\sqrt{23}}e^{\frac{t}{6}}\sin\frac{\sqrt{23}}{6}t$$

*As $t \to \infty$, $u(t) \to \pm\infty$*

## 3.4 Repeated Roots

$$ay'' + by' + cy = 0$$

For repeated roots:

$$b^2 - 4ac = 0$$
$$r_1 = r_2 = \frac{-b}{2a}$$
$$y_1(t) = e^{\frac{-bt}{2a}}$$

But how do we find the $2^{\text{nd}}$ solution? $\to$ *Method of d'Alembert (1717-1783)*. Our ansatz would be:

$$y(t) = v(t)y_1(t)$$

**Example 3.4.1**

$$9y'' + 6y' + y = 0$$

$$9r^2 + 6r + 1 = 0$$
$$r_1 = r_2 = -\frac{1}{3} \quad \to \quad ce^{\frac{-t}{3}}$$

$$\begin{aligned} y(t) &= v(t)y_1(t) \\ &= v(t)e^{\frac{-t}{3}} \\ y'(t) &= v'e^{\frac{-t}{3}} - \frac{1}{3}ve^{\frac{-t}{3}} \\ y''(t) &= v''e^{\frac{-t}{3}} - \frac{2}{3}v'e^{\frac{-t}{3}} + \frac{1}{9}ve^{\frac{-t}{3}} \end{aligned}$$

*Substitute into the original DE, we have*

$$\begin{aligned} 9v''e^{\frac{-t}{3}} &= 0 \\ v'' &= 0 \\ v' &= c \\ v &= c_1t + c_2 \end{aligned}$$

$\implies \quad y_2(t) = te^{\frac{-t}{3}}$

Generalize:

Assume: $b^2 - 4ac = 0$. So,

$$\begin{aligned} y_1(t) &= e^{\frac{-bt}{2a}} \\ y &= v(t)e^{\frac{-bt}{2a}} \\ y' &= v'e^{\frac{-bt}{2a}} - \frac{b}{2a}ve^{\frac{-bt}{2a}} \\ y'' &= v''e^{\frac{-bt}{2a}} - \frac{b}{2a}v'e^{\frac{-bt}{2a}} + \frac{b^2}{4a^2}ve^{\frac{-bt}{2a}} \end{aligned}$$

Substitute into $ay'' + by' + cy = 0$

$$\begin{aligned} \left\{a[y''] + b[y'] + cv\right\} e^{\frac{-bt}{2a}} &= 0 \\ av'' + (-b + b)v' + \left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right) v &= 0 \\ v'' &= 0 \\ v' &= c_1 \\ v &= c_1t + c_2 \end{aligned}$$

Thus,

$$y(t) = c_1te^{\frac{-bt}{2a}} + c_2e^{\frac{-bt}{2a}}$$

and the Wronskian is

$$\begin{aligned} W &= \begin{vmatrix} e^{\frac{-bt}{2a}} & te^{\frac{-bt}{2a}} \\ \frac{-b}{2a}e^{\frac{-bt}{2a}} & \left(1 - \frac{-bt}{2a}\right) e^{\frac{-bt}{2a}} \end{vmatrix} \\ &= e^{\frac{-bt}{a}} \quad \neq 0 \;\; \forall t \end{aligned}$$

**Example 3.4.2**

$$16y'' + 24y' + 9y = 0$$

*Char. Equation:*

$$16r^2 + 24r + 9 = 0$$
$$r = -\frac{3}{4}$$
$$y(t) = c_1 t e^{\frac{-3t}{4}} + c_2 e^{\frac{-3t}{4}}$$

**Note:**

If

$$r_1 = r_2 = 0$$

Then,

$$y'' = 0$$
$$y = c_1 t + c_2$$

## 3.5 Method of Underdetermined Coefficients

$$L[y] = y'' + p(t)y' + q(t)y = g(t) \quad (*)$$
$$L[y] = y'' + p(t)y' + q(t)y = 0 \quad (**)$$

**THEOREM 3.3**

If $Y_1$ and $Y_2$ are 2 solutions of (*), then their difference $Y_1$ - $Y_2$ is a solution of corresponding homogeneous equation

$$L[Y_1] - L[Y_2] = 0$$

If $y_1$ and $y_2$ are a fundamental set of solution, then

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t)$$

where $c_1$ and $c_2$ are certain constants.

**THEOREM 3.4**

The general solution of the nonhomogeneous equation (*) can be written in the form

$$y = \phi(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

where $y_1$ and $y_2$ are a fundamental set of solutions of the corresponding homogeneous equation (**) , $c_1$ and $c_2$ are arbitrary constants, and Y is some specific solution of the nonhomogeneous equation (*)

* g(t) is a polynomial, exponential, sin, cos, etc (not a ratio of some functions or tan)

**Example 3.5.1**

$$y'' - 5y' + 6y = -5e^{-t} \tag{7}$$

*1. Solve the corresponding homogeneous equation*

$$r^2 - 5r + 6 = 0$$
$$r_1 = 3, \quad r_2 = 2$$
$$y_c(t) = c_1 e^{3t} + c_2 e^{2t} \quad : \quad \textit{complementary solution}$$

*2. Find a particular solution*
*Ansatz:* $Y(t) = Ae^{-t}$

$$Y'(t) = -Ae^{-t}$$
$$Y''(t) = Ae^{-t}$$

*Substitute into Eq.(7)*

$$Ae^{-t} + 5Ae^{-t} + 6Ae^{-t} = -5e^{-t}$$
$$A = -\frac{5}{12}$$
$$Y(t) = -\frac{5}{12}e^{-t}$$

*3. Put everything together*

$$y(t) = c_1 e^{3t} + c_2 e^{2t} - \frac{5}{12}e^{-t}$$

**Example 3.5.2**

$$y'' + 2y' + 5y = 3\sin(2t)$$

*Char. Equation:*

$$r^2 + 2r + 5 = 0$$
$$r = -1 \pm 2i$$
$$y_c(t) = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$$

*Ansatz :* $Y(t) = A\sin 2t + B\cos 2t$ *(**note**:* $Y(t) = A\sin 2t$ *doesn't work)*

$$Y'(t) = 2A\cos 2t - 2B\sin 2t$$
$$Y''(t) = -4A\sin 2t - 4B\cos 2t$$

*Substitute into the original equation, we get:*

$$-4A\sin 2t - 4B\cos 2t + 4A\cos 2t - 4B\sin 2t + 5A\sin 2t + 5B\cos 2t = 3\sin 2t$$
$$(A - 4B)\sin 2t + (4A + B)\cos 2t = 3\sin 2t$$

*So,*

$$\begin{cases} A - 4B = 3 \implies A = \frac{3}{17}, \quad B = \frac{-12}{17} \\ 4A + B = 0 \end{cases}$$

$$y(t) = c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t + \frac{3}{17} \sin 2t - \frac{12}{17} \cos 2t$$

**Example 3.5.3**

$$2y'' + 3y' + y = t^2 + 3\sin t \tag{*}$$

*Solve char. equation*

$$2r^2 + 3r + 1 = 0$$
$$r_1 = -\frac{1}{2}, \quad r_2 = -1$$
$$y_c(t) = c_1 e^{\frac{-t}{2}} + c_2 e^{-t}$$

$$Y(t) = Y_1(t) + Y_2(t)$$
$$g(t) = g_1(t) + g_2(t)$$

*where* $g_1(t) = t^2$ *and* $g_2(t) = 3\sin t$*. For* $g_1(t)$ *:*

$$Y_{p_1(t)} = At^2 + Bt + C$$
$$Y'_{p_1(t)} = 2At + B$$
$$Y''_{p_1(t)} = 2A$$

*Sub into (*) but ignore* $3\sin t$

$$2(2A) + 3(2At + B) + At^2 + Bt + C = t^2$$

$$\begin{cases} A = 1 \\ B = -6 \\ C = 14 \end{cases}$$

$$Y_{p_1(t)} = t^2 - 6t + 14$$

*For* $p_2(t)$ *:*

$$Y_{p_2(t)} = D\sin t + E\cos t$$
$$Y'_{p_2(t)} = D\cos t - E\sin t$$
$$Y''_{p_2(t)} = -D\sin t - E\cos t$$

*Sub into (*) and ignore* $t^2$

$$\begin{cases} D = -\frac{3}{10} \\ E = -\frac{9}{10} \end{cases}$$

$$\begin{aligned} y(t) &= y_c + Y_{p_1} + Y_{p_2} \\ &= c_1 e^{-\frac{t}{2}} + c_2 e^{-t} + t^2 - 6t + 14 - \frac{3}{10}\sin t - \frac{9}{10}\cos t \end{aligned}$$

**Note:** If $Y(t)$ ansatz duplicates a term in $y_c$ then modify the ansatz by multiplying it by t. If doesn't work, then keep going with $t^2, t^3, \ldots$

## 3.6 Variation of Parameters

$$y'' + 4y = 3\csc 2t, \qquad 0 < t < \frac{\pi}{2}$$

can't use undetermined coefficients. For $y_c$ :

$$\begin{aligned} y'' + 4y &= 0 \\ r^2 + 4 &= 0 \\ r &= \pm 2i \end{aligned}$$

$$y_c = c_1 \cos 2t + c_2 \sin 2t$$

Basic idea here is to replace $c_1$ and $c_2$ with $u_1(t)$ and $u_2(t)$.

$$y = u_1(t)\cos 2t + u_2 \sin 2t$$

2 unknowns but only 1 equation $\implies$ underdetermined system. So Lagrange imposed another restriction

$$y'(t) = -2u_1 \sin 2t + u_1' \cos 2t + 2u_2 \cos 2t + u_2' \sin 2t$$

We have

$$u_1'(t)\cos 2t + u_2'(t)\sin 2t = 0 \tag{**}$$

So,

$$\begin{aligned} y' &= -2u_1 \sin 2t + 2u_2 \cos 2t \\ y'' &= -4u_1 \cos 2t - 2u_1' \sin 2t - 4u_2 \sin 2t + 2u_2' \cos 2t \end{aligned}$$

Sub into the original DE:

$$-2u_1' \sin 2t + 2u_2' \cos 2t = 3\csc 2t \tag{***}$$

Lagrange viewed (**) and (***) as a pair of linear algebraic equations for 2 unknowns

$$\begin{aligned} u_2' &= \frac{3}{2}\cot 2t \\ u_1' &= -\frac{3}{2} \\ u_1(t) &= -\frac{3}{2}t + c_1 \\ u_2(t) &= \frac{3}{4}\ln(\sin 2t) + c_2 \end{aligned}$$

$$\begin{aligned} y(t) &= \left(-\frac{3}{2}t + c_1\right)\cos 2t + \left(\frac{3}{4}\ln(\sin 2t) + c_2\right)\sin 2t \\ &= c_1 \cos 2t + c_2 \sin 2t - \frac{3}{2}t\cos 2t + \frac{3}{4}\sin 2t \ln(\sin 2t) \end{aligned}$$

$$y'' + p(t)y' + q(t)y = g(t)$$

where p, q, r are continuous. Assume:

$$y_c(t) = c_1y_1(t) + c_2y_2(t)$$

Then, our ansatz is $y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ and

$$y' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$$
$$u_1'y_1 + u_2'y_2 = 0$$
$$y' = u_1y_1' + u_2y_2'$$
$$y'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''$$

After lots of algebra,

$$u_1[y_1'' + py_1' + qy_1] + u_2[y_2'' + py_2' + qy_2] + u_1'y_1' + u_2'y_2' = g(t)$$

Since the first two term equal to 0, $u_1'y_1' + u_2'y_2' = g(t)$. We can deduce:

$$u_1'(t) = \frac{-y_2(t)g(t)}{W(y_1, y_2)(t)}$$
$$u_2'(t) = \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} \implies \begin{cases} u_1 & = -\int \frac{y_2g}{W}dt + C_1 \\ u_2 & = \int \frac{y_1g}{W}dt + C_2 \end{cases}$$

So,

$$Y(t) = -y_1\int \frac{y_2g}{W}dt + y_2\int \frac{y_1g}{W}dt$$

**Example 3.6.1**

$$y'' - 2y' + y = \frac{e^t}{1+t^2}$$

*Homogeneous Equation:*

$$y'' - 2y' + y = 0$$
$$r^2 - 2r + 1 = 0$$
$$r_1 = r_2 = 1$$
$$y_c = c_1te^t + c_2e^t$$

*where* $y_1 = te^t$ *and* $y_2 = e^t$ *and* $g(t) = \frac{e^t}{1+t^2}$. *The Wronskian determinant can be computed:*

$$W = \begin{vmatrix} te^t & e^t \\ e^t + te^t & e^t \end{vmatrix} = -e^{2t}$$

$$Y(t) = -te^t\int \frac{e^t\left(\frac{e^t}{1+t^2}\right)}{-e^{2t}}dt + e^t\int \frac{te^t\left(\frac{e^t}{1+t^2}\right)}{-e^{2t}}dt$$
$$= te^t\arctan t - e^t\left(\frac{1}{2}\ln\left(1+t^2\right)\right)$$

*Our final solution is*

$$y(t) = c_1te^t + c_2e^t + te^t\arctan t - \frac{1}{2}e^t\ln(1+t^2)$$

# 4 Series Solutions of Second Order Linear Equations

## 4.1 Review of Power Series

Power series:

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

converges at a point x if

$$\lim_{m \to \infty} \sum_{n=0}^{m} a_n (x - x_0)^n$$

exists for that x. It trivially converge for $x = x_0$.

$$\to \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

converges absolutely at point x if

$$\sum_{n=0}^{\infty} |a_n (x - x_0)^n| \quad converges$$

$\exists \rho \in \mathbb{R}$ (radius of convergence) such that $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ converges absolutely for $|x - x_0| < \rho$ and diverge for $|x - x_0| > \rho$

$\rho = 0$ only at $x_0$ if converges for all x and $\rho = \infty$. If $\rho > 0$ then the interval $|x - x_0| < \rho$ is called an interval of convergence.

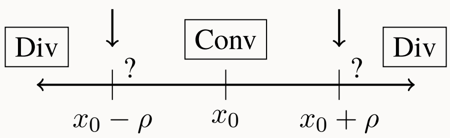


**Figure 7:** Interval of Convergence

**Example 4.1.1**

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2 (x+2)^n}{3^n}$$

*Ratio Test:*

$$\lim_{n \to \infty} \left| \frac{(n+1)^2 (x+2)^{n+1} 3^n}{3^{n+1} n^2 (x+2)^n} \right| = \frac{1}{3} |x+2|$$

*for the series to be absolutely convergent,*

$$\frac{1}{3}|x+2| < 1$$

$$-3 < x + 2 < 3$$

$$-5 < x < 1$$

*So,* $\rho = 3$. *For* $x = -5$ *:*

$$\sum_{n=0}^{\infty} \frac{(-1)^n n^2 (-3)^n}{3^n} = \sum_{n=1}^{\infty} n^2$$

*which is divergent. For* $x = 1$ *:*

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2 3^n}{3^n} = \sum_{n=1}^{\infty} (-1)^n n^2$$

*which is also divergent. Therefore, interval of convergence is* $(-5, 1)$.

We can observe that

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

converges to $f(x)$ and likewise

$$\sum_{n=0}^{\infty} b_n (x - x_0)^n$$

converges to $g(x)$ for $|x - x_0| < \rho$. Then, $g(x) \pm f(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x - x_0)^n$. Then,

$$f(x)g(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

where $c_n = \sum_{k=1}^{n} a_k b_{n-k}$ (Cauchy product)

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$
$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$$
$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}$$

Taylor Series for function f about $x - x_0$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n, \quad \rho > 0$$

f is analytic at $x = x_0$

**Example 4.1.2**

$$f(x) = x^{\frac{7}{3}}$$

*is not analytic at* $x_0 = 0$ *since* $f''(0)$ *d.n.e*

$$f(x) = |x - 1|$$

*is not analytic at* $x_0 = 1$ *since* $f'(x)$ *d.n.e*

**Reindexing:**

**Example 4.1.3**

$$\begin{aligned} x\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=1}^{\infty} n(n+1)a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} n(n+1)a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x_n \\ &= \sum_{n=0}^{\infty} \left[n(n+1)a_{n+1} + a_n\right] x^n \end{aligned}$$

## 4.2 Series Solutions Near An Ordinary Point (Part I)

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

P, Q, R are polynomial with no common factors.

- $x_0$ where $P(x_0) \neq 0$ is called an ordinary point
- $x_0$ where $P(x_0) = 0$ is called a singular point

Consider:

$$y'' + p(x)y' + q(x)y = 0$$

Ansatz: $y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ and assume series converges $|x - x_0| < \rho$ where $\rho > 0$. Let's look at:

$$y'' + xy' + 2y = 0, \qquad x_0 = 0 \tag{*}$$

$P(x) = 1 \;\; \forall x$, so $x_0$ is ordinary point. Therefore, there exists $\rho > 0$ such that $|x - 0| < \rho$ converges. Assume:

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x_n \\ y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \end{aligned}$$

Substitute into (*):

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} na_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x_n = 0$$
$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} na_n x^n + \sum_{n=0}^{\infty} 2a_n x_n = 0$$
$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + (n+2)a_n\right]x^n = 0$$
$$(n+2)(n+1)a_{n+2} + (n+2)a_n = 0$$

So, we obtain the following *recurrence relation*:

$$a_{n+2} = \frac{-a_n}{n+1}, \quad n = 0, 1, 2, \ldots$$

Let $a_0 = 1$, $a_1 = 0$ to generate one solution $y_1(x)$. So $a_1 = a_3 = a_5 = \ldots = 0$.

- For $n = 0 : a_2 = -a_0 = -1$
- For $n = 2 : a_4 = \frac{(-1)(-1)}{1\cdot 3} = \frac{1}{3}$
- For $n = 4 : a_6 = \frac{-a_4}{4+1} = \frac{-1}{1\cdot 3\cdot 5} = -\frac{1}{15}$
- For $n = 6 : a_8 = -\frac{96}{6+1} = \frac{1}{1\cdot 3\cdot 5\cdot 7} = \frac{1}{105}$

Thus,

$$a_{2n} = \frac{(-1)^n}{1\cdot 3\cdot 5\ldots(2n-1)}$$

and

$$y_1(x) = 1 - \frac{x^2}{1} + \frac{x^4}{1\cdot 3} - \frac{x^6}{1\cdot 3\cdot 5} + \frac{x^8}{1\cdot 3\cdot 5\cdot 7} + \ldots$$
$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n-1)!!}$$

For the second solution, let $a_0 = 0$ and $a_1 = 1 \to \quad a_0 = a_2 = a_4 = \ldots = 0$.

- $n = 1 : a_3 = -\frac{a_1}{2} = \frac{-1}{1\cdot 2}$
- $n = 3 : a_5 = \frac{-a_3}{4} = \frac{1}{1\cdot 2\cdot 4}$
- $n = 5 : a_7 = \frac{-a_5}{6} = \frac{-1}{1\cdot 2\cdot 4\cdot 6}$

Thus,

$$a_{2n+1} = \frac{(-1)^n}{2\cdot 4\cdot 6\ldots(2n)}$$

and

$$y_2(x) = x - \frac{x^3}{1\cdot 2} + \frac{x^5}{1\cdot 2\cdot 4} - \frac{x^7}{1\cdot 2\cdot 4\cdot 6} + \ldots$$
$$= x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!!}$$

**Example 4.2.1**

$$xy'' + y' + xy = 0, \quad x_0 = 1 \qquad (*)$$

*$x_0 = 1$ is an ordinary point. Assume:*

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n$$
$$y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

*Sub into (*)*

$$x \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} + \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + x \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

*Trick: $x = 1 + (x-1)$*

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} + \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-1} + \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + \sum_{n=0}^{\infty} a_n (x-1)^n + \sum_{n=0}^{\infty} a_n (x-1)^{n+1} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n + \sum_{n=1}^{\infty} (n+1) n a_{n+1} (x-1)^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n + \sum_{n=0}^{\infty} a_n (x-1)^n + \sum_{n=1}^{\infty} a_{n-1} (x-1)^n = 0$$

*We'll handle $n = 0$ separately*

$$\sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} + (n+1) n a_{n+1} + (n+1) a_{n+1} + a_n + a_{n-1}\right] (x-1)^n = 0$$

*So,*

$$a_{n+2} = \frac{-\left[(n+1)^2 a_{n+1} + a_n + a_{n-1}\right]}{(n+1)(n+2)} \quad \textit{for} \quad n \in \mathbb{Z}^+$$

*depends on 3 prior terms (very difficult to solve). For $n = 0$,*

$$(n+2)(n+1) a_{n+2} + (n+1) a_{n+1} + a_n = 0$$
$$2a_2 + a_1 + a_0 = 0$$
$$a_2 = \frac{-(a_1 + a_0)}{2}$$

*Take $a_0 = 1$ and $a_1 = 0$ to generate $y_1(x)$*

- $a_2 = -\frac{1}{2}$
- $a_3 = \frac{-(2^2 a_2 + a_1 + a_0)}{2 \cdot 3} = \frac{1}{6}$
- $a_4 = \frac{-(3^2 a_3 + a_2 + a_1}{3 \cdot 4} = -\frac{1}{12}$
- $a_5 = \frac{-(4^2 a_4 + a_3 + a_2)}{4 \cdot 5} = \frac{1}{12}$

$$\begin{aligned}y_1(x) &= a_0(x-1)^0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 \\ &= 1 - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{12}(x-1)^4 + \dots\end{aligned}$$

*To generate $y_2(x)$, let $a_0 = 0$ and $a_1 = 1$. Then,*

- $a_2 = -\frac{1}{2}$
- $a_3 = \frac{1}{6}$
- $a_4 = -\frac{1}{6}$

$$y_2(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{6}(x-1)^4 + \dots$$

## 4.3 Series Solutions Near An Ordinary Point (Part II)

$$P(x)y'' + Q(x)y' + R(x)y = 0 \qquad (*)$$

P, Q, R are polynomials. Assume there exists a solution $y = \phi(x)$

$$y = \phi(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n \qquad (**)$$

converges when $|x - x_0| < \rho, \quad \rho > 0$. Take (**) differentiate m times and set $x = x_0$ we get:

$$m!a_m = \phi^{(m)}(x_0)$$

Recall that Taylor Series Expansion:

$$a_m = \frac{f^{(m)}(x_0)}{(m!)}$$

and use this to compute $a_n$ in (**). If $y = \phi(x)$ is a solution to (**) satisfies ICs:

$$\begin{aligned}y(x_0) &= y_0 \\ y'(x_0) &= y_0'\end{aligned}$$

Then $a_0 = y_0$ and $a_1 = y_0'$ since

$$\begin{aligned}a_0 &= \frac{\phi(x_0)}{0!} = y_0 \\ a_1 &= \frac{\phi'(x_0)}{1!} = y_0'\end{aligned}$$

Since $\phi$ is a solution to (*),

$$\begin{aligned}P(x)\phi''(x) + Q(x)\phi'(x) + R(x)\phi(x) &= 0 \\ \phi''(x) + \frac{Q(x)}{P(x)}\phi'(x) + \frac{R(x)}{P(x)}\phi(x) &= 0 \\ \phi''(x) + p(x)\phi'(x) + q(x)\phi(x) &= 0 \\ \phi''(x) &= -p(x)\phi'(x) - q(x)\phi(x)\end{aligned}$$

Set $x = x_0$

$$\phi''(x_0) = -p(x_0)\phi'(x_0) + q(x_0)\phi(x_0)$$

Since $\phi''(x_0) = 2!a_n$

$$a_2 = \frac{-p(x_0)a_1 - q(x_0)a_0}{2!}$$
$$a_3 = \frac{-2!p(x_0)a_2 - [p'(x_0) + q(x_0)]a_1 - q_1'(x_0)\phi(x_0)}{3!}$$

$\implies$ There exists many derivative of p and q evaluated at $x_0$

$$p(x) = \sum_{n=0}^{\infty} p_n(x - x_0)^n$$
$$q(x) = \sum_{n=0}^{\infty} q_n(x - x_0)^n$$

If p and q are analytic at $x_0$ then $x_0$ is an ordinary point, otherwise it's a singular point.

**THEOREM 4.1**

If $x_0$ is an ordinary point of (*), then the general solution of (*) is

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0y_1(x) + a_1y_2(x)$$

where $a_0$ and $a_1$ are arbitrary and $y_1$ and $y_2$ are linearly independent.

Further: $\rho$ for each of the series solution, $y_1$ and $y_2$ is at least as large as the minimum of $\rho$ of the series of p and q.

**From Complex Analysis**

$$\rho_p = \text{dist}\,\{x_0, \ \text{the nearest zero of p}\}$$

**Example 4.3.1**

$$(1 + x^3)y'' + 4xy' + y = 0, \qquad x_0 = 0, \ \ x_0 = 2$$

*Here:* $P(x) = 1 + x^3$

$P(x) = 0 \to \quad x = -1, \ \ \frac{1}{2}, \ \ \frac{1}{2} \pm \frac{i\sqrt{3}}{2}$

- *For* $x_0 = 0$ *:*

$$dist\ \left\{0, \frac{1}{2} \pm \frac{i\sqrt{3}}{2}\right\} = 1$$
$$dist\ \ \{0, -1\} = 1$$
$$\implies \rho = 1$$

- *For* $x_0 = 2$ *:*

$$dist\ \ \{2, -1\} = 3$$
$$dist\ \left\{2, \frac{1}{2} \pm \frac{i\sqrt{3}}{2}\right\} = \sqrt{3}$$

$\implies \rho = \sqrt{3}$

**Example 4.3.2**

$$(\cos x)y'' + xy' - 2y = 0, \quad x_0 = 0$$

*$x_0$ is an ordinary point. Know:*

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \forall x$$

*Assume:*

$$y = \sum_{n=0}^{\infty} a_n x^n$$
$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

*Substitute into (*)*

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \cdot \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} a_n x^n n - \sum_{n=0}^{\infty} 2 a_n x^n = 0$$

*Let's look at the product of the two series (first term)*

- $x^0$:

$$(2a_2 - 2a_0)x^0$$

- $x^1$ :

$$n = 0 \quad \textit{for the 1st factor and} \quad n = 1 \quad \textit{for the second one}$$
$$(6a_3 - a_1)x^1$$

- $x^2$ :

$$n = 0 \quad \textit{for the 1st factor and } n = 2 \textit{ for the second one}$$
$$\underline{\textit{or}} \quad n = 1 \quad \textit{for the first factor and } n = 0 \textit{ for the second one}$$
$$(12a_4 - a_2)x^2$$

- $x^3$ :

$$n = 0, \quad n = 3 \to \quad 20a_5$$
$$n = 1, \quad n = 1 \to \quad -3a_3$$
$$(20a_5 - 2a_3)x^3$$

- $x^4$ :

$$n=0,\ \ n=4\rightarrow\quad 30a_6$$
$$n=2,\ \ n=0\rightarrow\quad \frac{1}{12}a_2$$
$$n=1,\ \ n=2\rightarrow\quad -4a_4$$
$$(30a_6+\frac{1}{12}a_2-4a_4)x^4$$

- $x^5$ :

$$n=2,\ \ n=1\rightarrow\quad \frac{1}{4}a_3$$
$$n=1,\ \ n=3\rightarrow\quad -7a_5$$
$$n=0,\ \ n=5\rightarrow\quad 42a_7$$
$$(42a_7+\frac{1}{4}a_3-7a_5)x^5$$

*Since the RHS is 0, all the coefficient must be 0.*

$$2a_2-2a_0=0 \implies \quad a_2=a_0$$
$$6a_3-a_1=0 \implies \quad a_3=\frac{1}{6}a_1$$
$$12a_4-a_2=0 \implies \quad a_4=\frac{a_0}{12}$$
$$20a_5-2a_3=0 \implies \quad a_5=-\frac{1}{60}a_1$$
$$30a_6+\frac{1}{12}a_2-4a_4=0 \implies \quad a_6=\frac{a_0}{120}$$
$$42a_7+\frac{1}{4}a_3-7a_5=0 \implies \quad a_7=\frac{1}{560}a_1$$

*For $y_1(x)$, let $a_0=1$, $a_1=0$*

$$a_2=1,\ \ a_3=a_5=a_7=\ldots=0$$
$$a_4=\frac{1}{12},\ \ a_6=\frac{1}{120}$$
$$y_1(x)=1+x^2+\frac{1}{12}x^4+\frac{1}{120}x^6+\ldots$$

*For $y_2(x)$, let $a_0=0$, $a_1=1$*

$$a_2=a_4=a_6=\ldots=0$$
$$a_3=\frac{1}{6},\ \ a_5=\frac{1}{60},\ \ a_7=\frac{1}{560}$$
$$y_2(x)=x+\frac{1}{6}x^3+\frac{1}{60}x^5+\frac{1}{560}x^7+\ldots$$

# 5 Laplace Transform

## 5.1 Definition of Laplace Transform

**Operational Calculus:**

$$F(s) = \int_{\alpha}^{\beta} K(s,t) f(t) dt$$

Transform: $f \to F$

$$K(s,t) = \quad \text{Kernel of the transformation}$$

$\to$ Laplace Transform:

$$\mathscr{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t) dt$$
$$K(s,t) = e^{-st}, \quad s \in \mathbb{C}$$
$$f(t), \quad t \geq 0$$

There is a diagram here that I still need to learn how to draw in tikz

**THEOREM 5.1**

Suppose:

1. $f$ is piecewise continuous on $0 \leq t \leq A$ for all $A \in \mathbb{R}$
2. $|f(t)| \leq ke^{at}$ where $t \geq M$; $a \in \mathbb{R}$; $K, M \in \mathbb{R}^+$ (exponential order)

Then, the Laplace Transform $\mathscr{L}\{f(t) = F(s)\}$ defined by $\int_0^{\infty} e^{-st} f(t) dt$ exists for $s \geq a$.

$\mathscr{L}$ is a linear operator ($\mathscr{L}^{-1}$ is a linear operator as well). Suppose that $f_1$ and $f_2$ whose Laplace transform exist $\mathscr{L}\{c_1 f_1(t) + c_2 f_2(t)\}$ = $\int_0^{\infty} e^{-st} \left[c_1 f_1(t) + c_2 f_2(t)\right] dt$ which is equal to:

$$= c_1 \int_0^{\infty} e^{-st} f_1(t) dt + c_2 \int_{-}^{\infty} e^{-st} f_2(t) dt$$
$$= c_1 \mathscr{L}\{f_1(t)\} + c_2 \mathscr{L}\{f_2(t)\}$$

## 5.2 IVP

$\mathscr{L}\{f'\}$ related to $\mathscr{L}\{f\}$ in a simple way.

**THEOREM 5.2**

Suppose $f$ is a continuous and $f'$ is piecewise continuous on $0 \leq t \leq A$. Also suppose $\exists k, a, M \in \mathbb{R}$ such that

$$|f(t)| \leq Ke^{at} \quad \text{for} \quad t \geq M$$

Then, $\mathscr{L}\{f'(t)\}$ exists for $s > a$ and

$$\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t) - f(0)\}$$
$$\mathscr{L}\{f''(t)\} = s^2 \mathscr{L}\{f(t)\} - sf(0) - f'(0)$$

**Corollary** *Suppose $f, f', f'' \dots f^{(n-1)}$ are continuous and $f^{(n)}$ is piecewise continuous on $0 \le t \le A$. Suppose $\exists k, a, M \in \mathbb{R}$ such that*

$$\begin{gathered} |f(t)| \le ke^{at}, \quad |f'(t)| \le ke^{at}, \dots \\ \left| f^{(n-1)}(t) \right| \le ke^{at}, \quad t \ge M \end{gathered}$$

*Then, $\mathscr{L}\{f^{(n)}(t)\}$ exists for $s > a$ and we can generalize*

$$\begin{gathered} \mathscr{L}\{f^{(n)}(t)\} = s^n \mathscr{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) \dots - s f^{(n-2)}(0) - f^{(n-1)}(0) \\ \mathscr{L}^{-1}\{y(s)\} = \phi(t) = y(t) \end{gathered}$$

*Note*: we can use partial fraction to find $\mathscr{L}^{-1}$. If we know complex analysis:

$$y(t) = \frac{1}{2\pi i} \int_{y+i\infty}^{y-i\infty} e^{st} Y(s) ds, \quad t > 0, \quad y \in \mathbb{R}$$

There exists a 1-1 correspondence between $f$ and $F$.

**Example 5.2.1**

$$\textit{Find} \quad \mathscr{L}^{-1}\{F(s)\}, \quad F(s) = \frac{2}{s^2 + 3s - 4}$$

$$\begin{aligned} F(s) = \frac{2}{(s+4)(s-1)} &= \frac{A}{s+4} + \frac{B}{s-1} \\ &= \frac{-\frac{2}{5}}{s+4} + \frac{\frac{2}{5}}{s-1} \\ &= \frac{2}{5}\left(\frac{1}{s-1}\right) - \frac{2}{5}\left(\frac{1}{s+4}\right) \end{aligned}$$

*Thus,*

$$f(t) = \frac{2}{5} e^t - \frac{2}{5} e^{-4t}$$

**Example 5.2.2**

$$\textit{Find} \quad \mathscr{L}^{-1}\{F(s)\}, \quad F(s) = \frac{8^2 - 4s + 12}{s(s^2 + 4)}$$

$$\begin{aligned} F(s) = \frac{3}{5} + \frac{5s - 4}{s^2 + 4} &= \frac{3}{s} + \frac{5s}{s^2 + 4} - \frac{4}{s^2 + 4} \\ &= 3\left(\frac{1}{s}\right) + 5\left(\frac{s}{s^2 + 2^2}\right) - 2\left(\frac{2}{s^2 + 2^2}\right) \\ f(t) &= 3 + 5\cos 2t - 2\sin 2t \end{aligned}$$

**Example 5.2.3**

$$y^{(4)} - y = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 1, \quad y'''(0) = 0$$

*Let $\mathscr{L}\{y\} = Y(s)$*

$$\begin{aligned} \mathscr{L}\{y^{(4)}\} &= s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) \\ &= s^4 Y(s) - s^3 - s - Y(s) \end{aligned}$$

*Know:* $\mathscr{L}\{0\} = 0$

$$\begin{aligned}
s^4Y(s) - s^3 - s - Y(s) &= 0 \\
(s^4 - 1)Y(s) &= s^3 + s \\
Y(s) &= \frac{s^3+s}{s^4-1} = \frac{s}{s^2-1} \\
\implies y(t) &= \cosh t
\end{aligned}$$

**Example 5.2.4**

$$y'' + 2y' + y = 4e^{-t}, \;\; y(0) = 2, \;\; y'(0) = -1$$

$$\begin{aligned}
(s^2+2s+1)Y(s) - 2s + 1 - 4 &= \frac{4}{s+1} \\
Y(s) &= \frac{4}{(s^2+1)^3} + \frac{2(s+1)}{(s+1)^2} + \frac{1}{(s+1)^2} \\
Y(s) &= 2\left(\frac{2!}{(s+1)^3}\right) + 2\left(\frac{1}{s+1}\right) + \frac{1}{(s+1)^2} \\
y(t) &= 2t^2e^{-t} + 2e^{-t} + te^{-t}
\end{aligned}$$

**Example 5.2.5**

$$\textit{Find} \;\; \mathscr{L}^{-1}\{\frac{s-1}{s^2+\frac{1}{2}s+3}\}$$

$$\begin{aligned}
F(s) &= \frac{1}{2}\frac{s-1}{s^2+\frac{1}{2}s+3} \\
&= \frac{1}{2}\frac{s-1}{\left(s+\frac{1}{4}\right)^2 + \left(\frac{\sqrt{47}}{4}\right)^2} \\
&= \frac{1}{2}\left[\frac{s+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{47}{16}} - \frac{\frac{5}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{47}{16}}\right] \\
f(t) &= \frac{1}{2}e^{-\frac{t}{4}}\cos\left(\frac{\sqrt{47}t}{4}\right) - \frac{5}{2\sqrt{47}}e^{-\frac{t}{4}}\sin\left(\frac{\sqrt{47}t}{4}\right)
\end{aligned}$$

## 5.3 Step Function

Unit step function $\equiv U_c$ , $c \in \{\mathbb{R}^+ \cup 0\}$

$$u_c(t) = \begin{cases} 0, & t < c, \;\; c \geq 0 \\ 1, & t \geq c \end{cases}$$

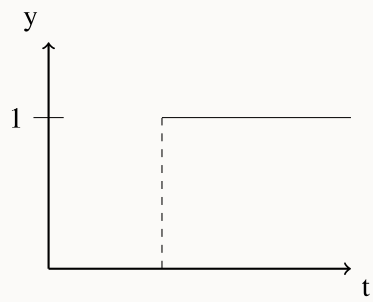


**Figure 8**

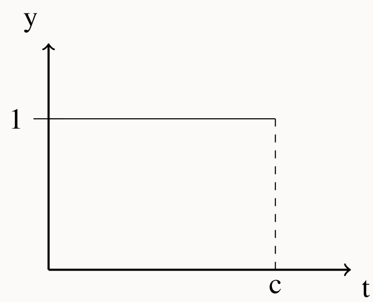


**Figure 9:** $y(t) = 1 - u_c(t)$

Given function $f$ , defined for $t \geq 0$

$$y = g(t) = \begin{cases} 0, & t < c \\ f(t-c), & t \geq c \end{cases}$$

represents a translation of f a distance c in the positive direction.

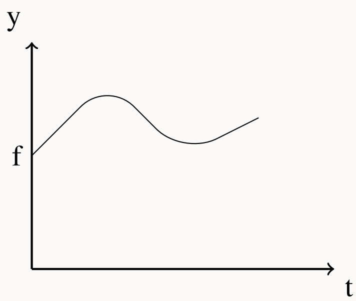


**Figure 10**

**Example 5.3.1**

$$f(t) = u_1(t) + 2u_3(t) - 6u_4(t)$$

$$f(t) = \begin{cases} 0 + 2 \cdot 0 - 6 \cdot 0 = 0, & 0 \leq t \leq 1 \\ 1 + 2 \cdot 0 - 6 \cdot 0 = 1, & 1 \leq t \leq 3 \\ 1 + 2 \cdot 1 - 6 \cdot 0 = 3, & 3 \leq t \leq 4 \\ 1 + 2 \cdot 1 - 6 \cdot 1 = -3, & 4 \leq t \end{cases}$$

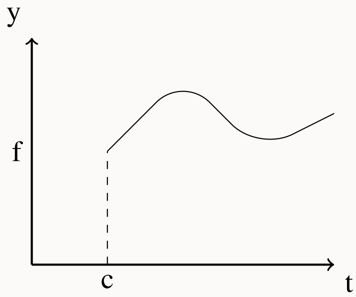


**Figure 11**

$$\begin{aligned}
\mathscr{L}\{u_c(t)\} &= \int_0^\infty e^{-st}u_c(t)dt \\
&= \int_0^c e^{-st}\cdot 0dt + \int_c^\infty e^{-st}\cdot 1dt \\
&= \int_c^\infty e^{-st}dt \\
&= \lim_{M\to\infty}\int_c^M e^{-st}dt \\
&= \lim_{M\to\infty}\frac{-e^{-st}}{s}\bigg|_c^M \\
&= \lim_{M\to\infty}\frac{-e^{-sM}+e^{-cs}}{s} \\
&= e^{\frac{-cs}{s}}
\end{aligned}$$

Look at the relationship between $\mathscr{L}\{f(t)\}$ and $\mathscr{L}\{u_c(t)f(t-c)\}$.

**THEOREM 5.3**

If $F(s) = \mathscr{L}\{f(t)\}$ exists for $s > a \geq 0$ and if $c \in \mathbb{R}^+$ then

$$\mathscr{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathscr{L}\{f(t)\} = e^{-cs}F(s), \quad s > a$$

Conversely, if $f(t) = \mathscr{L}^{-1}\{F(s)\}$, then

$$u_c(t)f(t-c) = \mathscr{L}^{-1}\{e^{-cs}F(s)\}$$

**THEOREM 5.4**

If $F(s) = \mathscr{L}\{f(t)\}$ exists for $s > a \geq 0$ and if $c \in \mathbb{R}$, then

$$\mathscr{L}\{e^{ct}f(t)\} = F(s-c), \quad s > a + c$$

Conversely, if $f(t) = \mathscr{L}^{-1}\{F(s)\}$, then

$$e^{ct}f(t) = \mathscr{L}^{-1}\{F(s-c)\}$$

**Example 5.3.2**

$$F(s) = \frac{(s-2)e^{-s}}{s^2-4s+3}, \quad \textit{Find} \;\; \mathscr{L}^{-1}$$

$$\begin{aligned} G(s) &= \frac{s-2}{s^2-4s+3} \\ &= \frac{s-2}{(s-2)^2-1} \end{aligned}$$

$$\begin{aligned} \mathscr{L}^{-1}[G(s)] &= e^{2t}\cosh t \\ \mathscr{L}^{-1}[F(s)] &= e^{2(t-1)}\cosh(t-1)u_1(t) \end{aligned}$$

**Example 5.3.3**

$$F(s) = \frac{e^{-3s}}{s^2+9}, \quad \textit{Find} \;\; \mathscr{L}^{-1}$$

$$\begin{aligned} G(s) &= \frac{1}{s^2+9} \\ &= \frac{1}{s^2+3^2} \end{aligned}$$

$\to \mathscr{L}^{-1}\{G(s)\} = \frac{\sin 3t}{3}$

$$\begin{aligned} \mathscr{L}^{-1}\{F(t)\} &= \frac{\sin 3(t-3)}{3}u_3(t) \\ &= \frac{\sin(3t-9)}{3}u_3(t) \end{aligned}$$

**Rectangular Window Function:**

$$\prod_{a,b}(t) = \begin{cases} 0, & t < a \\ 1, & a < t < b \\ 0, & t > b \end{cases}$$

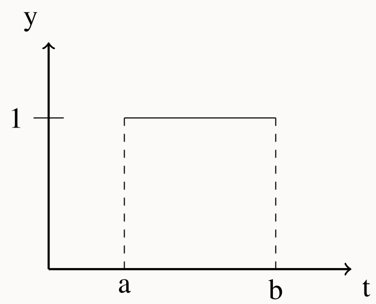


**Figure 12:** $= u_a(t-a) - u_b(t-b)$

**Example 5.3.4**

$$F(s) = e^{-s}\frac{3s^2-s+2}{(s-1)(s^2+1)}$$

*Consider:*

$$\begin{aligned} \frac{3s^2-s+2}{(s-1)(s^2+1)} &= \frac{A}{s-1} + \frac{Bx+C}{s^2+1} \\ &= \frac{2}{s-1} + \frac{s}{s^2+1} \end{aligned}$$

$$\begin{aligned}
&\mathscr{L}^{-1}\{\frac{2e^{-s}}{s-1}\}(t)+\mathscr{L}^{-1}\{\frac{e^{-s}s}{s^2+1}\}(t)\\
&=\left[2\mathscr{L}^{-1}\{\frac{1}{s-1}\}(t-1)+\mathscr{L}^{-1}\{\frac{s}{s^2+1}\}(t-1)\right]u_1(t)\\
&=\left[2e^{t-1}+\cos(t-1)\right]u_1(t)
\end{aligned}$$

## 5.4 Discontinuous Forcing Functions

**Example 5.4.1**

$$y''+y=u_{3\pi}(t),\quad y(0)=1,\quad y'(0)=0$$

$$\begin{aligned}
\mathscr{L}\{y''\}+\mathscr{L}\{y\}&=\mathscr{L}\{u_{3\pi}(t)\}\\
\left(s^2Y(s)-sY(0)-y'(0)+Y(s)\right)&=\frac{e^{-3\pi s}}{s}\\
(s^2+1)Y(s)&=s+\frac{e^{-3\pi s}}{s}\\
Y(s)&=\frac{s}{s^2+1}+\frac{e^{-3\pi s}}{s(s^2+1)}\\
Y(s)&=\frac{s}{s^2+1}+e^{-3\pi s}\left(\frac{1}{s}-\frac{s}{s^2+1}\right)\\
y(t)&=\cos t+u_{3\pi}(t)\left[1-\cos(t-3\pi)\right]
\end{aligned}$$

- *For* $0\le t<3\pi$ *:*

$$y(t)=\cos t$$

- *For* $t\ge 3\pi$ *:*

$$\begin{aligned}
y(t)&=\cos t+1-\cos(t-3\pi)\\
&=2\cos t+1
\end{aligned}$$

Let's look deeper into the above example. For $0\le t<3\pi$

$$\begin{aligned}
y(t)&=\cos t\\
y'(t)&=-\sin t\\
y''(t)&=-\cos t
\end{aligned}$$

For $t\ge 3\pi$ :

$$\begin{aligned}
y(t)&=2\cos t+1\\
y'(t)&=-2\sin t\\
y''(t)&=-2\cos t
\end{aligned}$$

$$\lim_{t\to 3\pi^-} \cos t = \cos 3\pi = -1$$
$$\lim_{t\to 3\pi^+} (\cos 2t + 1) = 2(-1) + 1 = -1$$

For 1st derivative:

$$\lim_{t\to 3\pi^-} -\sin t = 0$$
$$\lim_{t\to 3\pi^+} (-2\sin t) = 0$$

For 2nd derivative:

$$\lim_{t\to 3\pi^-} -\cos t = 1$$
$$\lim_{t\to 3\pi^+} -2\cos t = 2$$

which shows the limit does not exist. So $y''$ is discontinuous at $t = 3\pi$

**Example 5.4.2**

$$y'' + 4y = \sin t + u_\pi(t)\sin(t-\pi), \quad y(0) = 0, \quad y'(0) = 0$$

$$\begin{aligned}
\mathscr{L}\{y''\} + 4\mathscr{L}\{y\} &= \mathscr{L}\{\sin t\} + \mathscr{L}\{u_\pi(t)\sin(t-\pi)\} \\
s^2Y(s) - sy(0) - y'(0) + 4Y(s) &= \frac{1}{s^2+1} + e^{-\pi s}\frac{1}{s^2+1} \\
Y(s) &= \left(1 + e^{-\pi s}\right)\frac{1}{(s^2+1)(s^2+4)} \\
Y(s) &= \left(1 + e^{-\pi s}\right)\left(\frac{\frac{1}{3}}{s^2+1} - \frac{\frac{1}{3}}{s^2+4}\right) \\
Y(s) &= \left(1 + e^{-\pi s}\right)\left[\frac{1}{3}\left(\frac{1}{s^2+1}\right) - \frac{1}{6}\left(\frac{2}{s^2+2^2}\right)\right]
\end{aligned}$$

*Let* $H(s) = \frac{1}{3}\left(\frac{1}{s^2+1}\right) - \frac{1}{6}\left(\frac{2}{s^2+2^2}\right)$.

$$\begin{aligned}
\mathscr{L}^{-1}\{H(s)\} &= \frac{1}{3}\sin t - \frac{1}{6}\sin 2t \\
\mathscr{L}\{e^{-\pi s}H(s)\} &= u_\pi(t)\left[\frac{1}{3}\sin(t-\pi) - \frac{1}{6}\sin\left(2(t-\pi)\right)\right] \\
&= -u_\pi(t)\left[\frac{1}{3}\sin t + \frac{1}{6}\sin 2t\right]
\end{aligned}$$

*Putting Together*

$$y(t) = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t - u_\pi(t)\left(\frac{1}{3}\sin t + \frac{1}{6}\sin 2t\right)$$

# 6 PDE - Heat Equation - Fourier Series

## 6.1 Intro to PDE - Heat Conduction in a Rod

Review: $u_t = \frac{\partial u}{\partial t}$ , $u_{xx} = \frac{\partial^2 u}{\partial x^2}$

$$u = f(t, x, y)$$
$$u_t = u_{xx} + u_{yy}$$

which is known as the 2 dimensional heat equation. **Order of PDE:**

$$u_t = u_{xx} \;\; : \;\; 2^{\text{nd}} \text{ order}$$
$$u_t = uu_{xxx} + \sin x \;\; : \;\; 3^{\text{rd}} order$$

**Number of Variables:**

$$u_t = u_{xx} \;\; : \;\; 2 \text{ vars}$$
$$u_x = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{tt} \;\; : \;\; 3 \text{ vars}$$

**$2^{\text{nd}}$ order linear PDE in 2 variables:**

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

where A,B,. . . , G are constants or function of x and y.

**Example 6.1.1** *Nonlinear PDE:*

$$uu_{xx} + u_t = 0$$
$$xu_x + yu_y + u^2 = 0$$

There are 3 basic types of linear equation:

1. Parabolic Equation: $B^2 - 4AC = 0$ (heat equation, diffusion)
2. Hyperbolic Equation: $B^2 - 4AC > 0$ (vibrating system, wave equation)
3. Elliptic Equation: $B^2 - 4AC < 0$ (steady-state)

Heat Equation:

$$\begin{cases} \text{PDE} \\ \text{BC} \\ \text{IC} \end{cases}$$

Extend superposition to $\infty$ (infinite linear combination)

From fig. 13, let's assume heat constant in any given cross-section and no heat lost to the side.

$$\alpha^2 u_{xx} = u_t, \;\; 0 < x < L, \;\; t > 0 \qquad (*)$$
$$\alpha^2 = \frac{\kappa}{\rho \cdot s}$$

where $\kappa$ is thermal conductivity and $\rho$ is the density of the object and $s$ is the specific heat

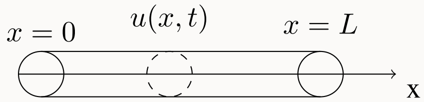


**Figure 13:** A rod in Heat Conduction Problem

IC:

$$u(x,0) = f(x), \ \ 0 \le x \le L$$

Assume $T_1$ at $x = 0$, $T_2$ at $x = L$ and $T_1 = T_2 = 0$. The boundary condition (BC) is:

$$u(0,t) = 0, \ \ u(L,t) = 0, \ \ t > 0$$

Now, our ansatz is (based on separation of variables):

$$u(x,t) = X(x)T(t)$$
$$u(x,t) = XT$$
$$u_{xx} = X''T, \ \ u_t = XT'$$

Sub into (*), we obtain:

$$\alpha^2 X''T = XT'$$
$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -\sigma, \ \ \sigma > 0$$

Thus, we can observe that we can split a PDE into a system of ODEs:

$$X'' + \sigma X = 0$$
$$T' + \alpha^2 \sigma T = 0$$

We also need to solve BC based from our ansatz

$$u(0,t) = X(0)T(t) = 0$$
$$X(0) = 0, \ \ T(t) = 0 \ \ \forall t$$

We must have $X(0) = 0$ by same arg $X(L) = 0$ (2 pts BVP). First, let $\sigma = \lambda^2$ to avoid radical sign

$$X'' + \sigma X = 0$$
$$X'' + \lambda^2 X = 0$$
$$X(x) = k_1 \cos(\lambda x) + k_2 \sin(\lambda x)$$

The 1st BC: $X(0) = 0$

$$X(0) = k_1 \cos 0 + k_2 \sin 0 \quad \rightarrow \quad k_1 = 0$$
$$X(x) = k_2 \sin(\lambda x)$$

The 2nd BC: $X(L) = 0$

$$k_2 \sin(\lambda L) = 0$$
$$\sin(\lambda L) = 0$$
$$\lambda = \frac{n\pi}{L}, \;\; n \in \mathbb{Z}^+$$
$$\lambda^2 = \frac{n^2\pi^2}{L^2}$$

The value of $\sigma$ that yield non-trivial solution are called *eigenvalues* of BVP (boundary value problem)

$$X(x) = \sin\left(\frac{n\pi x}{L}\right)$$

are called *eigenfunction*. Substitute $\sigma$:

$$T' + \alpha^2\sigma T = 0 \;\; \text{yield:}$$
$$T' + \left(\frac{n^2\pi^2\alpha^2}{L^2}\right) T = 0$$
$$T(t) = e^{-\frac{n^2\pi^2\alpha^2 t}{L^2}}$$
$$u_n(x,t) = X(x)T(t)$$
$$u_n(x,t) = e^{-\frac{n^2\pi^2\alpha^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right), \;\; n \in \mathbb{Z}^+$$

which is the fundamental solution of heat conduction. Extending this using principle of superposition to $\infty$ , we obtain:

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t)$$

Unless:

$$f(x) = b_1 \sin\left(\frac{\pi x}{L}\right) + b_2 \sin\left(\frac{2\pi x}{L}\right) + \ldots + b_m \sin\left(\frac{m\pi x}{L}\right)$$

**Example 6.1.2**

$$\textit{PDE:} \;\; \alpha^2 u_{xx} = u_t, \;\; 0 < x < L, \;\; t > 0$$
$$\textit{IC:} \;\; u(x,0) = f(x), \;\; 0 \le x \le L$$
$$\textit{BC:} \;\; u(0,t) = 0, \;\; u(L,t) = 0$$

*Ansatz:* $u(x,t) = X(x)T(t)$ , $t > 0$. *Then fundamental solution of heat conduction is*

$$u_n(x,t) = e^{\frac{-n^2\pi^2\alpha^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right), \;\; n \in \mathbb{Z}^+$$

*We also have:*

$$u(x,t) = \sum_{n=1}^{m} c_n u_n(x,t)$$

*where Fourier series would determined* $c_n$ *, the sine series, unless:*

$$f(x) = b_1 \sin\left(\frac{n\pi x}{L}\right) + b_2 \sin\left(\frac{2\pi x}{L}\right) + \ldots + b_m \sin\left(\frac{m\pi x}{L}\right)$$

**Example 6.1.3**

$$\textit{PDE:}\ \ 100u_{xx} = u_t,\ \ 0 < x < 1,\ \ t > 0$$
$$\textit{IC:}\ \ u(x,0) = \sin(2\pi x) - \sin(5\pi x),\ \ 0 \le x \le 1$$
$$\textit{BC:}\ \ u(0,t) = 0,\ \ u(1,t) = 0,\ \ t > 0$$

*Soln:* $u_n(x,t) = e^{-100n^2\pi^2 t}\sin(n\pi x)$

$$\textit{IC:}\ \ u(x,0) = \sin(2\pi x) - \sin(5\pi x),\ \ 0 \le x \le 1$$

*when* $t = 0$.

$$u_n(x,0) = \sin(n\pi x)\ \ \rightarrow\ \ \textit{need}\ \ n = 2,\ \ n = 5$$

$$\begin{aligned} u(x,0) &= c_2 u_2(x,t) + c_5 u_5(x,t) \\ &= c_2 \sin 2\pi x + c_5 \sin 5\pi x \end{aligned}$$
$$\implies c_2 = 1,\ \ c_5 = -1$$

*So, our final solution is:*

$$u(x,t) = e^{-400\pi^2 t}\sin 2\pi x - e^{-2500\pi^2 t}\sin 5\pi x$$

## 6.2 Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}\right) \tag{*}$$

Solve for $a_m$ and $b_m$ cab be very complicated.

$$f(x) = \cos\pi x + \frac{1}{2}\cos 13\pi x + \frac{1}{4}\cos 169\pi x + \frac{1}{8}\cos 2197\pi x + \dots$$

which is convergent and continuous $\forall x$ but it's never differentiable $\rightarrow$ pathological function.

**Periodicity of sin/cos function** : $f$ is periodic with $T > 0$

$$f(x+T) = f(x),\ \ \forall x\ \in\ \text{dom(f)}$$
$$\sin\frac{m\pi x}{L}, \cos\frac{m\pi x}{L}, T = \frac{2L}{m}$$

Orthogonality of sin and cos function inner product $(u,v)$ defined $\alpha \le x \le \beta$

$$(u,v) = \int_{\alpha}^{\beta} u(x)v(x)dx = 0$$

if u and v are orthogonal

- $\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx = \begin{cases} 0, & \text{if}\ \ m \ne n \\ L, & \text{if}\ \ m = n \end{cases}$

- $\int_{-L}^{L} \cos\frac{n\pi x}{L} \sin\frac{n\pi x}{L} dx = 0 \; \forall m, n$
- $\int_{-L}^{L} \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L} dx = \begin{cases} 0, & \text{if } m \neq n \\ L, & \text{if } m = n \end{cases}$

1. Multiply (*) by $\cos\frac{n\pi x}{L}$ when n fixed ($n > 0$ )
2. Integrate with respect to x from -L to L.

$$\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = \frac{a_0}{2}\int_{-L}^{L}\cos\frac{n\pi x}{L}dx + \sum_{m=1}^{\infty} a_m \int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx + \sum_{m=1}^{\infty} b_m \int_{-L}^{L}\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx$$

*Euler - Fourier Formulas:*

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, 3 \ldots$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n \in \mathbb{Z}^+$$

**Example 6.2.1**

$$f(x) = \begin{cases} x + L, & -L \leq x \leq 0 \\ L, & 0 < x \leq L \end{cases}$$

*Fourier Series:*

$$f(x) = \frac{3L}{4} + \sum_{n=1}^{\infty}\left[\frac{2L\cos\left(\frac{(2n-1)\pi x}{L}\right)}{(2n-1)^2\pi^2} + \frac{(-1)^{n-1}\sin\left(\frac{n\pi x}{L}\right)}{n\pi}\right]$$

## 6.3 The Fourier Convergence Theorem

**THEOREM 6.1**

Suppose that $f$ and $f'$ are piecewise continuous on the interval $-L \leq x < L$. Furthermore, suppose that f is defined outside the interval $-L \leq x < L$ so that it is periodic with period 2L. Then $f$ has a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m\cos\frac{m\pi x}{L} + b_m\sin\frac{m\pi x}{L}\right)$$

whose coefficients are given as

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}dx, \quad m = 0, 1, 2, \ldots$$
$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}dx, \quad m = 1, 2, \ldots$$

The Fourier series converges to $f(x)$ at all points where $f$ is continuous and to $[f(x+) + f(x-)]/2$ at all points where $f$ is discontinuous.

*Note*:

$$f(x+) = \lim_{x\to x_0^+} f(x), \quad f(x-) = \lim_{x\to x_0^-} f(x)$$

As n increases, partial sum $s_n \to f(x)$ as $n \to \infty$ happens converges smoothly where $f(x)$, but at points of discontinuity, partial converges smoothly to the new value which tends to overshoot. (Gibbs Phenomenon)

$$\lim_{n\to\infty} S_n = \frac{f(x_0^-) + f(x_0^+)}{2}$$

There exists a way to remove Gibbs phenomenon called Lanczos sigma factor

$$\frac{a_0}{2} + \sum_{n=0}^{m} \sin\left(\frac{n\pi}{2m}\right)\left[a_n \cos\frac{n\pi x}{2} + b_n \sin\frac{n\pi x}{L}\right]$$

## 6.4 Even and Odd Functions

Recall:

$$\text{Even: } f(-x) = f(x)$$
$$\text{Odd: } f(-x) = -f(x)$$

Elementary Properties:

1. Sum(difference) and product (quotient) of 2 even functions are even.
2. Sum (difference) of 2 odd functions is odd. But the product (quotient) of 2 odd functions are even.
3. Sum (difference) of an odd function and an even function is neither. The product (quotient) of an odd and even function is odd.
4. If $f(x)$ is even, then $\int_{-L}^{L} f(x)dx = 2\int_0^L f(x)dx$
5. If $f(x)$ is odd, then $\int_{-L}^{L} f(x)dx = 0$

Cosine Series:

$$f : \begin{cases} \text{even} \\ \text{periodic (2L)} \end{cases}$$

$\to f(x)\cdot\cos\left(\frac{n\pi x}{L}\right)$ is even and $f(x)\cdot\sin\left(\frac{n\pi x}{L}\right)$ is odd. Fourier coefficient of f:

$$a_m = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0,1,2,3,\ldots$$
$$b_n = 0$$
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

Sine Series:

$$f : \begin{cases} \text{odd} \\ \text{periodic (2L)} \end{cases}$$

$f(x) \cdot \cos\left(\frac{n\pi x}{L}\right)$ is odd, and $f(x) \cdot \sin\left(\frac{n\pi x}{L}\right)$ is even.

$$a_n = 0, \quad n = 0, 1, 2 \dots$$
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n \in \mathbb{Z}^+$$
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

**Even and Odd Extensions:**

- For an even periodic extension, define $g$ of period 2L such that

$$g(x) = \begin{cases} f(x), & 0 \le x \le L \\ f(-x), & -L < x < 0 \end{cases}$$

$\rightarrow$ Fourier cosine series

- For an odd periodic extension, define $h$ of periodic 2L such that

$$h(x) = \begin{cases} f(x), & 0 < x < L \\ 0, & x = 0, L \\ -f(-x), & -L < x < 0 \end{cases}$$

$\rightarrow$ Fourier sine series

**Example 6.4.1**

$$f(x) = L - x, \quad 0 < x < L$$

*Find the Fourier Sine series of period 2L. For a sine series:*

$$a_n = 0, \quad n = 0, 1, 2, \dots$$

$$\begin{aligned}
b_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \\
&= \frac{2}{L}\int_0^L (L-x)\sin\frac{n\pi x}{L}dx \\
&= \frac{2}{L}\left[\int_0^L L\sin\frac{n\pi x}{L}dx - \int_0^L x\sin\frac{n\pi x}{L}dx\right] \\
&\vdots \\
&= \frac{-2L}{n\pi}(\cos n\pi - \cos 0) + \frac{2}{n\pi}(L\cos n\pi - 0) + \frac{2}{L}\left(\frac{L}{n\pi}\right)^2 \sin\frac{n\pi x}{L}\Big|_0^L \\
&= \frac{2L}{n\pi}
\end{aligned}$$

$$f(x) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi x}{L}\right)}{n}$$

## 6.5 Example of Solving a Complete Heat Conduction in a rod Problem:

Let's look at

$$\text{PDE:}\;\; u_{xx} = u_t,\;\; 0 < x < 1,\;\; t > 0$$
$$\text{BC:}\;\; u(0,t) = 0,\;\; u(1,t) = 0,\;\; t > 0$$
$$\text{IC:}\;\; u(x,0) = 1,\;\; 0 < x < 1$$

Here $\alpha = 1$ , $L = 1$

$$u_n(x,t) = e^{-n^2\pi^2 t}\sin(n\pi x)$$

Since IC: $u(x,0) = 1,\;\; 0 < x < 1$

$$u_n(x,0) = \sin(n\pi x) = 1$$
$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) = 1$$

$c_n$ is coefficient of the Fourier sine series of $f(x) = 1$

$$\begin{aligned} c_n &= 2\int_0^1 f(x)\sin(n\pi x)dx \\ &= 2\int_0^1 \sin(n\pi x)dx,\;\; n \in \mathbb{Z}^+ \\ &= -\frac{2}{n\pi}(\cos n\pi - 1) \end{aligned}$$

- If n is even, $c_n = 0$
- If n is odd, $c_n = \frac{4}{n\pi}$

Generally, $c_{2n-1} = \frac{4}{(-1+2n)\pi,\;\; c_{2n}=0}$. Or

$$\frac{4}{\pi}\left[\sin\pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x\right] = 1$$
$$u(x,t) = \frac{4}{\pi}\left[e^{-\pi^2 t}\sin\pi x + \frac{1}{3}e^{-(3\pi)^2 t}\sin 3\pi x + \frac{1}{5}e^{-(5\pi)^2 t}\sin 5\pi x + \dots\right]$$
$$u(x,t) = \sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}e^{-(2\pi-1)^2\pi^2 t}\sin\left[(2n-1)\pi x\right]$$

Now, we can solve for the PDE + BC + IC,

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$
$$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}dx\right)$$

# 7 Boundary Value Problem

**Regular Sturm - Louisville Problem:**

- $\exists$ an $\infty$ numbers of $\mathbb{R}$ eigenvalues that can be arranged in increasing order $\lambda_1 < \lambda_2 < \ldots < \lambda_n$ such that $\lambda_n \to \infty$ as $n \to \infty$
- For each $\lambda$, there exists a unique eigenfunction
- Eigenfunction corresponding to different eigenvalues are linearly independent.
- The set of eigenfunctions correspond to the set of eigenvalues is orthogonal with respect to the weight $p(x)$ on the interval $I$, For us, $p(x) = 1$

# 8 System of First Order Linear Equations

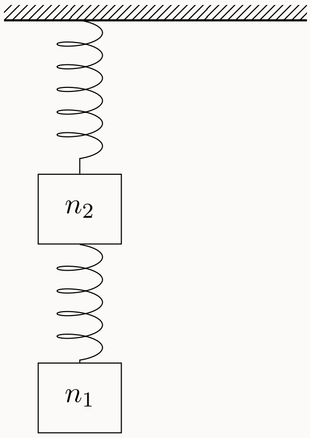


**Figure 14:** A mechanical Spring with Multiple Nodes

$$t^2u'' + tu' + (t^2 - 0.25)u = 0$$
$$u'' = -\frac{1}{t}u' - \left(1 - \frac{1}{4t^2}\right)u$$

Set $x_1 = u$ and $x_2 = u' \to x_1' = x_2$

$$x_2' = u'' = -\frac{1}{t}u' - \left(1 - \frac{1}{4t^2}\right)u$$

$$\begin{cases} x_1' = x_2 \\ x_2' = -\left(1 - \frac{1}{4t^2}\right)x_1 - \frac{1}{t}x_2 \end{cases}$$

$$x_1' = -2x_1 + x_2, \;\; x_2' = x_1 - 2x_2$$
$$(x_1' + 2x_1)' = x_1 - 2(x_1' + 2x_1)$$
$$x_1'' + 2x_1' = x_1 - 2x_1' - 4x_1$$
$$x_1'' + 4x_1' + 3x_1 = 0$$

which can be solved from the characteristics equation.

## 8.1 Homogeneous Linear Systems (Constant Coefficient)

$$\vec{x}' = \vec{A}\vec{x}, \quad A = n \times n \tag{*}$$

For $n = 1$: system reduces to $\frac{dx}{dt} = ax$, solution is $x = ce^{at}$ in section 3 that we saw. Notice that $\lambda = 0$ is the only equilibrium solution if $a \neq 0$

- If $a < 0$ - asymptotically stable $\to$ sink
- $a > 0$ - asymptotically unstable $\to$ source

For n = 2, this is important if it has visualization in the $x_1$ and $x_2$ plane called a phase plane. Evaluate $\vec{A}\vec{x}$ at a large number of points and plot the resulting vector yields a direction field of tangent vector to the solution of the system. To (*), ansatz solns will involve $e^{rt}$. Also, (*) are vector so we multiply $e^{rt}$ by a constant vector.

$$\vec{x} = \xi e^{rt} \tag{**}$$

Sub into (*), we have:

$$\begin{aligned} r\xi e^{rt} &= \vec{A}\xi e^{rt} \\ (\vec{A} - r\vec{I}\xi &= \vec{0} \end{aligned} \tag{***}$$

The problem of determining the eigenvalues and eigenvectors of $\vec{A}$ provided r - av eigenvalue and $\xi = a_n$ associated eigenvector.

**Example 8.1.1**

$$\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \vec{x}$$

*Ansatz:* $\vec{x} = \xi e^{rt}$ *From (***),*

$$(\vec{A} - r\vec{I})\xi = \vec{0}$$

$$\begin{pmatrix} 1-r & 1 \\ 4 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

det $(\vec{A} - r\vec{I}) = 0$,

$$\begin{vmatrix} 1-r & 1 \\ 4 & -2-r \end{vmatrix} = (1-r)(-2-r) - 4$$

So, $r^2 + r - 6 = 0 \to r_1 = 2$ , $r_2 = -3$ are eigenvalues

- $r_1 = 2$

$$\begin{pmatrix} -\xi_1 + \xi_2 \\ 4\xi_1 - 4\xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\xi_1 = \xi_2$$

$$\xi^{(1)} = (1, 1)^T$$

- $r_2 = 3$

$$\begin{pmatrix} 4\xi_1 + \xi_2 \\ 4\xi_1 + \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\xi^{(2)} = (1, -4)^T$$

Therefore,

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}$$

Breaking apart the general soln:

$$\vec{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}, \ \vec{x}^{(2)} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}$$

The Wronskian is:

$$W[\vec{x}^{(1)}, \vec{x}^{(2)}](t) = \begin{vmatrix} e^{2t} & e^{-3t} \\ e^{2t} & -4e^{-3t} \end{vmatrix}$$
$$= -5e^{-t} \neq 0 \quad \forall t$$

So the solution forms a fundamental set of solution

- For $\vec{x}^{(1)}(t)$ : the scalar form

$$x_1 = c_1 e^{2t}, \ \ x_2 = c_1 e^{2t}$$

eliminate $c_1$ , t $\to x_1 = x_2$. Solution lives on the straight line $x_2 = x_1$ in quadrant I for $c_1 > 0$ and QII for $c_1 < 0$. In either case, solution depart from the origin as t increases.

- For $\vec{x}^{(2)}(t)$: scalar form

$$x_1 = c_2 e^{-3t}, \ \ x_2 = -4c_2 e^{-3t}$$
$$x_1 = -\frac{1}{4} x_2 \quad \to \quad \text{soln in QIV for } c_2 > 0$$
$$\text{and QII for } \ c_2 < 0$$

In both cases, it moves towards the origin. For large t, the term $c_1\vec{x}^{(1)}(t)$ is dominant and term $c_2\vec{x}^{(2)}(t)$ become negligible.

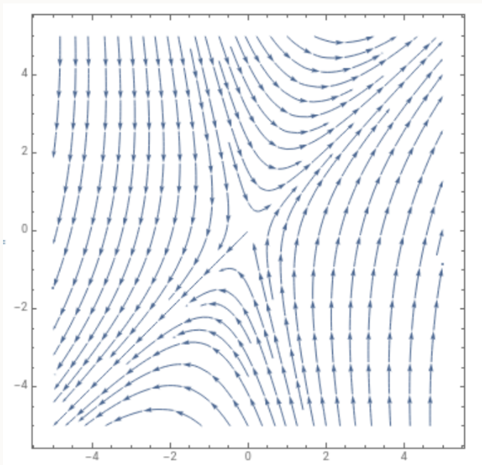


**Figure 15:** The direction field

**Example 8.1.2**

$$\vec{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \vec{x}$$

*Ansatz:* $\vec{x} = \vec{\xi} e^{rt}$

$$(\vec{A} - r\vec{I})\vec{\xi} = \vec{0}$$

$$\begin{pmatrix} 1-r & -2 \\ 3 & -4-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$det \left( \vec{A} - r\vec{I} \right) = 0$$

$$-(1-r)(4+r) + 6 = 0$$

$$r_1 = -1, \quad r_2 = -2$$

- *If* $r_1 = -1$ *:*

$$\begin{pmatrix} 2\xi_1 - 2\xi_2 \\ 3\xi_1 - 3\xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\xi_1 = \xi_2$$

$$\xi^{(1)} = (1, 1)^T$$

- *If* $r_2 = -2$ :

$$\begin{pmatrix} 3\xi_1 - 2\xi_2 \\ 3\xi_1 - 2\xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$3\xi_1 = 2\xi_2$$

$$\vec{\xi}^{(2)} = (2, 3)^T$$

*General solution:*

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t}$$

*which has original stable node*

**Example 8.1.3**

$$\vec{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \vec{x}$$

*Ansatz:* $\vec{x} = \vec{\xi} e^{rt}$

$$\begin{pmatrix} 1-r & 1 & 2 \\ 1 & 2-r & 1 \\ 2 & 1 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

$$r^3 - 4r^2 - r + 4 = 0$$

$$r_1 = 4, \quad r_2 = 1, \quad r_3 = -1$$

- $r_1 = 4$

$$\begin{pmatrix} -3\xi_1 + \xi_2 + 2\xi_3 \\ \xi_1 - 2\xi_2 + \xi_3 \\ 2\xi_1 + \xi_2 - 3\xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\vec{\xi}^{(1)} = (1, 1, 1)^T$$

- $r_2 = 1$

$$\begin{pmatrix} \xi_2 + 2\xi_3 \\ \xi_1 + \xi_2 + \xi_3 \\ 2\xi_1 + \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\vec{\xi}^{(2)} = (1, -2, 1)^T$$

- $r_3 = -1$

$$\begin{pmatrix} 2\xi_1 + \xi_2 + 2\xi_3 \\ \xi_1 + 3\xi_2 + \xi_3 \\ 2\xi_1 + \xi_2 + 2\xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\vec{\xi}^{(3)} = (1, 0, -1)^T$$

*General Soln:*

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t}$$

## 8.2 Complex Eigenvalues

$$\vec{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \vec{x}$$
$$\begin{pmatrix} -1-r & -4 \\ 1 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$r^2 + 2r + 5 = 0$$
$$r = -1 \pm 2i$$

- $r_1 = -1 + 2i$

$$\begin{pmatrix} -2i\xi_1 - 4\xi_2 \\ \xi_1 - 2i\xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\vec{\xi}^{(1)} = (2i, 1)^T$$

- $r_2 = -1 - 2i$

$$\begin{pmatrix} 2i\xi_1 - 4\xi_2 \\ \xi_1 + 2i\xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\vec{\xi}^{(2)} = (-2i, 1)^T$$

$$\vec{x} = c_1 \begin{pmatrix} 2i \\ 1 \end{pmatrix} e^{(-1+2i)t} + c_2 \begin{pmatrix} -2i \\ 1 \end{pmatrix} e^{(-1-2i)t}$$

Breaking apart the solution, we get:

$$\begin{aligned} \vec{x}^{(1)}(t) &= \begin{pmatrix} 2i \\ 1 \end{pmatrix} e^{-t}(\cos 2t + i \sin 2t) \\ &= \begin{pmatrix} -2e^{-t}\sin 2t \\ e^{-t}\cos 2t \end{pmatrix} + i \begin{pmatrix} 2e^{-t}\cos 2t \\ e^{-t}\sin 2t \end{pmatrix} \end{aligned}$$

So,

$$\vec{x} = c_1 e^{-t} \begin{pmatrix} -2\sin 2t \\ \cos 2t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 2\cos 2t \\ \sin 2t \end{pmatrix}$$

Let's then calculate the Wronskian

$$\vec{u}(t) = e^{-t} \begin{pmatrix} -2\sin 2t \\ \cos 2t \end{pmatrix}$$
$$\vec{v}(t) = e^{-t} \begin{pmatrix} 2\cos 2t \\ \sin 2t \end{pmatrix}$$
$$W(\vec{u}, \vec{v})(t) = \begin{vmatrix} -2e^{-t}\sin 2t & 2e^{-t}\cos 2t \\ e^{-t}\cos 2t & e^{-t}\sin 2t \end{vmatrix} = -2e^{-2t} \neq 0$$

which forms the fundamental set of solutions (spiral point stable)

**Example 8.2.1**

$$\vec{x}' = \begin{pmatrix} 0 & -5 \\ 1 & \alpha \end{pmatrix} \vec{x}$$

*a) Determine the eigenvalue in term of $\alpha$*

$$\begin{pmatrix} -r & -5 \\ 1 & \alpha - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$r^2 - \alpha r + 5 = 0$$
$$r_1 = \frac{\alpha}{2} + \frac{1}{2}\sqrt{\alpha^2 - 20}, \quad r_2 = \frac{\alpha}{2} - \frac{1}{2}\sqrt{\alpha^2 - 20}$$

*b) Find the critical value of $\alpha$ where the qualitative nature of the phase portrait changes.*

$$\textit{The roots are complex when:} \quad |\alpha| < \sqrt{20}$$

- $\alpha \in (-\sqrt{20}, 0) \to$ *negative real part*
- $\alpha \in (0, \sqrt{20}) \to$ *positive real part*
- $\alpha = 0 \to$ *pure imaginary eigenvalues (center)*
- $\alpha^2 > 20 \to$ *roots are* $\mathbb{R}$ *and distinct*

*Finally,* $\alpha = \sqrt{20}$

# 9 Nonlinear Systems

**Predator - Prey System**:

$$x(t) = \text{prey}, \quad y(t) = \text{predator}$$

$$x'(t) = x(2 - 3x) - 4xy \tag{1}$$

$$y'(t) = -y + 3xy \tag{2}$$

*Note*: $xy$ represents the rate at which predator eats prey and term like $2 - 3x$ tells us about the reproductive rate. If $y(0) = 0$ ($y'(t) = 0$ )

$$x'(t) = 2x - 3x^2 = 0 \implies x = 0, \; x = \frac{2}{3}$$

So $(0, 0)$, $(\frac{2}{3}, 0)$ are equilibrium points. If $y \neq 0$ , then (2) becomes:

$$-y + 3xy = 0$$

$$-1 + 3x = 0 \implies x = \frac{1}{3}$$

Sub $x = \frac{1}{3}$ into (1)

$$x(2 - 3x) - 4xy = 0$$

$$y = \frac{1}{4}$$

$(\frac{1}{3}, \frac{1}{4})$ is the $3^{\text{rd}}$ equilibrium point

# 10 Schrodinger's Equation

We had a talk/lecture about Schrodinger's Equation from Dr. Callas (he is a project manager at NASA's Mars Exploration Rover Project and also a math professor at PCC) in June, and we got to learn about the derivation of the equation and different aspects of it from a more scientific viewpoint like physics/chemistry.