Math 142 – Mathematical Modeling

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This is math 142 – Mathematical Modeling taught by Professor Huang. We meet weekly on MWF from 9:00am – 9:50am for lecture. There is one textbook used for the class, which is *Mathematical Models* by *Haberman*. You can find other lecture notes at my blog site. Please let me know through my email if you spot any mathematical errors/typos.

Contents

§1 Lec 1: Sep 24, 2021

§1.1 Intro to Mathematical Modeling

First, let's examine the following question

Question 1.1. Why do we learn mathematical modeling?

There are lots of question that math may provide some explanation so that we could understand the question deeply.

Example 1.1
1. How is Covid-19 spread? How can we control the spread of Covid-19?
2. How to control the spreading of the forest fire and how to reduce the loss?
3. How does the population of human evolve over time?

So,

Question 1.2. What is a mathematical model and how can we create the model?

Definition 1.2 (Mathematical Model) — A mathematical model is a description of a system using mathematical concepts and language. The process of developing a mathematical model is called mathematical modeling.

To create a mathematical model, we

1. formulate the problem: approximations and assumptions based on experiments and observations
2. solve the problem that is formulated above
3. interpret the mathematical results in the context of the problem

Let's now explain the three steps above in more details.

1. Formulation
   a) State the question: If the question is vague, then make it to be precise. If the question is too "big", then subdivide it into several simple and manageable parts.
   b) Identify factors: Decide important quantities and assign some notation to the corresponding quantity. Then, we need to determine the relationship between the quantities and represent each relationship with an equation.
2. Solve the problem above: This may entail calculations that involve algebraic equations, some ODE, PDE, etc; provide some theorems or doing some simulations, etc.
3. Interpretation/Evaluation: We need to translate the mathematical result in step 2 back to the real world situations and evaluate whether the model is good or not by asking the following questions:
   a) Has the model explained the real-world observations?
   b) Are the answers we found accurate enough?
   c) Were our assumptions good?

d) What are the strengths and weaknesses of our model?

e) Did we make any mistake in step 2?

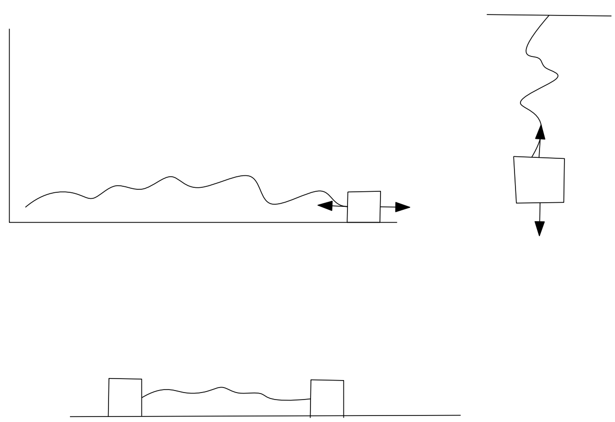
If the answer to any of the above question is not favorable, we need to go back to step 1 and go through all the steps again until we get some satisfying results.

§2 Lec 2: Sep 27, 2021

§2.1 An Example of Modeling a Mass-Spring System

Consider the following question

Question 2.1. How does the spring-mass system move/work?

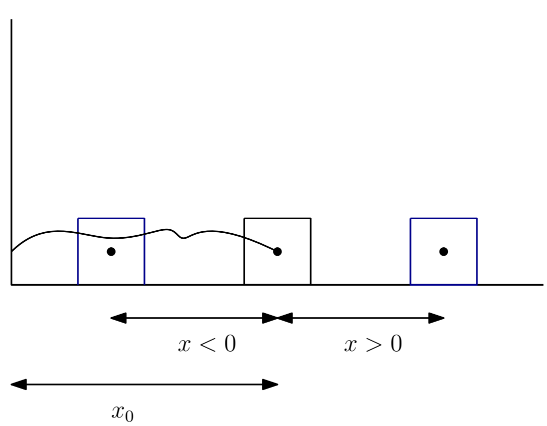


Formulation:

a) State the question: What formula can describe how the spring-mass system work?

b) Identify factors:

   (a) initial position x_0 (called natural length)
   (b) the spring constant k
   (c) friction f_c
   (d) mass of the object m
   (e) position x
   (f) velocity v
   (g) acceleration a
   (h) force F



Now, we try to find some relations between factors we listed above. First, let's describe our observations. If we contract the spring ($x < 0$), there is some force to push the spring outward ($F > 0$). If we stretch the spring ($x > 0$), there is some force that restores the initial shape of the spring ($F < 0$). So, we can observe that

$$F \cdot x < 0$$

The relation between F and x can be summarized by the Hooke's Law

$$F = -kx \tag{*}$$

Next, let's find the relation between the force and the movement of the object (F, m, v, a) by assuming that the movement of the object only depends on the force of the spring (not on other factors). This can be summarized by Newton's second law of motion.

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = m\frac{d}{dt}\left(\frac{d\vec{x}}{dt}\right) = m\frac{d^2\vec{x}}{dt^2} \tag{**}$$

By (*) and (**), we deduce

$$F = -kx = m\frac{d^2x}{dt^2}$$

Mathematical analysis: we need to find the solution of the ODE:

$$mx'' + kx = 0$$

To solve the ODE, we want to find the solution to the characteristic equation

$$m\lambda^2 + k = 0 \implies x = \pm\sqrt{\frac{k}{m}}i$$

Thus,

$$\begin{aligned} x(t) &= c_1 e^{t\sqrt{\frac{k}{m}}i} + c_2 e^{-t\sqrt{\frac{k}{m}}i} \\ &= (c_1 + c_2)\cos\left(\sqrt{\frac{k}{m}}t\right) + (c_1 - c_2)i\sin\left(\sqrt{\frac{k}{m}}t\right) \\ &= c_3\cos\left(\sqrt{\frac{k}{m}}t\right) + c_4\sin\left(\sqrt{\frac{k}{m}}t\right) \end{aligned}$$

§3 Lec 3: Sep 29, 2021

§3.1 An Example (Cont'd)

Recall that we have

$$x(t) = c_3 \cos\left(\sqrt{\frac{k}{m}}t\right) + c_4 \sin\left(\sqrt{\frac{k}{m}}t\right)$$

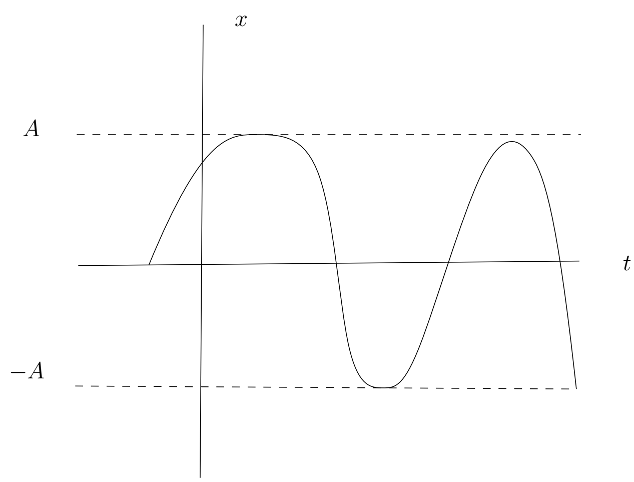
Let $\theta_2 = \sqrt{\frac{k}{m}}t$. Then,

$$x(t) = \sqrt{c_3^2 + c_4^2}\left(\frac{c_3}{\sqrt{c_3^2+c_4^2}}\cos(\theta_2) + \frac{c_4}{\sqrt{c_3^2+c_4^2}}\sin(\theta_2)\right)$$

Let $\sin\theta_1 = \frac{c_3}{\sqrt{c_3^2+c_4^2}}$ and $\cos\theta_1 = \frac{c_4}{\sqrt{c_3^2+c_4^2}}$ with $\tan\theta_1 = \frac{c_3}{c_4}$ or $\theta_1 = \arctan\left(\frac{c_3}{c_4}\right)$. So,

$$\begin{aligned} x(t) &= \sqrt{c_3^2+c_4^2}\sin(\theta_1+\theta_2) \\ &= \sqrt{c_3^2+c_4^2}\left(\sqrt{\frac{k}{m}} + \theta_1\right) \end{aligned}$$

Evaluation of $x(t) = A\sin(\omega t + \theta)$



From the figure above, we know $x(t)$ is periodic with period $T = \frac{2\lambda}{\omega} = 2\lambda\sqrt{\frac{m}{k}}$

$$\max_t x(t) = A, \qquad \min_t x(t) = -A$$

where A is the amplitude and $\omega t + TBA$

Since $x(t)$ is a periodic function, this means the spring will oscillate forever. However, in practice, it is impossible. Thus, we need to modify our model by removing or adding some assumption.

Now, we may consider the case that there is friction when spring oscillates.

$$F_f = -c\frac{dx}{dt}$$

Then,

$$m\frac{d^2x}{dt^2} = -kx - c\cdot\frac{dx}{dt}$$

§3.2 Population Dynamics

Consider the following question

Question 3.1. Can we predict whether a species or its population will thrive or go extinct?

In order to answer it, let's first investigate an example.

Example 3.1

How many people will there be in the U.S. in the next 4 years?

First let's reformulate the question in the example to be more specific:

Question 3.2. Can we build a math model to predict the number of people in the U.S. in 1, 2, 3, 4 year?

Assumption	Factor
the death and birth rate are constant	birth rate: b
the counting period (of the population) is fixed	death rate: d
the growth of the population only depends on	the period
the death and birth rate	initial population: N_0
	the distribution of the population: $N^{(a)}$
	migration rate
	the # of years from the current time: t
	the # of population at time t: $N(t)$
	the growth rate: R

To study $N(t)$ we need to consider the relation between $N(t)$ and $N(t+\Delta t)$

$$\begin{aligned} N(t+\Delta t) &= N(t) + \#\text{ of new birth at } [t, t+\Delta t] - \#\text{ of death at } [t, t+\Delta t] \\ &= N(t) + (b-d)\Delta t \cdot N(t) \\ &= (1 + (b-d)\Delta t) \cdot N(t) \end{aligned}$$

Thus,

$$\begin{aligned} N(t+\Delta t) &= (1 + R\Delta t)\, N(t) \\ N(1) &= (1+R)N_0 \\ N(2) &= (1+R)N(1) = (1+R)^2 N_0 \\ N(3) &= (1+R)N(2) = (1+R)^3 N_0 \\ N(4) &= (1+R)N(3) = (1+R)^4 N_0 \end{aligned}$$

§4 Lec 4: Oct 1, 2021

§4.1 Population Dynamics (Cont'd)

Example 4.1

$N_0 = 300$ millions, $R = 0.6\%$, $\Delta t = 1$

$$\begin{aligned} N(1) &= (1+r)N_0 = (1+0.6\%)\cdot 300 \\ &= 300 + 1.8 = 301.8 \text{ millions} \\ N(2) &= (1+r)^2 N_0 = (1+0.6\%)\cdot 300 \\ &= 301.8 \cdot 100.6\% \\ N(3) &= (1+R)^3 N_0 = (1+0.6\%)^3 \cdot 300 \\ N(4) &= (1+R)^4 \cdot N_0 = (1+0.6\%)^4 \cdot 300 \end{aligned}$$

Consider:

$$N(t+\Delta t) = (1 + R\cdot \Delta t)\cdot N(t)$$

where $t_0 = 0$, $t_1 = \Delta t$, $t_2 = 2\Delta t, \ldots$, $t_n = n\Delta t$

$$\implies N(n\cdot \Delta t) = (1+R\cdot \Delta t)N((nt)\Delta t) = \ldots = (1+R\Delta t)^n N_0$$

We have

$$(1+R\Delta t)^{\frac{1}{\Delta t R}\cdot Rn\Delta t}\cdot N_0 = (1+R\Delta t)^{\frac{1}{R\Delta t}Rt} N_0$$

Set $\Delta t \to 0$, we obtain $(1+R\Delta t)^{\frac{1}{R\Delta t}} \to e$. Then,

$$N(t) = e^{Rt}N_0 \text{ as } \Delta t \to 0$$

Next, let's analyze the property of the model above:

$$N(n\Delta t) = (1+R\Delta t)^n N_0$$

1. $1 + R\Delta t > 1$, then $N(n\Delta t) \to +\infty$, as $n \to +\infty$
2. $0 < 1 + R\Delta t < 1$, then $N(n\Delta t) \to 0$ as $n \to +\infty$

Conclusion: When $0 < 1 + R\Delta t < 1$, the model is acceptable; however, when $1 + R\Delta t > 1 (R > 0)$, the model should be modified. Thus, we may change our assumption: the growth rate is constant (e.g., the growth rate depends on the population itself)

§4.2 Continuous Population Model

Have:

$$N(t) = e^{Rt}N_0$$

Let's start from the previous lecture

$$N(t+\Delta t) = N(t) + R\Delta t \cdot N(t)$$

So

$$\frac{N(t+\Delta t)-N(t)}{\Delta t}=R\cdot N(t)$$
$$\lim_{\Delta t\to 0}\frac{N(t+\Delta t)-N(t)}{\Delta t}=\lim_{\Delta t\to 0}R\cdot N(t)=R\cdot N(t)$$
$$\frac{dN(t)}{dt}=R\cdot N(t)$$
$$\int\frac{dN(t)}{N(t)}=\int Rdt$$
$$\ln(N(t))=Rt+C$$
$$N(t)=e^Ce^{Rt}=N_0e^{Rt}$$

Evaluate the continuous model $N(t)=e^{Rt}N_0$

1. $0<R<1$: $N(t)\to\infty$ as $t\to\infty$ and $N(t)\uparrow$ as $t\uparrow$
2. $-1<R<0$: $N(t)\to 0$ as $t\to\infty$ and $N(t)\downarrow$ as $t\uparrow$

<u>Conclusion</u>: When $R<0$, the model is acceptable; however, when the growth rate $R>0$,the individuals (of a species) will compete each other as the resource is limited, $N(t)\to\infty$ as $t\to\infty$. Now, let's consider the density-dependent growth. Assumption:

- The growth rate is density dependent, i.e., $R(t)=R\left(N(t)\right)$
- If the population is small, then the influence of the environment is small, then we hope that the population has exponential growth.
- As $N(t)$ gets large enough, we don't expect the growth of $N(t)$. In other word, the growth rate $R\left(N(t)\right)\leq 0$ when $N(t)$ is large enough (since $R(t)$ is usually assume to be smooth, $R\left(N(t)\right)=0$ when $N(t)$ is large enough)

$$\frac{dN}{dt}=R\left(N(t)\right)\cdot N(t)$$

From our assumption, $R\left(N(t)\right)$ should be a constant when $N(t)$ is small and $R\left(N(t)\right)=0$ as $N(t)$ is large enough. So we can consider $R\left(N(t)\right)$ of the form

$$R\left(N(t)\right)=a-bN(t)$$

Thus, the model becomes

$$\frac{dN}{dt}=(a-bN)N$$

This is known as the logistic model.

> **Remark 4.2.** The discrete-time population model is called Beverton-Holt model.
>
> $$\begin{cases}N(t\cdot\Delta t)=\frac{R_0(N(t-1)\cdot\Delta t)}{1+N((t-1)\Delta t)/M}\\ R(N)=\frac{R_0}{1+N((t-1)\cdot\Delta t)/M}\end{cases}$$

§5 Lec 5: Oct 4, 2021

§5.1 Continuous and Discrete Population Models

Recall the continuous logistic population model

$$\frac{dN}{dt} = N\left(a - bN\right)$$

Let's manipulate this

$$\begin{aligned}
\frac{dN}{N(a-bN)} &= dt \\
\int \frac{1}{aN} + \frac{b}{a(a-bN)}dN &= \int dt \\
\frac{1}{a}\ln N - \frac{1}{a}\ln|a-bN| &= t + c \\
\ln\left|\frac{N}{a-bN}\right| &= at + \tilde{c} \\
\frac{N}{a-bN} &= e^{at+\tilde{c}} = Ce^{at} \\
N &= \frac{a}{b + Ce^{-at}}
\end{aligned}$$

Since $N(0) = N_0 \implies N_0 = \frac{a}{b+C}$, we have

$$N(t) = \frac{a}{b + \left(\frac{a}{N_0} - b\right)e^{-at}}$$

Let's now consider the relation between continuous logistic population and discrete-time logistic model for $\Delta t = 1$. For the discrete case,

$$\begin{cases} N(t) = \frac{R_0 N(t-1)}{1+N(t-1)/M} \\ R\left(N(t)\right) = \frac{R_0}{1+N(t-1)/M} \end{cases}$$

For the continuous case,

$$N(t) = \frac{a}{b + \left(\frac{a}{N_0} - b\right)e^{-at}}$$

Then,

$$N(t-1) = \frac{a}{b + \left(\frac{a}{N_0} - b\right)e^{-at}e^{a}}$$

Notice that

$$\begin{aligned}
\frac{1}{N(t)} &= \frac{b}{a} + \left(\frac{a}{N_0} - b\right)e^{at}/a \\
e^a \cdot \frac{1}{N(t-1)} &= \left(\frac{b}{a} + \left(\frac{a}{N_0} - b\right)e^{at}e^{-a}/a\right)\cdot e^a \\
\frac{1}{N(t)} - \frac{e^a}{N(t-1)} &= \frac{b}{a} - \frac{b}{a}e^a
\end{aligned}$$

For the continuous model, as $t \to \infty$, we can see that $N(t) \to \frac{a}{b}$ which is a good model.

§5.2 Discrete One-Species Model with an Age Distribution

Motivation: The birth and death rates will vary a lot if state A has more young citizens than state B.
Let's consider the period $\Delta t = 1$ year, define variables for a population at each age

$$\begin{aligned}
N_0(t) &= \#\text{ individuals whose age } < 1\\
N_1(t) &= \#\text{ of individuals one year old}\\
N_2(t) &= \#\text{ of individuals two years old}\\
&\vdots\\
N_M(t) &= \#\text{ of individuals M years old}
\end{aligned}$$

where M is the oldest age with proper population. Suppose

$$\begin{aligned}
b_m &= \text{ birth rate for a population that is m years old}\\
d_m &= \text{ death rate for a population that is m years old}
\end{aligned}$$

Let's consider the population $N_m(t+1)$

$$\begin{aligned}
N_0(t+1) &= b_0 N_0(t) + b_1 N_1(t) + \ldots + b_M N_M(t)\\
N_1(t+1) &= N_0(t) - d_0 N_0(t) = (1-d_0)N_0(t)\\
N_2(t+1) &= N_1(t) - d_1 N_1(t) = (1-d_1)N_1(t)\\
&\vdots\\
N_M(t+1) &= N_{M-1}(t) - d_{M-1}N_{M-1}(t) = (1-d_{M-1})\,N_{M-1}(t)
\end{aligned}$$

In matrix notation,

$$\vec{N}(t) = \begin{bmatrix} N_0(t)\\ N_1(t)\\ N_2(t)\\ \vdots\\ N_M(t) \end{bmatrix}$$

Then,

$$\begin{bmatrix} N_0(t+1)\\ N_1(t+1)\\ \vdots\\ N_M(t+1) \end{bmatrix} = \begin{bmatrix} b_0 & b_1 & \ldots & b_M\\ 1-d_0 & 0 & \ldots & 0\\ 0 & 1-d_1 & \ldots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & \ldots & 1-d_{M-1} & 0 \end{bmatrix} \begin{bmatrix} N_0(t)\\ N_1(t)\\ \vdots\\ N_M(t) \end{bmatrix}$$

$\implies \vec{N}(t+1) = L\vec{N}(t)$ – the matrix is called Leslie matrix.

§6 Lec 6: Oct 6, 2021

§6.1 Stable Age Distribution

Definition 6.1 (Stable Age Distribution) — A stable age distribution exists if the populations approach an age distribution that is independent of time as time increases, i.e., $\frac{1}{\|\vec{N}(t)\|_1}\vec{N}(t) \to \vec{v}$ as $t \to \infty$ where

$$\|\vec{N}(t)\|_1 = \sum_{i=0}^{M} |N_i(t)|$$

Assume that the Leslie matrix

$$L = \begin{bmatrix} 2 & 1 \\ 0.44 & 0 \end{bmatrix}$$

and

$$\vec{N}(0) = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$

Let's track the evolution of the population age groups. We have

$$\vec{N}(t+1) = L \cdot \vec{N}(t)$$
$$\vec{N}(1) = L\vec{N}(0) = \begin{bmatrix} 2 & 1 \\ 0.44 & 0 \end{bmatrix}\begin{bmatrix} 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 300 \\ 44 \end{bmatrix}$$
$$\vec{N}(2) = L\vec{N}(1) = \begin{bmatrix} 2 & 1 \\ 0.44 & 0 \end{bmatrix}\begin{bmatrix} 300 \\ 44 \end{bmatrix} = \begin{bmatrix} 644 \\ 132 \end{bmatrix}$$

Continue this process we obtain

$$\vec{N}(3) = \begin{bmatrix} 1420 \\ 2834 \end{bmatrix}, \quad \begin{bmatrix} 3123.4 \\ 624.8 \end{bmatrix}, \dots$$

<u>Observation</u>: The population appears to grow over time without bound.a The ratio $\frac{N_0(t+1)}{N_0(t)}$ and $\frac{N_1(t+1)}{N_1(t)}$

$$\frac{N_0(1)}{N_0(0)} = \frac{300}{100} = 3 \qquad \frac{N_0(2)}{N_0(1)} = \frac{644}{300} = 2.1467$$
$$\frac{N_0(3)}{N_0(2)} = \frac{1420}{300} = 2.2050 \qquad \frac{N_0(4)}{N_0(3)} = 2.1996$$

Apply the same process to N_1 and we can notice that they both approach 2.2, i.e.,

$$\begin{bmatrix} N_0(t+1) \\ N_1(t+1) \end{bmatrix} \approx 2.2 \begin{bmatrix} N_0(t) \\ N_1(t) \end{bmatrix}$$

The fraction of the population in age 0 and fraction of the population in age 0 is 1.

$$\frac{N_0(0)}{N_0(0)+N_1(0)} = \frac{100}{100+100} = \frac{1}{2} \qquad \frac{N_0(1)}{N_0(1)+N_1(1)} = \frac{300}{344} \approx 0.872$$
$$\frac{N_0(2)}{N_0(2)+N_1(2)} \approx 0.8407 \qquad \frac{N_0(3)}{N_0(3)+N_1(3)} \approx 0.8336 \qquad \dots$$

With these calculations, we can see that

$$\frac{N_0(t)}{N_0(t)+N_1(t)} \to 0.833 \implies \frac{N_1(t)}{N_0(t)+N_1(t)} \to 0.167$$

So

$$\frac{1}{\|\vec{N}(t)\|}\begin{bmatrix} N_0(t) \\ N_1(t) \end{bmatrix} \to \begin{bmatrix} 0.833 \\ 0.167 \end{bmatrix}$$

Recall that

$$\begin{bmatrix} N_0(t+1) \\ N_1(t+1) \end{bmatrix} = L\begin{bmatrix} N_0(t) \\ N_1(t) \end{bmatrix} \approx 2.2\begin{bmatrix} N_0(t) \\ N_1(t) \end{bmatrix}$$

Claim 6.1. 2.2 is one eigenvalue of the Leslie matrix L.

<u>Guess</u>: $\begin{bmatrix} 0.833 \\ 0.167 \end{bmatrix}$ is an eigenvector of the Leslie matrix L. Let's check.

$$\begin{aligned} \det(L-\lambda I) &= \det\left(\begin{bmatrix} 2 & 1 \\ 0.44 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \\ &= (2-\lambda)(-\lambda) - 0.44 \\ &= (\lambda - 2.2)(\lambda + 0.2) \end{aligned}$$

Thus, $\lambda = 2.2$, $\lambda = -0.2$ which verifies our claim. When $\lambda = 2.2$, we can find the corresponding eigenvector as follows

$$\begin{aligned} L - 2.2I &= \begin{bmatrix} 2 & 1 \\ 0.44 & 0 \end{bmatrix} - \begin{bmatrix} 2.2 & 0 \\ 0 & 2.2 \end{bmatrix} \\ &= \begin{bmatrix} -0.2 & 1 \\ 0.44 & -2.2 \end{bmatrix} \end{aligned}$$

We need to find the null space of $L - 2.2I$, i.e.

$$\begin{bmatrix} -0.2 & 1 \\ 0.44 & -2.2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5x_2 \\ x_2 \end{bmatrix} = 6x_2\begin{bmatrix} \frac{5}{6} \\ \frac{1}{6} \end{bmatrix}$$

Thus, $\begin{bmatrix} \frac{5}{6} \\ \frac{1}{6} \end{bmatrix} \approx \begin{bmatrix} 0.833 \\ 0.167 \end{bmatrix}$ is the corresponding eigenvector (of 2.2).
From this example, we may guess in order to find the stable age distribution, we need to find the maximum eigenvalue of the Leslie matrix and then find the corresponding normalized eigenvector. Now, we will try to check our guess for the general Leslie model.

$$\vec{N}(t+\Delta t) = L\vec{N}(t)$$

with

$$\vec{N}(t) = \begin{bmatrix} N_0(t) \\ N_1(t) \\ \vdots \\ \vec{N}_M(t) \end{bmatrix} \quad \text{and} \quad L \in \mathbb{R}^{(M+1)\times(M+1)}$$

being a non-negative. Let's assume that $\vec{N}(0) = \vec{N}_0$, then we have $\vec{N}(n\cdot\Delta t) = L\vec{N}\left((n-1)\cdot\Delta t\right) = \ldots = L^n \cdot \vec{N}_0$. Suppose that the Leslie matrix L is diagonalizable, i.e., there are $M+1$ eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_{M+1}$ and $M+1$ linearly independent eigenvectors $\vec{v}_1, \ldots, \vec{v}_{M+1}$.

§7 Lec 7: Oct 7, 2021

§7.1 Stable Age Distribution (Cont'd)

Assume that $\vec{N}(0) = \vec{N}_0$, then we have $\vec{N}(n \cdot \Delta t) = L\vec{N}((n-1) \cdot \Delta t) = \ldots = L^n \vec{N}_0$. Suppose that the Leslie matrix L is diagonalizable, i.e., there are $M+1$ eigenvalues $\lambda_1 \geq \ldots \geq \lambda_{M+1}$ and $M+1$ linearly indep. eigenvectors $\vec{v}_1, \ldots, \vec{v}_{M+1}$.

$$L = VDV^{-1}$$

where

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{M+1} \end{bmatrix}, \quad V = \begin{bmatrix} \vec{v}_1 & \ldots & \vec{v}_{M+1} \end{bmatrix}$$

Since $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{M+1}$ are linearly independent, $\{\vec{v}_1, \ldots, \vec{v}_{M+1}\}$ is a basis for $\mathbb{R}^{M+1}$. Then, there exists $c_1, c_2, \ldots, c_{M+1}$ s.t.

$$\vec{N}_0 = \sum_{i=1}^{M+1} c_i \vec{v}_i$$

Thus,

$$\begin{aligned} \vec{N}(n \cdot \Delta t) &= L^n \vec{N}_0 \\ &= L^n \left(\sum_{i=1}^{M+1} c_i \vec{v}_i \right) \\ &= \sum_{i=1}^{M+1} c_i (L^n \vec{v}_i) \\ &= \sum_{i=1}^{M+1} c_i \lambda_i^n \vec{v}_i \\ &= c_1 \vec{v}_1 + \sum_{i=2}^{M+1} c_i \left(\frac{\lambda_i}{\lambda_1} \right)^n \vec{v}_i \end{aligned}$$

If $|\lambda_1| > |\lambda_i|$ for $i \geq 2$, then $\frac{|\lambda_i|}{|\lambda_1|} < 1$ which means

$$\left| \frac{\lambda_i}{\lambda_1} \right|^n \to 0 \text{ as } n \to \infty \text{ for } i \geq 2$$

Therefore, we have

$$\frac{1}{\lambda_1^n} \vec{N}(n \cdot \Delta t) = c_1 \vec{v}_1 + \sum_{i=2}^{M+1} c_i \left(\frac{\lambda_i}{\lambda_1} \right)^n \vec{v}_i \to c_1 \vec{v}_1$$

as $n \to \infty$. Thus, for large value of n, we can approximate $\vec{N}(n \cdot \Delta t)$ by $c_1 \lambda_1^n \vec{v}_1$.
The process to find "stable age distribution":

1. Find the maximum eigenvalue of the Leslie matrix L

$$\det(L - \lambda I) = 0$$

2. $|\lambda_1| > |\lambda_i|$

3. Find one corresponding eigenvector $\vec{v}_i$ associated to λ_1

4. Normalize $\vec{v}_1$: $\frac{\vec{v}_1}{\|\vec{v}_1\|}$

§7.2 Logistic Equations with Phase Plane Solution

Definition 7.1 (Phase Plane) — A phase plane is a visual display of certain characteristics of certain kinds of differential equations. A coordinate plane with axes being the values of two variables.

Logistic Equation:

$$\frac{dN}{dt} = N \cdot (a - bN)$$

Notice that this is an autonomous differential equation. One important thing for autonomous DE is the stability of the equilibrium points.

$$N(a - bN) = 0 \implies N = 0, \quad N = \frac{a}{b}$$

We can observe that the equilibrium point $N(t) = \frac{a}{b}$ is stable and $N(t) = 0$ is unstable. Now, let's show the stability of equilibrium points from an analytical aspect. We will first analyze the solution in the neighborhood of $N = \frac{a}{b}$. Let's consider the Taylor's expansion of $f(N) = N(a - bN)$ at $N = \frac{a}{b}$.

$$\begin{aligned}
f(N) &= N \cdot (a - bN) \\
&= f\left(\frac{a}{b}\right) + \frac{d}{dN} f(N)\Big|_{N=\frac{a}{b}\left(N-\frac{a}{b}\right)} + \frac{d^2 f(N)}{dN^2}\Big|_{N=\frac{a}{b}\cdot\frac{1}{2}\left(N-\frac{a}{b}\right)^2} \\
&= 0 + (-a)\left(N - \frac{a}{b}\right) + (-b)\left(N - \frac{a}{b}\right)^2 \\
&\approx -a \cdot \left(N - \frac{a}{b}\right)
\end{aligned}$$

Therefore,

$$\frac{dN}{dt} = N \cdot (a - bN) \approx (-a)\left(N - \frac{a}{b}\right)$$

near the neighborhood of $N = \frac{a}{b}$.

$$\frac{dN}{dt} = -a\left(N - \frac{a}{b}\right)$$

Let $y = N - \frac{a}{b} \implies \frac{dy}{dt} = \frac{dN}{dt}$

$$\begin{aligned}
\frac{dy}{dt} &= -ay \implies y = Ce^{-at} \\
N - \frac{a}{b} &= Ce^{-at} \\
N(t) &= \frac{a}{b} + Ce^{-at}
\end{aligned}$$

as $t \to \infty$, we have $N(t) \to \frac{a}{b}$. Thus, $N(t) = \frac{a}{b}$ is stable.

§8 Lec 8: Oct 11, 2021

§8.1 Logistic Equation with Phase Plane Solution (Cont'd)

We'd like to illustrate $N(t) = \frac{a}{b}$ is stable from perturbation analysis point of view. Let $N(t) = \frac{a}{b} + \varepsilon \cdot N_1(t)$ by assuming that

$$|\varepsilon N_1(t)| \ll \frac{a}{b}$$

Let's substitute $N(t) = \frac{a}{b} + \varepsilon N_1(t)$ into the original DE:

$$\begin{aligned}
\frac{dN}{dt} &= N\left(a - bN\right) \\
\frac{d}{dt}\left(\frac{a}{b} + \varepsilon N_1(t)\right) &= \varepsilon \frac{d}{dt} N_1(t) \\
&= \left(\frac{a}{b} + \varepsilon N_1(t)\left(a - \left(a + \varepsilon b N_1(t)\right)\right)\right) \\
&= -\frac{a}{b}\varepsilon b N_1(t) - \varepsilon^2 b N_1^2(t) \\
&= -a\varepsilon N_1(t) - \varepsilon^2 b N_1^2(t) \\
\frac{d}{dt} N_1(t) &= -aN_1(t) - \varepsilon b N_1^2(t) \\
&\approx -aN_1(t)
\end{aligned}$$

Thus, $N_1(t) = Ce^{-at} \to 0$ as $t \to \infty$ and $N(t) \to \frac{a}{b}$ as $t \to \infty$. So, $N(t) = \frac{a}{b}$ is stable.

§8.2 SIR Model

The SIR model was first used by Kermack and McKendrick in 1947. Now this model is popularly used to study the spread of infectious disease such as measles, Covid 19, etc. It consists of three parts:

- S: the number of susceptible individuals
- I: the number of infected individuals
- R: the number of recovered individuals

The process of the spread of the infectious disease is at the beginning where all the individuals are susceptible. The some of them become infectious and then become recovered individuals.

$$\boxed{S} \longrightarrow \boxed{I} \longrightarrow \boxed{R}$$

We assume that the total population

$$N = S + I + R$$

is fixed. Let β be the contact rate (individuals who come into contact with each other). Let γ be the recovery rate for the infected individuals.

§9 Lec 9: Oct 13, 2021

§9.1 SIR Model (Cont'd)

SIR model without vital dynamics

- We assume that the course of the infection is short.
- The birth and death can be ignored.
- The total number N can be treated as a constant.

Observation: The more interactions between the people in S and I the more individuals in S will "transfer" to I.

$$\frac{dS}{dt} = -\beta \cdot S \cdot I/N \tag{1}$$

The change of I will involve two parts: $S \to I$ which will increase I, and $I \to R$ which will decrees I

$$\frac{dI}{dt} = \beta \cdot S \cdot I/n - \gamma \cdot I \tag{2}$$

$$\frac{dR}{dt} = \gamma I \tag{3}$$

Let's combine the three equations.

$$\begin{cases} \frac{dS}{dt} = \frac{-\beta SI}{N} \\ \frac{dI}{dt} = \frac{\beta SI}{N} - \gamma I \\ \frac{dR}{dt} = \gamma I \end{cases}$$

with $S + I + R = N$ being a constant. Thus, to understand the model, we only need to understand

$$\begin{cases} \frac{dS}{dt} = -\frac{\beta SI}{N} \\ \frac{dI}{dt} = \frac{\beta SI}{N} - \gamma I \end{cases}$$

Let's normalize S, I, R first by setting

$$s = \frac{S}{N}, \quad i = \frac{I}{n}, \quad r = \frac{R}{N}$$

$$\frac{ds}{dt} = \frac{1}{N}\frac{dS}{dt} = \frac{1}{N}\left(\frac{-\beta SI}{N}\right) = -\beta si$$

$$\frac{di}{dt} = \frac{1}{N}\frac{dI}{dt} = \frac{1}{N}\left(\frac{\beta SI}{N} - \gamma I\right) = \beta si - \gamma i$$

and we know $r = 1 - i - s$.

Remark 9.1. $s \in [0,1]$, $i \in [0,1]$, $r \in [0,1]$.

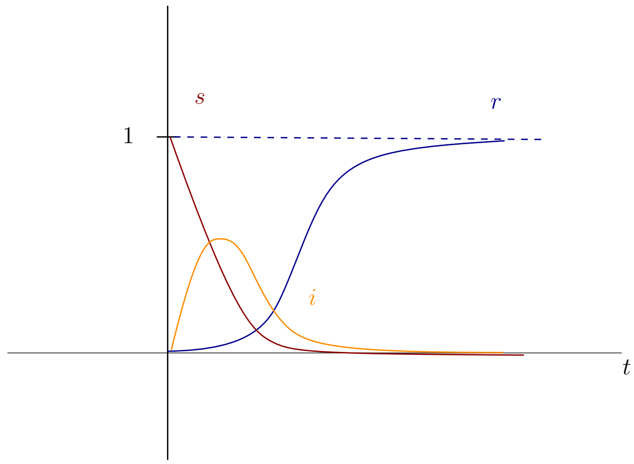
Next, let's analyze the new model

$$\begin{cases} \frac{ds}{dt} = -\beta si \\ \frac{di}{dt} = \beta si - \gamma i = (\beta s - \gamma) i \end{cases}$$

Observe that

1. $\frac{ds}{dt} = -\beta si \le 0 \implies s \downarrow$
2. $\frac{di}{dt} = (\beta s - \gamma) i = 0 \implies i = 0,\ s = \frac{\gamma}{\beta}$. When $\frac{di}{dt} > 0$, we know that $s > \frac{\gamma}{\beta}$. Similarly, when $\frac{di}{dt} < 0$, $s < \frac{\gamma}{\beta}$.

Let's draw the graph for s, i, r together.

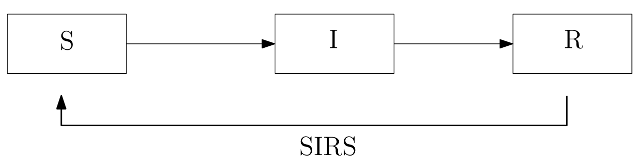


SIR Model with Vital Dynamics:
For this model, the disease will last for a long period. It is not reasonable to ignore the birth and death rate. It is not a reasonable assumption that $S + I + R = N$ where N is a constant. For this case, let's introduce new parameters birth rate b and death rate d.

$$\frac{dS}{dt} = \frac{-\beta SI}{N} + bN - dS$$
$$\frac{dI}{dt} = \frac{\beta SI}{N} - \gamma I - dI$$
$$\frac{dR}{dt} = \gamma I - dR$$

§9.2 SIRS Model

SIRS Model without Vital Dynamics:



$$\begin{cases} \frac{dS}{dt} = -\frac{\beta SI}{N} + \alpha R \\ \frac{dI}{dt} = \frac{\beta SI}{N} - \gamma I \\ \frac{dR}{dt} = \gamma I - \alpha R \end{cases} \qquad \text{and } S + I + R = N \text{ fixed}$$

SIRS with Vital Dynamics: Similar to SIR with vital dynamics, we need to take the birth and death rate into account.

$$\begin{cases} \frac{dS}{dt} = -\frac{\beta SI}{N} + \alpha R + bN - dS \\ \frac{dI}{dt} = \frac{\beta SI}{N} - \gamma I - dI \\ \frac{dR}{dt} = \gamma I - \alpha R - dR \end{cases} \qquad \text{and } N(t) = S + I + R \text{ not fixed}$$

Intro to Two-Species Models: There are several different relations: competition, predator and prey, symbiosis, mutualism.

§10 Lec 10: Oct 15, 2021

§10.1 Solutions to System of Differential Equations

Theorem 10.1

If $(\lambda, \vec{v})$ is an eigen pair of M, then $e^{\lambda t}\vec{v}$ is a solution of $\frac{d\vec{y}(t)}{dt} = M\vec{y}(t)$.

Proof. Set $\vec{y}(t) = e^{\lambda t}\vec{v}$. Then we have

$$\frac{d}{dt}\vec{y}(t) = \frac{d}{dt}\left(e^{\lambda t}\vec{v}\right) = \left(\frac{d}{dt}e^{\lambda t}\right)\vec{v} = \lambda e^{\lambda t}\vec{v} \tag{1}$$

and

$$\begin{aligned} M\vec{y}(t) &= M\left(e^{\lambda t}\vec{v}\right) \\ &= e^{\lambda t}M\vec{v} \\ &= e^{\lambda t}(\lambda\vec{v}) \\ &= \lambda e^{\lambda t}\vec{v} \end{aligned} \tag{2}$$

Combining (1) and (2) we have $\vec{y}(t) = e^{\lambda t}\vec{v}$ is a solution of $\frac{d}{dt}\vec{y}(t) = M\vec{y}(t)$. $\square$

From the above theorem, we could find n solutions $e^{\lambda_1 t}\vec{v}_1, \ldots, e^{\lambda_n t}\vec{v}_n$.

Question 10.1. Are these n solutions linearly independent?

If $\sum_{i=1}^n c_i\vec{v}_i = \vec{0}$ where $c_i = 0$ in which $i = 1, \ldots, n$, then $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent. Know: $\sum_{i=1}^n c_i\vec{v}_i = \vec{0}$ and $M\vec{v}_i = \lambda_i\vec{v}_i$. We want to show $c_i = 0$ for all i. Let's use mathematical induction to show this.

- When $n = 1$, $c_1\vec{v}_1 = \vec{0} \implies c_1 = 0$ because $\vec{v}_1 \neq 0$
- Assume that the statement is correct when $n = k$.
- We want to show now that the statement also applies for the case $n = k + 1$. Have

  $$\sum_{i=1}^{k+1} c_i M\vec{v}_i = \sum_{i=1}^{k+1} c_i\lambda_i\vec{v}_i = \vec{0} \tag{3}$$

  Idea: get rid of one term so that we could use the induction assumption.

  $$\sum_{i=1}^{k+1} c_i\vec{v}_i = \vec{0} \implies \sum_{i=1}^{k+1} c_i\lambda_{k+1}\vec{v}_i = \vec{0} \tag{4}$$

  So (3) – (4),

  $$\begin{gathered} \sum_{i=1}^{k} c_i\left(\lambda_i - \lambda_{k+1}\right)\vec{v}_i = \vec{0} \\ c_i\left(\lambda_i - \lambda_{k+1}\right) = 0 \end{gathered}$$

  Thus, $c_i = 0$ since λ_i are distinct.

  $$\sum_{i=1}^{k+1} c_i\vec{v}_i = c_{k+1}\vec{v}_{k+1} = \vec{0} \implies c_{k+1} = 0$$

  Thus, the statement is true for $n = k + 1$.

Theorem 10.2

If M has n distinct eigenvalues $\lambda_1, \dots, \lambda_n$ with the corresponding eigenvectors $\vec{v}_1, \dots, \vec{v}_n$ then $\left\{e^{\lambda_1 t}\vec{v}_1, \dots, e^{\lambda_n t}\vec{v}_n\right\}$ are linearly independent.

Proof. Left as exercise. $\square$

Example 10.3

Solve the following ODE:

$$\begin{cases} \frac{dx}{dt} = 2x - 3y \\ \frac{dy}{dt} = x - 2y \end{cases}$$

Let's rewrite the ODE into the matrix vector form.

$$\vec{Y}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad M = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$$

Now, let's find the eigenvalues and the corresponding eigenvectors of M.

$$\begin{aligned} \det\left(M - \lambda I\right) &= \det \begin{bmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{bmatrix} \\ &= \lambda^2 - 1 = 0 \end{aligned}$$

So, $\lambda_{1,2} = \pm 1$.

- For $\lambda_1 = -1$,

$$\begin{aligned} (M + I)\vec{v}_1 &= \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \\ &= - \begin{bmatrix} x \\ y \end{bmatrix} \\ \implies \vec{v}_1 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

- For $\lambda = 1$, using the same process we obtain $\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Therefore,

$$\vec{Y}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

is the general solution for $\frac{d\vec{Y}(t)}{dt} = M\vec{Y}(t)$.

§11 Lec 11: Oct 18, 2021

§11.1 Solutions to System of Differential Equations (Cont'd)

Example 11.1 (Cont'd of the last example from last lecture)
Suppose that the initial conditions are $x(0) = 8$ and $y(0) = 4$. Find the explicit solution for the DE. Recall

$$\vec{Y}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

is the general solution. So,

$$\begin{aligned} c_1 e^{-0} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{0} \begin{bmatrix} 3 \\ 1 \end{bmatrix} &= \begin{bmatrix} 8 \\ 4 \end{bmatrix} \\ \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} 8 \\ 4 \end{bmatrix} \\ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 4 \end{bmatrix} &= \begin{bmatrix} 2 \\ 2 \end{bmatrix} \end{aligned}$$

Question 11.1. If there are some complex eigenvalues for the real matrix M, how can we find the general real solutions for $\frac{d\vec{Y}(t)}{dt} = M\vec{Y}(t)$?

Example 11.2
Find the real solution for the ODE

$$\begin{cases} \frac{dx}{dt} = x(t) - y(t) \\ \frac{dy}{dt} = x(t) + y(t) \end{cases}$$

Notice that

$$\vec{Y}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

First, let's find the eigenvalues and their corresponding eigenvectors of M.

$$\begin{aligned} \det(M - \lambda I) &= \det\left(\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) \\ &= \lambda^2 - 2\lambda + 2 = 0 \end{aligned}$$

So, $\lambda = 1 \pm i$.

- For $\lambda = 1 + i$, we have $\begin{bmatrix} i \\ 1 \end{bmatrix}$ is a corresponding eigenvector.
- For $\lambda = 1 - i$, we have $\begin{bmatrix} -i \\ 1 \end{bmatrix}$ is a corresponding eigenvector.

Thus,

$$\vec{Y}(t) = c_1 e^{(1+i)t} \begin{bmatrix} i \\ 1 \end{bmatrix} + c_2 e^{(1-i)t} \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

is the general solution for $\frac{d\vec{Y}(t)}{dt} = M\vec{Y}(t)$.

Question 11.2. How do we transform the general solution to general real solution?

Recall that

$$e^{ai} = \cos(a) + i\sin(a), \quad a \in \mathbb{R}$$

So,

$$\begin{aligned}
\vec{Y}(t) &= c_1 e^{(1+i)t}\begin{bmatrix} i \\ 1 \end{bmatrix} + c_2 e^{(1-i)t}\begin{bmatrix} -i \\ 1 \end{bmatrix} \\
&= c_1 e^t e^{ti}\begin{bmatrix} i \\ 1 \end{bmatrix} + c_2 e^t e^{-ti}\begin{bmatrix} -i \\ 1 \end{bmatrix} \\
&= c_1 e^t\left(\cos(t) + i\sin(t)\right)\begin{bmatrix} i \\ 1 \end{bmatrix} + c_2 e^t\left(\cos(-t) + i\sin(-t)\right)\begin{bmatrix} -i \\ 1 \end{bmatrix} \\
&= c_1 e^t\begin{bmatrix} \left(\cos(t) + i\sin(t)\right)i \\ \cos(t) + i\sin(t) \end{bmatrix} + c_2 e^t\begin{bmatrix} \left(\cos(-t) + i\sin(-t)\right)(-i) \\ \cos(-t) + i\sin(-t) \end{bmatrix} \\
&= c_1 e^t\begin{bmatrix} -\sin(t) + \cos(t)i \\ \cos(t) + \sin(t)i \end{bmatrix} + c_2 e^t\begin{bmatrix} -\sin(t) - \cos(t)i \\ \cos(t) - \sin(t)i \end{bmatrix} \\
&= (c_1 + c_2)e^t\begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} + (c_1 - c_2)ie^t\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}
\end{aligned}$$

Because c_1 and c_2 are arbitrary numbers we could choose $c_1 + c_2 = 1$ and $c_1 - c_2 = 0$ or $c_1 + c_2 = 0$ and $(c_1 - c_2)i = 1$.

$$e^t\begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}, \quad e^t\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

are two linearly independent real solutions of $\frac{d\vec{Y}(t)}{dt} = M\vec{Y}(t)$. The general real solutions can be represented by

$$\vec{Y}(t) = \tilde{c_1}e^t\begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} + \tilde{c_2}e^t\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

where $\tilde{c_1}, \tilde{c_2} \in \mathbb{R}$.

Method II: Exponential Method

When $n = 1$, we have ODE

$$\frac{dx}{dt} = mx \implies x(t) = e^{mt}x_0$$

is the solution of $\frac{dx}{dt} = mx$. Recall that

$$\begin{aligned}
e^{mt} &= \sum_{j=0}^{\infty}\frac{(mt)^j}{j!} \\
e^{Mt} &= \sum_{j=0}^{\infty}\frac{(Mt)^j}{j!} = \sum_{j=1}^{\infty}\frac{t^jM^j}{j!}
\end{aligned}$$

To get a clearer look at e^{Mt}, let's consider the case that M is diagonal, e.g., $M = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

$$\begin{aligned}
e^{Mt} &= \sum_{j=0}^{\infty} \frac{t^j M^j}{j!} \\
&= \sum_{j=0}^{\infty} \frac{t^j \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^j}{j!} \\
&= \sum_{j=0}^{\infty} \frac{t^j \begin{bmatrix} 2^j & 0 \\ 0 & 3^j \end{bmatrix}}{j!} \\
&= \begin{bmatrix} \sum_{j=0}^{\infty} \frac{(2t)^j}{j!} & 0 \\ 0 & \sum_{j=0}^{\infty} \frac{(3t)^j}{j!} \end{bmatrix} \\
&= \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix}
\end{aligned}$$

If M is diagonalizable, how can we compute e^{Mt}?

$$\begin{aligned}
M &= JDJ^{-1} \\
e^{Mt} &= \sum_{j=0}^{\infty} \frac{t^j M_j}{j!} \\
&= \sum_{j=0}^{\infty} \frac{t^j (JDJ^{-1})^j}{j!} \\
&= \sum_{j=0}^{\infty} \frac{t^j JD^j J^{-1}}{j!} \\
&= J \left(\sum_{j=0}^{\infty} \frac{t^j D^j}{j!} \right) J^{-1} \\
&= Je^{Dt} J^{-1}
\end{aligned}$$

§12 Lec 12: Oct 22, 2021

§12.1 Asymptotic Properties of Solutions to Linear ODE System

Consider:

$$\begin{cases} \frac{dx}{dt} = ax + by \\ \frac{dy}{dt} = cx + dy \end{cases}$$

Then,

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \vec{Y}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

So,

$$\begin{aligned} \det(M - \lambda I) &= \det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} \\ &= (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 - (a+d)\lambda + ad - bc \end{aligned}$$

Set $p = a + d$, $q = ad - bc$. Then,

$$\begin{aligned} \det(M - \lambda I) = \lambda^2 - p\lambda + q = 0 \\ \Delta = p^2 - 4q \end{aligned}$$

Thus the eigenvalues distribution of the matrix M are as follows

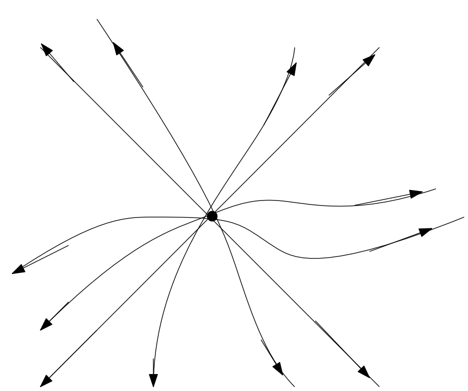
1. $\Delta > 0$, the eigenvalues are real and distinct (node or saddle)
2. $\Delta = 0$, repeated real eigenvalues (improper node)
3. $\Delta < 0$, the eigenvalues are complex (spiral)

First, let's consider the case where we have two real roots: $\Delta > 0$.

a) positive real roots $p > 0$, $q > 0$

$$\vec{Y}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$

Since $\lambda_1, \lambda_2 > 0 \implies e^{\lambda_1 t} -> \infty$



Example 12.1

Consider

$$M = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$$

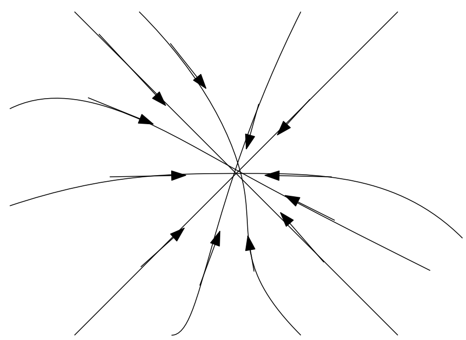
then

$$\det(M - \lambda I) = \lambda^2 - 6\lambda + 7 = 0$$
$$\lambda = 3 \pm \sqrt{2} > 0$$
$$\vec{Y}(t) = c_1 e^{(3+\sqrt{2})t} \begin{bmatrix} 1 \\ \sqrt{2} - 1 \end{bmatrix} + c_2 e^{(3-\sqrt{2})t} \begin{bmatrix} 1 \\ -\sqrt{2} - 1 \end{bmatrix}$$

b) Two negative real solutions: $p < 0$, $q > 0$.

$$\vec{Y}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$$

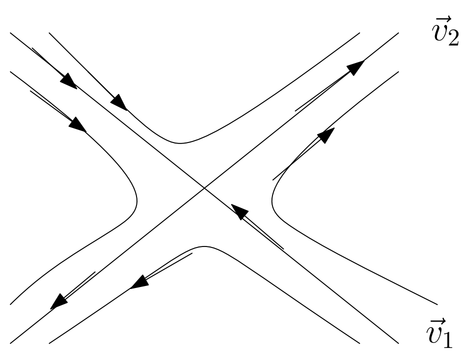
Since $\lambda_1, \lambda_2 < 0 \implies e^{\lambda_1 t} \to 0$, $e^{\lambda_2 t} \to 0$ as $t \to \infty$. So the equilibrium solution is stable.



c) $\lambda_1 < 0$ and $\lambda_2 > 0$ and so $q < 0$

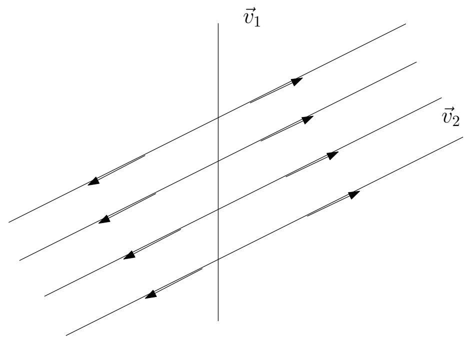
$$\vec{Y}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$$

Since $\lambda_1 < 0 \implies e^{\lambda_1 t} \to 0$ as $t \to \infty$ and $\lambda_2 > 0 \implies e^{\lambda_2 t} \to \infty$ as $t \to \infty$.



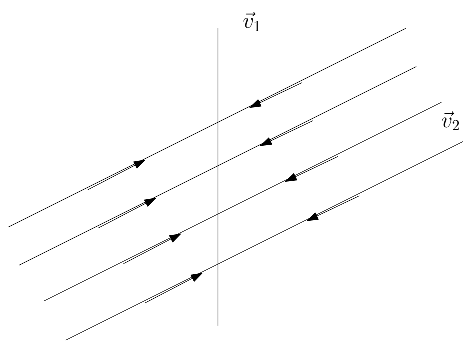
d) One root is 0: $q = 0$ and another root is positive: $p > 0$. Let's assume that $\lambda_1 = 0$, $\lambda_2 > 0$

$$\vec{Y}(t) = c_1 \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$$



e) One root is 0: $q = 0$, and another root is negative: $p < 0$

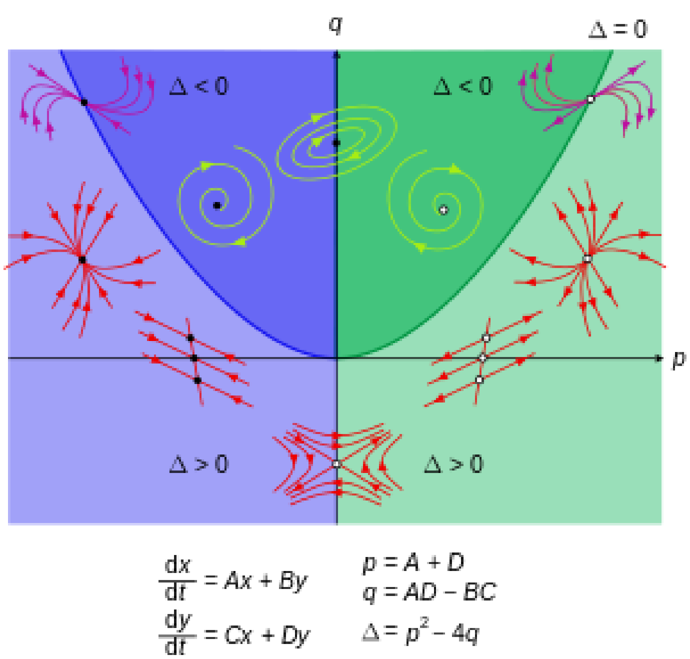
$$\vec{Y}(t) = c_1\vec{v}_1 + c_2e^{\lambda_2 t}\vec{v}_2$$



§13 Lec 13: Oct 25, 2021

§13.1 Asymptotic Properties (Cont'd)

2. Real and equal: $\Delta = 0$
   a) Both are positive, $p > 0$, the equilibrium point is unstable because $\vec{Y}(t) = (c_1 + c_2 t)e^{\lambda t} \to \infty$
   b) Both negative, $p < 0$, the equilibrium point is stable because $\vec{Y}(t) = (c_1 + c_2 t)e^{\lambda t} \to 0$ as $t \to \infty$
   c) Both zero, $p = 0$, the equilibrium point is unstable.
3. Complex roots for $\lambda^2 - p\lambda + q = 0$:
$$\Delta := p^2 - 4q < 0$$
   Then we have
$$\lambda = \frac{p \pm \sqrt{\Delta}}{2} = \frac{p \pm i\sqrt{-\Delta}}{2} = \mu \pm vi$$
   a) Real part is positive: $p > 0$ then we could write
$$\vec{Y}(t) = e^{\mu t}\left(c_1 \sin(vt)\vec{v}_1 + c_2 \cos(vt)\vec{v}_2\right)$$
   since $\mu > 0$, $e^{\mu t} \to \infty$ as $t \to \infty$. Therefore, the equilibrium point is unstable.
   b) Real part is negative: $p < 0$
$$\vec{Y}(t) = e^{\mu t}\left(c_1 \sin(vt)\vec{v}_1 + c_2 \cos(vt)\vec{v}_2\right)$$
   where $e^{\mu t} \to 0$ as $t \to \infty$. Thus, the equilibrium point is stable.
   c) The real part is zero: $p = 0$. Then the solution can be written as
$$\vec{Y}(t) = c_1 \sin(bt)\vec{v}_1 + c_2 \cos(bt)\vec{v}_2$$
   Notice that for any fixed constants, $\vec{Y}(t)$ is a cyclic function of t. We call the equilibrium point is neutrally stable.



Question 13.1. Why do we spend so much time to learn how to solve linear ODE and study their asymptotic properties?

Let's introduce a new section to answer this.

§13.2 Introduction to Two-Species Models

Let's consider a simple model between two species by assuming that the population of these two species are only depending on their population. First, let us denote the populations of these two species as N_1 and N_2. By our assumption that the change of the populations N_1, N_2 only depends on N_1, N_2, i.e., we just ignore the other environmental factors.

$$\begin{cases} \frac{dN_1}{dt} = g(N_1, N_2) \\ \frac{dN_2}{dt} = f(N_1, N_2) \end{cases} \tag{*}$$

If we assume that there is no migration of these two species

$$g(0, N_2) = 0, \quad f(N_1, 0) = 0$$

For a non-linear ODE, we're interested in the stability of the equilibrium points. Recall the definition of equilibrium point: constants solutions for the original DE. Thus, we could find the equilibrium point by solving

$$\begin{cases} g(N_{1e}, N_{2e}) = 0 \\ f(N_{1e}, N_{2e}) = 0 \end{cases} \tag{**}$$

Now let's assume that (N_{1e}, N_{2e}) is a solution for (**). Our goal is to study the stability of (N_{1e}, N_{2e}). We consider small perturbations on (N_{1e}, N_{2e}), i.e.,

$$\begin{cases} N_1(t) = N_{1e} + \varepsilon N_{11}(t) \\ N_2(t) = N_{2e} + \varepsilon N_{21}(t) \end{cases}$$

Let's substitute $N_1(t)$, $N_2(t)$ back to the original DE system

$$\begin{cases} \frac{dN_1}{dt} = g(N_1, N_2) \\ \frac{dN_2}{dt} = f(N_1, N_2) \end{cases}$$

Then we have

$$\begin{cases} \frac{d}{dt}\left(N_1e + \varepsilon N_{11}(t)\right) = \varepsilon \frac{dN_{11}(t)}{dt} = g\left(N_1e + \varepsilon N_{11}(t), N_2e + \varepsilon N_{21}(t)\right) \\ \frac{d}{dt}\left(N_2e + \varepsilon N_{21}(t)\right) = \varepsilon \frac{dN_{21}(t)}{dt} = f\left(N_1e + \varepsilon N_{11}(t), N_2e + \varepsilon N_{21}(t)\right) \end{cases}$$

Since ε is pretty small, we can consider the Taylor expansion of the function g, f at (N_{1e}, N_{2e}). Recall the Taylor expansion with two variables

$$\begin{aligned} \phi\left(x + \Delta x, y + \Delta y\right) &= \phi(x, y) + \left(\frac{\partial \phi}{\partial x}(x, y) \cdot \Delta x + \frac{\partial \phi}{\partial y}(x, y) \cdot \Delta y\right) \\ &\quad + \frac{1}{2!}\left(\frac{\partial^2 \phi}{\partial x^2}(x, y)\Delta x^2 + 2\frac{\partial^2 \phi}{\partial x \partial y}\Delta x \Delta y + \frac{\partial^2 \phi}{\partial y^2}(x, y)\Delta y^2\right) + \ldots \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\partial}{\partial x}\Delta x + \frac{\partial}{\partial y} \cdot \Delta y\right)^n \cdot \phi(x, y) \end{aligned}$$

§14 Lec 14: Oct 27, 2021

§14.1 Two-Species Models (Cont'd)

The Taylor expansion function for g at the equilibrium point is

$$\begin{aligned} g(N_{1e}+\varepsilon N_{11}(t), N_{2e}+\varepsilon N_{21}(t)) &= g(N_{1e}, N_{2e}) + \frac{\partial}{\partial N_1} g(N_{1e}, N_{2e})\varepsilon N_{11}(t) \\ &\quad + \frac{\partial}{\partial N_2} g(N_{1e}, N_{2e})\varepsilon N_{21}(t) + \mathcal{O}(\varepsilon^2) \\ &= \frac{\partial}{\partial N_1} g(N_{1e}, N_{2e})\varepsilon N_{11}(t) + \frac{\partial}{\partial N_2} g(N_{1e}, N_{2e})\varepsilon N_{21}(t) + \mathcal{O}(\varepsilon^2) \end{aligned}$$

Similarly, we have

$$f\left(N_{1e}+\varepsilon N_{11}(t), N_{2e}+\varepsilon N_{21}(t)\right) = \frac{\partial}{\partial N_1} f\left(N_{1e}, N_{2e}\right)\varepsilon N_{11}(t) + \frac{\partial}{\partial N_2} f\left(N_{1e}, N_{2e}\right)\varepsilon N_{21}(t) + \mathcal{O}(\varepsilon^2)$$

Substitute the Taylor expansion of g and f back to the differential equation system we have

$$\begin{cases} \varepsilon \frac{dN_{11}}{dt} = \frac{\partial}{\partial N_1} g\left(N_{1e}, N_{2e}\right)\varepsilon N_{11}(t) + \frac{\partial}{\partial N_2} g\left(N_{1e}, N_{2e}\right)\varepsilon N_{21}(t) + \mathcal{O}(\varepsilon^2) \\ \varepsilon \frac{dN_{21}}{dt} = \frac{\partial}{\partial N_1} f\left(N_{1e}, N_{2e}\right)\varepsilon N_{11}(t) + \frac{\partial}{\partial N_2} f\left(N_{1e}, N_{2e}\right)\varepsilon N_{21}(t) + \mathcal{O}(\varepsilon^2) \end{cases}$$

Since ε is very small, we could ignore the term $\mathcal{O}(\varepsilon)$. Therefore, analyzing the stability of the equilibrium point is equivalent to the analysis of the asymptotic properties ($t \to \infty$) of the linear ODE system:

$$\begin{cases} \frac{dN_{11}}{dt} = \frac{\partial}{\partial N_1} g\left(N_{1e}, N_{2e}\right) N_{11}(t) + \frac{\partial}{\partial N_2} g\left(N_{1e}, N_{2e}\right) N_{21}(t) \\ \frac{dN_{21}}{dt} = \frac{\partial}{\partial N_1} f(N_{1e}, N_{2e}) N_1(t) + \frac{\partial}{\partial N_2} f\left(N_{1e}, N_{2e}\right) N_{21}(t) \end{cases}$$

§14.2 Predator-Prey Models

For this model, we can consider the relations between sharks and the small fish as an example. Let

- F : number of a certain species of fish in a specific region of the sea
- S : number of sharks in the same area

Assume that the area in bounded such that there is no migration, and the food for fish is unlimited. Therefore, the model can be represented by the two species model.

$$\begin{cases} \frac{dF}{dt} = g(F,S) \\ \frac{dS}{dt} = f(F,S) \end{cases}$$

Since this model is about the relation between the predator and the prey, we should expect some properties of g and f.
Observations: Since the food for the fish is unlimited, we can expect the increase of number of fish.

- $F \uparrow \implies S \uparrow$ (the sharks have enough food to maintain a large population)
- $S \uparrow \implies F \downarrow$ (the demand of the food of the shark increases)
- $F \downarrow \implies S \downarrow$ (the decrease of the food of the shark results in the fact that there is not a sufficient amount of food for sharks to maintain a large population)
- $S \downarrow \implies F \uparrow$

The observation above continues periodically. One popular simple model for the predator-prey is called Lotka-Volterra model. Recall the model for one species: by our assumption, the food for the fish is unlimited, we should expect the exponential growth of the fish, i.e.,

$$\frac{dF}{dt} = aF$$

If the population growth of the fish stops growing at some point, we should consider the logistic model.

$$\frac{dF}{dt} = aF - bF^2$$

Next, let's consider one species model for sharks

$$\frac{dS}{dt} = -kS$$

Now let's consider the interaction of fish and shark: the growth rate of the shark increases when fishes appear. The growth rate of fish decreases when sharks appear.

$$\begin{cases} \frac{dF}{dt} = aF - bF^2 - cFS \\ \frac{dS}{dt} = -kS + \lambda FS \end{cases}$$

where a, b, c, k, λ are some positive constants. This is known as the Lotka-Volterra model.

§15 Lec 15: Oct 29, 2021

§15.1 Predator-Prey Models (Cont'd)

Our goal now is to analyze the Lotka-Volterra model by asking the following questions:

Question 15.1. 1. Is this model reasonable?

2. Can the solution of the Lotka-Volterra model be consistent with our observation?

Exercise 15.1. Consider the case $b \neq 0$.

Here we will consider the case $b = 0$.

$$\begin{cases} \frac{dF}{dt} = aF - cFS = (a - cS)F \\ \frac{dS}{dt} = -kS + \lambda sF = (-k + \lambda F)S \end{cases}$$

- Without any predator, $S = 0 \implies \frac{dF}{dt} = aF$, prey(fish) will increase exponentially.
- Without any prey, $F = 0 \implies \frac{dS}{dt} = -kS$, predator(sharks) will decrease exponentially.
- $-cFS$ describes the effect of the predator on prey. The predator reduces the prey's growth with rate cS.
- λsF describes the effect of the prey on the predator. The prey makes some contributions to the growth of the predator by rate λF.

Next, we will consider

1. the equilibrium population for the Lotka-Volterra model and its stability
2. understand the relation between F and S by considering the trajectories of the solution of

$$\frac{dF}{dS} = \frac{dF/dt}{dS/dt} = \frac{(a - cS)F}{(-k + \lambda F)S}$$

First, let's consider the equilibrium points

$$\begin{cases} (a - cS)F = 0 \\ (-k + \lambda F)S = 0 \end{cases}$$

$$\begin{aligned} (a - cS)F = 0 &\implies F = 0, \quad S = \frac{a}{c} \\ F = 0 &\implies S = 0 \\ S = \frac{a}{c} &\implies F = \frac{k}{\lambda} \end{aligned}$$

Thus, we have two equilibrium points

$$\begin{aligned} (F, S) &= (0, 0) \\ (F, S) &= \left(\frac{k}{\lambda}, \frac{a}{c}\right) \end{aligned}$$

The stability of these equilibrium points are

1. $(F,S) = (0,0)$: Set
$$\begin{aligned} g(F,S) &= aF - cFS \\ f(F,S) &= -kS + \lambda FS \end{aligned}$$
Then,
$$\begin{aligned} M &= \begin{bmatrix} \frac{\partial g}{\partial F}\Big|_{(0,0)} & \frac{\partial g}{\partial S}\Big|_{(0,0)} \\ \frac{\partial f}{\partial F}\Big|_{(0,0)} & \frac{\partial f}{\partial S}\Big|_{(0,0)} \end{bmatrix} \\ &= \begin{bmatrix} a & 0 \\ 0 & -k \end{bmatrix} \end{aligned}$$
Because $a > 0$ and a is one of the eigenvalues of the matrix M, $(0,0)$ is not stable. In order to find the relation between F and S near $(0,0)$ we can consider
$$\begin{aligned} \frac{dF}{dS} &= \frac{aF}{-kS} = -\frac{a}{k}\frac{F}{S} \\ \frac{dF}{F} &= -\frac{a}{k}\frac{dS}{S} \\ \int \frac{dF}{F} &= -\frac{a}{k}\int \frac{dS}{S} \\ F &= \tilde{c}S^{-\frac{a}{k}} \end{aligned}$$

2. $\left(\frac{k}{\lambda}, \frac{a}{c}\right)$:
$$\begin{aligned} M &= \begin{bmatrix} \frac{\partial g}{\partial F}\Big|_{\left(\frac{k}{\lambda}, \frac{a}{c}\right)} & \frac{\partial g}{\partial S}\Big|_{\left(\frac{k}{\lambda}, \frac{a}{c}\right)} \\ \frac{\partial f}{\partial F}\Big|_{\left(\frac{k}{\lambda}, \frac{a}{c}\right)} & \frac{\partial f}{\partial S}\Big|_{\left(\frac{k}{\lambda}, \frac{a}{c}\right)} \end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{-ck}{\lambda} \\ \frac{a\lambda}{c} & 0 \end{bmatrix} \\ \det(M - tI) = 0 &\implies t = \pm\sqrt{ak}i \end{aligned}$$
We have two complex eigenvalues with real part equal to 0 for M. Thus, $\left(\frac{k}{\lambda}, \frac{a}{c}\right)$ is algebraically unstable. Next, let's consider the relations between F and S near $\left(\frac{k}{\lambda}, \frac{a}{c}\right)$ (use Taylor's expansion near it)

§16 Lec 16: Nov 1, 2021

§16.1 Predator-Prey Models (Cont'd)

Let's consider the relations between F and S near $\left(\frac{k}{\lambda}, \frac{a}{c}\right)$ (we can consider the Taylor's expansion of g and f near $\left(\frac{k}{\lambda}, \frac{a}{c}\right)$). By setting

$$\begin{cases} F = \frac{k}{\lambda} + \varepsilon F_1 \\ S = \frac{a}{c} + \varepsilon S_1 \end{cases}$$

And by ignoring the small ε, we can consider the constant coefficient DEs

$$\begin{cases} \frac{dF_1}{dt} = -\frac{ck}{\lambda} S_1 \\ \frac{dS_1}{dt} = \frac{a\lambda}{c} F_1 \end{cases}$$

So

$$\begin{aligned} \frac{dF}{dt} = \varepsilon \frac{dF_1}{dt} &= a\left(\frac{k}{\lambda} + \varepsilon F_1\right) - c\left(\frac{k}{\lambda} + \varepsilon F_1\right)\left(\frac{a}{c} + \varepsilon S_1\right) \\ &= -\varepsilon \frac{ck}{\lambda} S_1 + \mathcal{O}(\varepsilon^2) \end{aligned}$$

In order to find F_1, or S_1, we can consider

$$\begin{aligned} \frac{d^2 F_1}{dt^2} = \frac{d}{dt}\left(\frac{dF_1}{dt}\right) &= \frac{d}{dt}\left(-\frac{ck}{\lambda} S_1\right) \\ &= -\frac{ck}{\lambda}\frac{dS_1}{dt} \\ &= -\frac{ck}{\lambda}\frac{a\lambda}{c} F_1 \\ \frac{d^2 F_1}{dt^2} + akF_1 = 0 \end{aligned}$$

The corresponding characteristic polynomial is

$$t^2 + ak = 0 \implies t = \pm\sqrt{ak} i$$

$$\begin{cases} F_1 = c_1 \cos\left(\sqrt{ak} t\right) + c_2 \sin\left(\sqrt{ak} t\right) \\ S_1 = c_3 \cos\left(\sqrt{ak} t\right) + c_4 \sin\left(\sqrt{ak} t\right) \end{cases}$$

Remark 16.1. We could also use the formula for linear ODE system with complex eigenvalues directly.

By considering the initial condition, we have

$$\begin{cases} F_1 = F_{10} \cos(wt) - \frac{cw}{a\lambda} S_{10} \sin(wt) \\ S_1 = S_{10} \cos(wt) + \frac{a\lambda}{cw} F_1 0 \sin(wt) \end{cases}$$

where $w = \sqrt{ak}$. We can see that S_1 and F_1 are periodic functions with period $T = \frac{2\lambda}{w} = \frac{2\lambda}{\sqrt{ak}}$.

Remark 16.2. The period property only holds near the equilibrium point $\left(\frac{k}{\lambda}, \frac{a}{c}\right)$, i.e., $\varepsilon \ll 1$.

Goal: Find the phase plane of F and S

$$\begin{cases} \frac{dF}{dt} = aF - cFS = aF\left(1 - \frac{c}{a}S\right) \\ \frac{dS}{dt} = -kS + \lambda FS = kS\left(-1 + \frac{\lambda}{k}F\right) \end{cases}$$

Set $u = \frac{c}{a}S$, $v = \frac{\lambda}{k}F$ (make the equilibrium points to be independent of the parameters). Then we have

$$\begin{cases} \frac{dv}{dt} = \frac{\lambda}{k}\frac{dF}{dt} = \frac{\lambda}{k}aF\left(1 - \frac{c}{a}S\right) = av(1-u) \\ \frac{du}{dt} = \frac{c}{a}\frac{ds}{dt} = \frac{c}{a}ks\left(-1 + \frac{\lambda}{k}F\right) = ku(-1+v) \end{cases}$$

To study the relation between F and S, we only need to study the relation between u and v.

$$\begin{aligned} \frac{du}{dv} &= \frac{ku(-1+v)}{av(1-u)} \\ \frac{1-u}{u}du &= \frac{k}{a}\frac{v-1}{v}dv \\ \int\left(\frac{1}{u} - 1\right)du &= \frac{k}{a}\int\left(1 - \frac{1}{v}\right)dv \\ \ln(u) - u &= \frac{k}{a}(v - \ln v) + c \\ ue^{-u} &= \tilde{c}v^{-\frac{k}{a}} \cdot e^{\frac{k}{a}v} \end{aligned}$$

because $u = \frac{c}{a}S$ and $v = \frac{\lambda}{k}F$.

$$\begin{aligned} \frac{c}{a}Se^{-\frac{c}{a}S} &= \tilde{c}\left(\frac{\lambda}{k}\right)^{-\frac{k}{a}}F^{-\frac{k}{a}}e^{\frac{\lambda}{a}F} \\ F^{-k}e^{\lambda F} &= \hat{c}S^a e^{-cS} = Z \end{aligned}$$

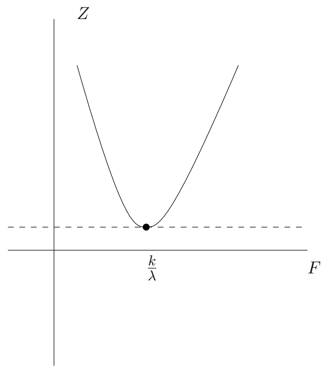
Next, let's sketch the relation between F and S. To implement, we introduce a new variable Z by setting

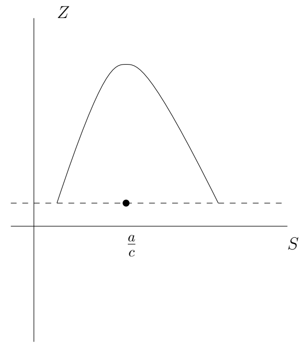
$$\begin{cases} Z = F^{-k}e^{\lambda F} \\ Z = \hat{c}e^{-cS}S^a \end{cases}$$

Idea: Let's study the relation between Z and F, Z and S. These relations are much easier than the relation between F and S because Z is explicitly represented in terms of F or S.

$$\begin{aligned} \frac{dZ}{dF} &= -kF^{-k-1}e^{\lambda F} + F^{-k}\left(e^{\lambda F}\lambda\right) \\ &= F^{-k}e^{\lambda F}\left(-\frac{k}{F} + \lambda\right) = 0 \\ F &= \frac{k}{\lambda} \end{aligned}$$

- When $F > \frac{k}{\lambda}$, $\frac{dZ}{dF} > 0$
- When $F < \frac{k}{\lambda}$, $\frac{dZ}{dF} < 0$





We have

$$\begin{aligned}\frac{dZ}{dS} &= \hat{c}e^{-cS} \cdot (-c)S^a + \hat{c}e^{-cS} \cdot (aS^{a-1}) \\ &= \hat{c}e^{-cS} \cdot S^a \left(-c + \frac{a}{S}\right) = 0 \\ S &= \frac{a}{c}\end{aligned}$$

- When $S > \frac{a}{c}$, $\frac{dZ}{dS} < 0$
- When $S < \frac{a}{c}$, $\frac{dZ}{ds} > 0$

§17 Lec 17: Nov 3, 2021

§17.1 Cooperation Model

Many organisms cooperate to perform some tasks that they cannot achieve individually.

Example 17.1
- With species cooperation: raising young, gathering food, predator protection or defense, etc
- Between-species cooperation: remoras and sharks. Remoras remove parasites, dead skin from the sharks. Sharks will also provide protection for the remoras.

Cooperation is interesting but tricky. The cooperation will involve the process: contributing to the common good, but the individuals might lose something.
Aim: We will build cooperation model using game theory. There are two types of organism:

1. Cooperators
2. Cheaters

There are three interactions for these two types of organisms

i) Cooperator meets cooperator: they work together

$$\text{cost: } \frac{c}{2} \text{ for each,} \quad \text{benefit: } b \text{ for each}$$

ii) Cooperator meets cheater: only cooperator works, cheater doesn't contribute anything.

$$\text{cost: } c \text{ for cooperator,} \quad 0 \text{ for cheater,} \quad \text{benefit: } b \text{ for each}$$

iii) Cheater meets cheater: both cheats and do nothing $\implies$ no cost, no benefit.

Assume that the total population is N. All organisms are equally likely to die at a rate d. Assume that for each birth, there is one death occurs. We propose the model:

1. Rate of the changes of number of cooperators = the rate of cooperator births - rate of cooperator deaths.
2. Rate of the changes of numbers cheaters = rate of cheater births - rate of cheater deaths

Our next goal is to find the number of birth by finding the payoffs

total payoff to cooperator = # cooperators interact with ×payoff from the interaction with the cooperator+#cheaters

Assumption: The birth rate is proportional to the payoff.

$$\text{the rate of cooperator birth} = \#\text{ of cooperators } \times \text{total payoff to cooperators} \times k$$

where k is the proportional constant.

§18 Lec 18: Nov 5, 2021

§18.1 Cooperation Model (Cont'd)

Similarly for the cheater, we have the total payoff for the cheater = to the number of cooperators interact with · the payoff from the interaction with the cheater. From these interactions, we could construct a payoff matrix, that represents the hat benefit received by the organism depending on the choice of these organism

$$\begin{pmatrix} b-\frac{c}{2} & b-c \\ b & 0 \end{pmatrix}$$

which is a payoff matrix.
Observations:

- If the other organism is a cooperator then cheating will give a higher off.
- If the other organism is a cheater, then if $b > c$ it's better to cooperate if $b < c$ it's better to cheat.

Net, let's introduce some notion for the cooperation model

- N = total population of organisms
- x = fraction of organisms that are cooperators
- y = fraction of organisms that are cheaters. Note that $x + y = 1$
- b = benefit, c = cost
- k = proportional constant
- d = death rate

Assumption:

- Birth rate is proportional to the payoff from all its interaction.
- For each interaction, payoff is determined by the payoff matrix
- In each unit time, each orgasm will interact with other n randomly chosen organism.
- Offspring of cooperators are cooperators
- Offspring of cheater are cheaters

Therefore, we have the model

$$\begin{cases} \frac{d}{dt}(Nx) = Nx \cdot \left(R \cdot \left(nx \cdot \left(b - \frac{c}{2}\right) + ny \cdot (b-c)\right) - d\right) \\ \frac{d}{dt}(Ny) = Ny \cdot (k \cdot nx \cdot b - d) \end{cases}$$

Since N is fixed number and $x + y = 1$, we could cancel N in both of the equations above

$$\frac{dx}{dt} + \frac{dy}{dt} = 0$$

So

$$x\left(k \cdot \left(nx\left(b - \frac{c}{2}\right) + ny\,(b-c)\right) - d\right) + y\,(knxb - d) = 0$$
$$kx\left(nxb - \frac{nxc}{2} + nyb - nyc\right) - dx + kynxb - yd = 0$$

because $x + y = 1 \implies y = 1 - x$

$$kx\left(nb + \frac{nxc}{2} - nc\right) - d + k\left(1 - x\right) \cdot nxb = 0$$
$$knx\left(b - \frac{c}{2}\right)\left(2 - x\right) = d \geq 0$$
$$b - \frac{c}{2} \geq 0 \implies b \geq \frac{c}{2}$$

Next, let's subsided the expression for d to the ODEs above, we have

$$\begin{cases} \frac{dx}{dt} = knx\left(x\left(b - \frac{c}{2}\right) + y\left(b - c\right)\right) - xknx\left(b - \frac{c}{2}\right)\left(2 - x\right) \\ \frac{dy}{dt} = yknxb - yknx\left(b - \frac{c}{2}\right)\left(2 - x\right) \end{cases}$$

To do the simplification for the above ODE system, we can get

$$\begin{cases} \frac{dx}{dt} = knxy\left[\left(\frac{c}{2} - b\right)x + \left(b - c\right)\right] \\ \frac{dy}{dt} = knx\left(\frac{c}{2} - \left(b - \frac{c}{2}y\right)\right) \end{cases}$$

Recall that $x + y = 1$, we have $y = 1 - x$. Let's substitute $y = 1 - x$ to the expression of $\frac{dx}{dt}$, we obtain

$$\frac{dx}{dt} = \frac{knx(1-x)\left(\left(\frac{c}{2} - b\right)x + b - c\right)}{g(x)}$$

where $x \in [0, 1]$.

§19 Lec 19: Nov 8, 2021

§19.1 Cooperation Models (Cont'd)

Let's first find the equilibrium points by setting

$$g(x) = knx(1-x)\left(\left(\frac{c}{2}-b\right)x + (b-c)\right) = 0$$

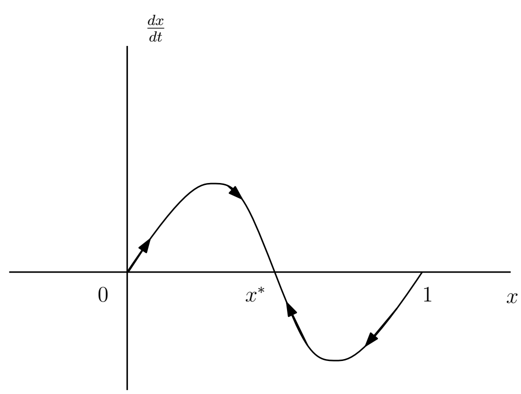
So $x = 0, 1, -\frac{b-c}{\frac{c}{2}-b}$ (require $\frac{c}{2}-b<0$). Since $x \in [0,1]$, we need to discuss whether $-\frac{b-c}{\frac{c}{2}-b} \in [0,1]$? Now, let's assume that

$$0 \le -\frac{b-c}{\frac{c}{2}-b} \le 1$$
$$0 \le b-c \le b-\frac{c}{2}$$
$$\implies b \ge c \ge 0$$

Therefore, we have the following cases for the equilibrium points

1. When $b > c$, we have three equilibrium points on $[0,1]$

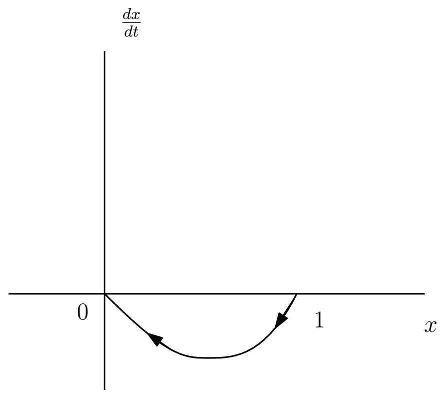
$$x = 0,\ 1,\ \frac{b-c}{b-\frac{c}{2}}$$



From the figure above, we can conclude that the equilibrium points $x = 0, 1$ are unstable. The equilibrium point $x = x^* = \frac{b-c}{b-\frac{c}{2}}$ is stable.

2. When $\frac{c}{2} < b \le c$, we only have two equilibrium points in $[0,1]$

$$x = 0,\ 1$$
$$x^* = \frac{b-c \le 0}{b-\frac{c}{2} > 0} \le 0$$



From the figure, we can conclude that $x = 0$ is stable but $x = 1$ is unstable.

§19.2 Stochastic Population Growth

Stochastic Process: family of random variables. Discrete and continuous time models predict the average behavior of a population. This can be treated as the average size of the population over many trials with the same environment. In real life, we may care more than the average.

Example 19.1

In stock, we may ask

Question 19.1. What's the probability that one specific stock will drop 1000 points in a day/week?

This will be more useful than the question "What's the average behavior in the stock market each day/week?"

Example 19.2

What's the probability that a population for one species goes extinct?

Therefore, realistic population model should take the randomness into account.

Example 19.3

Let's consider the cells division. It is more practical to consider the random division because the real birth rate varies between different periods.
Assumptions:

- Death rate is ignored.
- Census time is divided into subintervals Δt.
- b is birth rate per cell.
- time intervals of interest is $[0, T]$.

In each interval, each cell has a probability to divide and the probability is $b \cdot \Delta t$. How to simulate the stochastic process?

- At each time step, generate $N(t)$ random numbers from a uniform distribution on $[0, 1]$
- For each $X < b\Delta t$, it means that there is a cell to divide

$$\begin{aligned} N(t + \Delta t) &= N(t) + \#\text{ random numbers} < b \cdot \Delta t \\ &= N(t) + |\{X : X < b\Delta t\}| \end{aligned}$$

§20 Lec 20: Nov 10, 2021

§20.1 Stochastic Population Growth (Cont'd)

From the simulation, we can see that we get different random numbers for each run and hence a different sequence of population sizes: $N(0)$, $N(\Delta t)$, $N(2\Delta t)$,. . . , $N(T)$. This just captures random growth of populations. To analyze $N(t)$ itself is tricky because $N(t)$ varies for each simulation.

Question 20.1. What should we analyze for the stochastic birth model?

We could analyze the $P_N(t)$, where $P_N(t)$ represents the probability that the population equals N at time t. Let $b =$ birth rate where b can be considered as the probability of a birth per unit time. We set Δt to be small enough s.t. the probability for the case where there are more than 2 births can be negligible. Assume

$$P\,(\text{one birth}) = b \cdot \Delta t$$
$$P\,(\geq 2 \text{ birth}) \ll 1$$

Example 20.1

An average of 20 chickens hatch from a population of 600 hens in one hour. Then the birth rate

$$b = \frac{20}{600} = \frac{1}{30} \text{ per hour}$$

or the birth rate

$$b = \frac{20}{600 * 60} = \frac{1}{1800} \text{ per minute}$$

Let's now get back $P_N(t)$. To find the stochastic model for $P_N(t)$ we need to find the relation between $P_N(t+\Delta t)$ and $P_N(t)$. For sufficient small Δt, we should have

$$P_N(t+\Delta t) = \sigma N - 1 P_{N-1}(t) + \gamma_N P_N(t) \ldots \tag{*}$$

where $\sigma N - 1$ is the probability that exactly one birth occurs among $N-1$ individuals and γ_N is the probability that no birth among N individuals.

Question 20.2. σ_{N-1}? γ_N?

The probability of an individual giving birth in the time interval with length Δt is $b \cdot \Delta t$ because $P(\geq 2 \text{ birth}) \ll 1$. So the probability of not giving birth should be $1 - b \cdot \Delta t$. Therefore, the probability that N individuals will not give birth should be

$$(1 - b\Delta t)^N \implies \gamma_N = (1 - b\Delta t)^N \approx 1 - bN\Delta t$$

So

$$(1 - b\Delta t)^N = 1 + \binom{N}{1}(-b\Delta t) + \binom{N}{2}(-b\Delta t)^2 + \ldots$$

The probability of at least one birth among N individual is

$$\begin{aligned} 1 - \gamma_N &= 1 - (1 - b\Delta t)^N \\ &\approx 1 - (1 - Nb\Delta t) \\ &= Nb\Delta t \end{aligned}$$

$\implies \sigma_{N-1} \approx (N-1)b\Delta t$ because the case ≥ 2 birth can be ignored.

An alternative way to compute σ_{N-1},

$$\begin{aligned}
\sigma_{N-1} &= \binom{N-1}{1}(b\Delta t)\left(1-b\Delta t\right)^{N-2} \\
&= \left(N-1\right)\left(b\Delta t\right)\left(1+\binom{N-2}{1}\cdot\left(-b\Delta t\right)+\binom{N-2}{2}\left(-b\Delta t\right)^2+\dots\right) \\
&\approx (N-1)b\Delta t
\end{aligned}$$

We substitute the expression of γ_N, σ_{N-1} back into (*).

$$P_N\left(t+\Delta t\right) \approx b\cdot\left(N-1\right)\Delta t P_{N-1}(t)+\left(1-bN\Delta t\right)P_N(t)$$

§21 Lec 21: Nov 12, 2021

§21.1 Stochastic Population Growth (Cont'd)

Because Δt is sufficiently small, we could consider the Taylor expansion of $P_N(t+\Delta t)$ at t

$$\begin{aligned}
P_N\left(t+\Delta t\right) &= P_N(t) + \frac{dP_N(t)}{dt}\Delta t + \frac{1}{2}\frac{d^2P_N(t)}{dt^2}\Delta t^2 + \dots \\
&= P_N(t) + \frac{dP_N(t)}{dt}\Delta t + \mathcal{O}(\Delta t^2) \\
&\approx P_N(t) + \frac{dP_N(t)}{dt}\Delta t \quad (\Delta t \ll 1)
\end{aligned}$$

Plug in $P_N(t+\Delta t)$ into (*) from last lecture,

$$\begin{aligned}
P_N(t) + \frac{dP_N(t)}{dt}\Delta t &\approx b(N-1)\Delta t P_{N-1}(t) + \left(1 - bN\Delta t\right)P_N(t) \\
\frac{dP_N(t)}{dt}\Delta t &= b(N-1)\Delta t P_{N-1}(t) - bN\Delta t P_N(t) \\
\frac{dP_N(t)}{dt} &= b(N-1)P_{N-1}(t) - bNP_N(t) \qquad (**)
\end{aligned}$$

Let's solve (**). Assume that we have the initial condition

$$P_N(0) = \begin{cases} 0, & N \neq 1 \\ 1, & N = 1 \end{cases}$$

Let's rewrite (**)

$$\frac{dP_N(t)}{dt} + bNP_N(t) = b(N-1)P_{N-1}(t) \qquad (**)$$

We can see that to find $P_N(t)$, we need $P_{N-1}(t)$, to find $P_{N-1}(t)$, we need $P_{N-2}(t)$. Therefore, we need to solve

$$P_1(t) \to P_2(t) \to P_3(t) \to \dots \to P_{N-1}(t) \to P_N(t)$$

Now, let's consider $P_1(t)$

$$\frac{dP_1(t)}{dt} + bP_1(t) = 0$$

Since there is no death, we are only interested in populations $\geq N(0) = 1$, $P_0(t) = 0$.

$$\begin{aligned}
\frac{dP_1(t)}{dt} + bP_1(t) &= 0 \\
\frac{dP_1(t)}{P_1(t)} &= -bdt \\
\ln P_1(t) - \ln P_1(0) &= -bt \\
P_1(t) = P_1(0)e^{-bt} &= e^{-bt}
\end{aligned}$$

where $P_1(0)$ is the initial condition. Next, we can consider the case $N = 2$. We have

$$\frac{dP_2(t)}{dt} + 2bP_2(t) = bP_1(t) = be^{-bt} \qquad (\star)$$

In order to solve the DE above, we need to find the solution for the homogeneous DE:

$$\frac{dP_2(t)}{dt} + 2bP_2(t) = 0 \implies P_2(t) = Ce^{-2bt}$$

Then we consider the method of parameter C by setting C to be a function of t. Then

$$P_2(t) = C(t)e^{-2bt}$$

$\implies \frac{dP_2(t)}{dt} = C'(t)e^{-2bt} + C(t)\left(-2be^{-2bt}\right)$. So ($\star$) becomes

$$C'(t)e^{-2bt} - 2bC(t)e^{-2bt} + 2bC(t)e^{-2bt} = be^{-bt}$$

$$\begin{aligned} C'(t) &= be^{bt} \\ C(t) &= C + \int be^{bt}\,dt \\ &= C + e^{bt} \\ P_2(t) &= \left(C + e^{bt}\right)e^{-2bt} \\ 0 = P_2(0) &= (C+1)\cdot 1 = C + 1 \end{aligned}$$

So $C = -1$.

$$P_2(t) = \left(e^{bt} - 1\right)e^{-2bt} = \left(1 - e^{-bt}\right)e^{-bt}$$

Continue this process, we could get the following results

$$\begin{aligned} P_3(t) &= \left(1 - e^{-bt}\right)^2 e^{-bt} \\ P_4(t) &= \left(1 - e^{-bt}\right)^3 e^{-bt} \\ &\vdots \end{aligned}$$

The general formula then is

$$P_N(t) = \left(1 - e^{-bt}\right)^{N-1} e^{-bt}$$

Let's show the conclusion by mathematical induction.

1. Base case: $N = 1$

$$P_1(t_0 = e^{-bt} = e^{-bt}\left(1 - e^{-bt}\right) \checkmark$$

2. Let's assume that the results hold for the case $N = k$, i.e.,

$$P_k(t) = e^{-bt}\left(1 - e^{-bt}\right)^{k-1}$$

3. We need to show the results for the case $N = k + 1$

$$\frac{dP_N(t)}{dt} + bNP_N(t) = b(N-1)P_{N-1}(t)$$

We have,

$$\frac{dP_{k+1}(t)}{dt} + b(k+1)P_{k+1}(t) = bke^{-bt}\left(1 - e^{-bt}\right)^{k-1} \tag{$\circ$}$$

Similar to the process for $N = 2$, we first find the general solution for the homogeneous DE:

$$\begin{aligned} \frac{dP_{k+1}(t)}{dt} + b(k+1)P_{k+1}(t) &= 0 \\ P_{k+1}(t) &= Ce^{-b(k+1)t} \end{aligned}$$

Then we set

$$\begin{aligned} P_{k+1}(t) &= C(t)e^{-b(k+1)t} \\ \frac{dP_{k+1}(t)}{dt} &= C'(t)e^{-b(k+1)t} - C(t)b(k+1)e^{-b(k+1)t} \end{aligned}$$

Substitute $P_{k+1}(t)$, $\frac{dP_{k+1}(t)}{dt}$ into ($\circ$),

$$\begin{aligned}
C'(t)e^{-b(k+1)t} &= bke^{-bt}\left(1-e^{-bt}\right)^{k-1} \\
C(t) &= C + \int bke^{-bkt}\sum_{j=0}^{k-1}\binom{j}{k-1}\left(-e^{-bt}\right)^{j}1^{k-1-j} \\
&= C + \int bk\sum_{j=0}^{k-1}\binom{j}{k-1}(-1)^{j}e^{-bjt-bkt}
\end{aligned}$$

§22 Lec 22: Nov 22, 2021

§22.1 Stochastic Population Growth (Cont'd)

We have

$$\begin{aligned}
P_{k+1}(t) &= \left(c + \int bk \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j \cdot e^{-bjt-bkt}\right) e^{-b(k+1)t} \\
&= \left(c + e^{kbt}(1-e^{-bt})^k\right) \cdot e^{-b(k+1)t} \\
&\vdots
\end{aligned}$$

So the general formula is

$$P_N(t) = \left(1-e^{-bt}\right)^{N-1} e^{-bt}$$

Since $P_{k+1}(0) = 0$ for all $k \geq 1$, we have

$$\begin{aligned}
P_{k+1}(0) &= (c+0) \cdot 1 = 0 \implies c = 0 \\
P_{k+1}(t) &= e^{kbt} \left(1-e^{-bt}\right)^k \cdot e^{-b(k+1)t} \\
&= \left(1-e^{-bt}\right)^k \cdot e^{-bt}
\end{aligned}$$

Here, $P_N(t)$ is called the probability mass function. It gives the probability that the population is exactly equal to N at time t.
Properties of $P_N(t)$:

1. $\sum_{N=1}^{\infty} P_N(t) = 1$.

   Proof. Have

$$\begin{aligned}
\sum_{N=1}^{\infty} P_N(t) &= \sum_{N=1}^{\infty} e^{-bt} \left(1-e^{-bt}\right)^{N-1} \\
&= e^{-bt} \sum_{N=0}^{\infty} \left(1-e^{-bt}\right)^N \\
&= e^{-bt} \frac{1}{1-(1-e^{-bt})} \\
&= e^{-bt} \cdot e^{bt} = 1
\end{aligned}$$

   □

2. Expected (mean) population $E(t)$ at time t is e^{bt}

   Proof. Have

$$\begin{aligned}
E(t) &= \sum_{N=1}^{\infty} N \cdot P_N(t) \\
&= \sum_{N=1}^{\infty} N \cdot e^{-bt} \left(1-e^{-bt}\right)^{N-1} \\
&= e^{-bt} \frac{1}{\left(1-(1-e^{-bt})\right)^2} \\
&= e^{-bt} \cdot e^{2bt} \\
&= e^{bt}
\end{aligned}$$

   □

§22.2 Flow

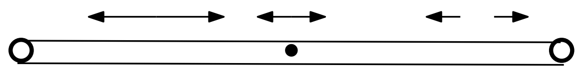
Random Walks: Let's imagine that a large group of bacteria that are swimming in a long, thin tube. Bacteria swim in a "run and tumble" way:

- A "run" propels a bacterium to the left or right.
- A "tumble" randomly change the moving directions of the bacteria.

Question 22.1. How far along the tube do bacteria swim by time t?

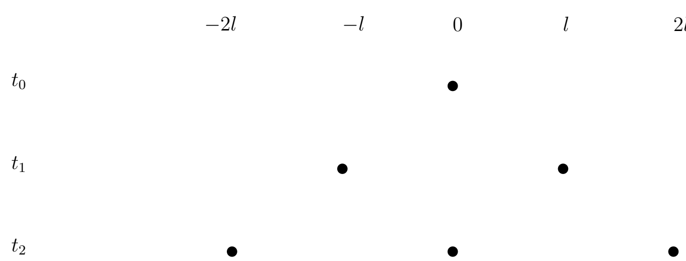
Assumptions:

- Tube is long and thin so that it can be modeled effectively as one dimensional.
- All bacteria introduced at $t = 0$ at the center of the tube.



$x = 0$

- Each "run" moves the bacteria a distance l along the tube.
- Bacterium run left with probability $\frac{1}{2}$ (more general case P) and right with probability $\frac{1}{2}$ (more general case $1 - P$)
- Move left to be negative direction or move right to be positive direction
- Break up time into t_0, t_1, t_2, …where each run happens during $[t_{k-1}, t_k)$, so t_k is the time where the bacterium ends its kth run and x_k is the location at time t_k.



This is a stochastic process with random variable x_k. These are some questions:

1. What is the average position $\overline{x}_k$ (or denote $\overline{x}_k$ by $E(x_k)$.
2. What is the distribution for x_k? E.g., the probability that bacterium is at $\pm 2l$, $\pm l$, 0, etc.

§23 Lec 23: Nov 24, 2021

§23.1 Flow (Cont'd)

First, let's consider the relation positions at t_k and t_{k+1}

$$x_{k+1} = x_k + d_{k+1}$$

where d_{k+1} is the directional distance by the bacterium in its $(k+1)$st

$$d_{k+1} = \begin{cases} l, & \text{with probability } \frac{1}{2}(1-p) \\ -l, & \text{with probability } \frac{1}{2}(p) \end{cases}$$

Next, we will find $E(x_{k+1})$ by considering

$$\begin{aligned} E\left(x_{k+1}\right) &= E\left(x_k + d_{k+1}\right) \\ &= E(x_k) + E(d_{k+1}) \end{aligned}$$

Notice that

$$\begin{gathered} E(d_{k+1}) = \frac{1}{2}l + \frac{1}{2}(-l) = 0 \\ E(x_{k+1}) = E(x_k) = \ldots = E(x_1) = E(x_0) = 0 \end{gathered}$$

Since this is a random process, we are interested in the variance of x_{k+1}

$$\begin{aligned} \operatorname{var}\left(x_{k+1}\right) &= E\left((x_{k+1} - E(x_{k+1}))^2\right) \\ &= E\left((x_{k+1} - 0)^2\right) \\ &= E(x_{k+1}^2) \\ &= E\left((x_k + d_{k+1})^2\right) \\ &= E(x_k^2) + 2E(x_k d_{k+1}) + E(d_{k+1}^2) \end{aligned}$$

Since x_k and d_{k+1} are independent, we have $E(x_k d_{k+1}) = E(x_k)E(d_{k+1}) = 0$. Notice that

$$E\left(d_{k+1}^2\right) = \frac{1}{2}l^2 + \frac{1}{2}(-l)^2 = l^2$$

So

$$\begin{aligned} \operatorname{var}(x_{k+1}) &= E(x_k^2) + l^2 \\ &= \operatorname{var}(x_k) + l^2 \\ &= \operatorname{var}(x_{k-1}) + l^2 + l^2 \\ &= \operatorname{var}(x_{k-1}) + 2l^2 \\ &\vdots \\ &= \operatorname{var}(x_0) + (k+1)l^2 \\ &= 0 + (k+1)l^2 = (k+1)l^2 \end{aligned}$$

Next, let's answer the second question: For $l = 1$, we want to find $P_N(d)$ the probability of being at location d at time t_N (after N step). When $l = 1$, then the possible d at time t_N is 0, $\pm 1, \pm 2, \ldots, \pm N$. In order to think about the "run" direction, we could correct this with the coin flip: head $\implies$ right, tail $\implies$ left. Therefore, for the N coin flip, we have total 2^N outcome (because for each flip, it has two possibilities and each flip is independent). Thus, we can make the table for the outcomes.

figure here
When $N = 4$, it has 16 possibility. Let's assume that we have d_1 head and d_2 tail.

$$\begin{cases} d_1 + d_2 = 4 \\ d_1 - d_2 = d \end{cases} \implies d_1 = \frac{4+d}{2}$$

Since d_1, d_2 are positive integers, we have $d_1 = 0, 2, 4$. Therefore, for the general N, we have

$$\begin{cases} d_1 + d_2 = N \\ d_1 - d_2 = d \end{cases}$$
$$\implies \begin{cases} d_1 = \frac{N+d}{2} \\ d_2 = \frac{N-d}{2} \end{cases}$$

Since d_1 and d_2 are positive integers, we have that N and d have the same odd or even properties. Notice that for each outcome, the probability should be $\frac{1}{2^N}$. Therefore, we have

$$P_N(d) = \binom{N}{\frac{N+d}{2}} \cdot \left(\frac{1}{2}\right)^N$$

Question 23.1. $P_N(d)$ if the right moving probability $1 - p$ and the left has probability p.

§23.2 Diffusion Equation

The continuous hypothesis: Let's treat distance from the origin as a continuous variable x (no longer restricted to multiple of l). Our goal is to derive an equation for the density of the run $\rho(x, t)$. Density is defined: for some interval with length Δx around the point $\hat{x}$ $\left(\left[\hat{x} - \frac{\Delta x}{2}, \hat{x} + \frac{\Delta x}{2}\right]\right)$, the number of the runner at time τ is $\rho(\hat{x}, z) \cdot \Delta x$.

§24 Lec 24: Nov 29, 2021

§24.1 Diffusion Equation (Cont'd)

We can model how the number of runners in interval $\left[\hat{x}-\frac{\Delta x}{2},\hat{x}+\frac{\Delta x}{2}\right]$ changes between time points t and $t+\Delta t$. The number of runners in $\left[\hat{x}-\frac{\Delta x}{2},\hat{x}+\frac{\Delta x}{2}\right]$ at time t is $\rho(\hat{x},t)\cdot\Delta x$. The number of runners in $\left[\hat{x}-\frac{\Delta x}{2},\hat{x}+\frac{\Delta x}{2}\right]$ at time $t+\Delta t$ is $\rho(\hat{x},t+\Delta t)\cdot\Delta x$. Thus, the change in number of runners in $[t,t+\Delta t]$ is $\rho(\hat{x},t+\Delta t)\cdot\Delta x-\rho(\hat{x},t)\cdot\Delta x=(\rho(\hat{x},t+\Delta t)-\rho(\hat{x},t))\cdot\Delta x$.
Let $q(x,t)$ be the flow rate ("flux") of the runners at point x at time $[t,t+\Delta t]$: change in number of runners in interval $\left[\hat{x}-\frac{\Delta x}{2},\hat{x}+\frac{\Delta x}{2}\right]$ is

$$\begin{bmatrix}\text{flow in net \# crossing}\\ \hat{x}-\frac{\Delta x}{2}\in[t,t+\Delta t]\end{bmatrix}-\begin{bmatrix}\text{flow in net \# crossing}\\ \hat{x}+\frac{\Delta x}{2}\in[t,t+\Delta t]\end{bmatrix}=q\left(\hat{x}-\frac{\Delta x}{2},t\right)\cdot\Delta t-q\left(\hat{x}+\frac{\Delta x}{2},t\right)\cdot\Delta t$$

Since the above two perspectives describe the same phenomenon (the number of runners that change over $\left[\hat{x}-\frac{\Delta x}{2},\hat{x}+\frac{\Delta x}{2}\right]$ over $[t,t+\Delta t]$, we have

$$(\rho(\hat{x},t+\Delta t)-\rho(\hat{x},t))\cdot\Delta x=\left(q\left(\hat{x}-\frac{\Delta x}{2},t\right)-q\left(\hat{x}+\frac{\Delta x}{2},t\right)\right)\cdot\Delta t$$

or

$$\frac{\rho(\hat{x},t+\Delta t)-\rho(\hat{x},t)}{\Delta t}=\frac{q\left(\hat{x}-\frac{\Delta x}{2},t\right)-q\left(\hat{x}+\frac{\Delta x}{2},t\right)}{\Delta x}$$

Set $\Delta x\to 0,\Delta t\to 0$, we thus have

$$\frac{\partial\rho}{\partial t}(\hat{x},t)=-\frac{\partial q}{\partial x}(\hat{x},t)\implies\frac{\partial\rho}{\partial t}+\frac{\partial q}{\partial x}=0 \tag{*}$$

(*) is called the continuity equation where $\rho(x,t)$ is the density of runners (bacteria) and $q(x,t)$ is the flow rate (flux).

Fact 24.1. Equation (*) holds for any system, where mass is conserved (no creation or destruction).

Notice that (*) involves two functions ρ and q which make it difficult to analyze the solution of (*). So we want to check whether ρ and q have some connections so that (*) can be described as a DE with only ρ or q.
Let $\Delta x=l$ (step length) and Δt is the time for one step. Let $p(x,t)$ be the probability that a runner is at position $x=ml$ at time $t=n\cdot\Delta t$. Then

$$p(x,t+\Delta t)=\frac{1}{2}p(x-\Delta x,t)+\frac{1}{2}p(x+\Delta x,t) \tag{**}$$

because $\Delta t,\Delta x$ are pretty small, we can consider the Taylor expansion of $p(x,t+\Delta t)$, $p(x-\Delta x,t)$, $p(x+\Delta x,t)$ at (x,t)

1. $p(x,t+\Delta t)$

$$\begin{aligned}p(x,t+\Delta t)&=p(x,t)+\frac{\partial p}{\partial t}\cdot\Delta t+\frac{1}{2}\frac{\partial^2 p}{\partial t^2}(\Delta t)^2+\dots\\&=p(x,t)+\frac{\partial p}{\partial t}\Delta t\end{aligned}$$

2. $p(x-\Delta x,t)$ and $p(x+\Delta x,t)$

$$\begin{aligned}p(x-\Delta x,t)&=p(x,t)+\frac{\partial p}{\partial x}(-\Delta x)+\frac{1}{2}\frac{\partial^2 p}{\partial x^2}(-\Delta x)^2+\dots\\p(x+\Delta x,t)&=p(x,t)+\frac{\partial p}{\partial x}\Delta x+\frac{1}{2}\frac{\partial^2 p}{\partial x^2}(\Delta x)^2+\dots\end{aligned}$$

Thus,

$$\frac{1}{2}p(x-\Delta x,t)+\frac{1}{2}p(x+\Delta x,t)=p(x,t)+\frac{1}{2}\frac{\partial^2 p}{\partial x^2}(\Delta x)^2+\mathcal{O}\left((\Delta x)^4\right)$$

Substitute 1. and 2. into (**), we obtain

$$p(x,t)+\frac{\partial p}{\partial t}\Delta t+\mathcal{O}(\Delta t^2)=p(x,t)+\frac{1}{2}\frac{\partial^2 p}{\partial x^2}(\Delta x)^2+\mathcal{O}(\Delta t^4)$$

Set $\Delta t \to 0$, $\Delta x \to 0$

$$\frac{\partial p}{\partial t}=\frac{1}{2}\frac{\partial^2 p}{\partial x^2}\cdot D$$

where $D=\lim_{\Delta x,\Delta t\to 0}\frac{(\Delta x)^2}{\Delta t}$.

§25 Lec 25: Dec 1, 2021

§25.1 Diffusion Equations (Cont'd)

Since $\rho(x,t) = \frac{Np(x,t)}{\triangle x}$ where N is the total population of runners (bacteria), substitute $p(x,t) = \frac{\Delta x}{N}\rho(x,t)$ back into

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial x^2}\cdot D$$

we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\Delta x}{N}\rho(x,t)\right) &= \frac{p}{2}\frac{\partial}{\partial x^2}\left(\frac{\Delta x}{N}\rho(x,t)\right) \\
\frac{\Delta x}{N}\cdot\frac{\partial}{\partial t}\rho(x,t) &= \frac{\Delta x}{N}\frac{D}{2}\frac{\partial}{\partial x^2}\rho(x,t) \\
\frac{\partial \rho}{\partial t} &= \frac{D}{2}\cdot\frac{\partial^2 \rho}{\partial x^2} \\
&= \tilde{D}\frac{\partial^2 \rho}{\partial x^2} \qquad (*)
\end{aligned}$$

Here (*) is called the diffusion equation. $\tilde{D}$ is called the diffusion coefficient. Also, (*) is called the heat equation because it can describe the distribution of the heat over time.
Let's compare $\frac{\partial}{\partial t}\rho = D\frac{\partial^2}{\partial x^2}\rho$ and

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0$$

Since these two DEs describe the same situation, we have

$$\begin{aligned}
\tilde{D}\frac{\partial^2}{\partial x^2}\rho &= -\frac{\partial q}{\partial x} \\
\frac{\partial}{\partial x}\left(\tilde{D}\frac{\partial}{\partial x}\rho\right) + \frac{\partial}{\partial x}q &= 0 \\
\frac{\partial}{\partial x}\left(\tilde{D}\frac{\partial}{\partial x}\rho + q\right) &= 0
\end{aligned}$$

Therefore, we have $\tilde{D}\frac{\partial}{\partial x}\rho + q = C(t)$. If $C(t) = 0$, $q = -\tilde{D}\frac{\partial}{\partial x}\rho$, which is called Fick's Law.
Find the solution for the heat equation

$$\frac{\partial p}{\partial t} = D\cdot\frac{\partial^2}{\partial x^2}p$$

We will apply the Fourier transformation on p with respect to x. First, let's review the Fourier Transformation on a function $f(x)$. The Fourier Transformation on $f(x)$ is denoted by $\hat{f}(s)$ and it is defined as

$$\hat{f}(s) = \int_{-\infty}^{\infty} f(x)e^{-2\lambda isx}\,dx$$

The Fourier Transformation on the derivative of $f(x)$ satisfies

$$\begin{aligned}
\hat{f'}(s) &= \int_{-\infty}^{\infty} f'(x)e^{-2\lambda isx}\,dx \\
&= \int_{-\infty}^{\infty} e^{-2\lambda isx}df(x) \\
&= e^{-2\lambda isx}f(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\cdot(-2\lambda is)e^{-2\lambda isx}\,dx \\
&= 2\lambda is\hat{f}(s)
\end{aligned}$$

Here we need to assume that $\lim_{x\to\infty} f(x) = 0$

$$\begin{aligned}\hat{f''}(s) &= 2\lambda is\hat{f'}(s)\\ &= (2\lambda is)^2\hat{f}(s)\\ &= 4\lambda^2 s^2\hat{f}(s)\end{aligned}$$

The inverse Fourier Transform on $\hat{f}(s)$ is defined to be $\int_{-\infty}^{\infty}\hat{f}(s)e^{2\lambda isx}\,ds$ which equal to $f(x)$ itself, i.e.,

$$f(x) = \int_{-\infty}^{\infty}\hat{f}(s)e^{2\lambda isx}\,ds$$

Now we are ready to solve

$$\frac{\partial\rho}{\partial t} = D\frac{\partial^2}{\partial x^2}\rho$$

by applying Fourier Transform on both sides w.r.t. x. Then we have

$$\begin{aligned}\widehat{\left(\frac{\partial\rho}{\partial t}\right)} &= D\widehat{\left(\frac{\partial^2}{\partial x^2}\rho\right)}\\ \frac{\partial\hat{\rho}}{\partial t} &= -4\lambda^2 s^2 D\hat{\rho}\\ \hat{\rho}(s,t) &= ce^{-4\lambda^2 s^2 Dt}\end{aligned}$$

To get the expression of $\rho(x,t)$, we need to take the inverse Fourier Transform of $\hat{\rho}(s,t)$ w.r.t. s. Then we have

$$\begin{aligned}\rho(x,t) &= \int_{-\infty}^{\infty}\hat{\rho}(s,t)e^{2\lambda isx}\,ds\\ &= \int_{-\infty}^{\infty} ce^{-4\lambda^2 s^2 Dt}e^{2\lambda isx}\,ds\\ &= c\int_{-\infty}^{\infty} e^{-4\lambda^2 s^2 Dt+2\lambda isx}\,ds\\ &= c\int_{-\infty}^{\infty} e^{-\left(2\lambda s\sqrt{Dt}-\frac{xi}{2\sqrt{Dt}}\right)^2-\frac{x^2}{4tD}}\,ds\\ &= ce^{-\frac{x^2}{4tD}}\int_{-\infty}^{\infty} e^{-\left(2\lambda s\sqrt{Dt}-\frac{xi}{2\sqrt{Dt}}\right)^2}\,ds\end{aligned}$$

Set $y = 2\lambda s\sqrt{Dt} - \frac{xi}{2\sqrt{Dt}} \implies dy = 2\lambda\sqrt{Dt}ds$

$$\begin{aligned}\rho(x,t) &= ce^{-\frac{x^2}{4tD}}\int_{-\infty}^{\infty} e^{-y^2}\frac{1}{2\lambda\sqrt{Dt}}\,dy\\ &= \frac{c}{2\lambda\sqrt{Dt}}e^{-\frac{x^2}{4tD}}\int_{-\infty}^{\infty} e^{-y^2}\,dy\\ &= \frac{c}{2\lambda\sqrt{Dt}}e^{-\frac{x^2}{4tD}}\sqrt{\lambda}\\ &= \frac{c}{2\sqrt{\lambda Dt}}e^{-\frac{x^2}{4tD}}\end{aligned}$$

Notice that

- $\lim_{t\to 0^+}\rho(0,t) = \lim_{t\to 0^-}\rho(0,t) = \infty$. This is consistent with our assumption: all bacterial are at $x=0$ at $t=0$.
- $\lim_{x\to\infty}\rho(x,t) = \lim_{x\to-\infty}\rho(x,t) = 0$
- $\rho(0,t) = \frac{c}{2\sqrt{\lambda Dt}} \implies$ density at origin decay over time.

§25.2 Diffusion on a Bounded Domain

<u>Motivation</u>: In real case, we would like to model dynamics on a finite domain. In this case, we need to consider the influence at the boundaries. So we need to impose the boundary conditions (along with the initial conditions) to ensure our PDE problem to be well-posed.
Suppose the length of the bar is L and the bar is perfectly insulated on the outsides (except possibly from the ends $x = 0$, $x = L$). Since the bar is perfectly insulated, we will not gain or lose energy anywhere except the boundary $\implies$ conservation of energy. This system satisfies the continuity equation

$$\frac{\partial}{\partial t}(Tc\rho) = -\frac{\partial q}{\partial x} \tag{1}$$

where $c =$ specific heat, $\rho =$ material density, $Tc\rho$ stands for the heat energy and q represents the flux (here c and ρ are some fixed number).
Repeat the process of the density function for the running process, we have that the flux of the heat should satisfy the Fick's Law:

$$q(x,t) = -k \cdot \frac{\partial T}{\partial x} \tag{2}$$

where k is the thermal conductivity of the material. Let's substitute (2) to (1), we have

$$\begin{aligned}
\frac{\partial}{\partial t}(c\rho T) &= -\frac{\partial}{\partial x}\left(-k\frac{\partial T}{\partial x}\right) \\
c\rho\frac{\partial}{\partial t}T &= k\frac{\partial^2 T}{\partial x^2} \\
\frac{\partial T}{\partial t} &= \underbrace{\frac{k}{c\rho}}_{D}\frac{\partial^2 T}{\partial x^2}
\end{aligned}$$

where $0 < x < L$, $t > 0$ and it satisfies the initial condition $T(x,0) = T_0(x)$. In a finite domain, we also need to consider temperature dynamics at the boundaries.
Popular boundary conditions:

1. Temperature on the boundaries is constant

   $$T(0,t) = \alpha_0 \text{ and } T(L,t) = \alpha_L$$

   This is called "Dirichlet" or "fixed" conditions.

2. Flux on boundaries is constant:

   $$q(0,t) = \beta_0, \quad q(L,t) = \beta_L$$

   This is called "Neumann" or "fixed-flux" boundary condition. Because $q(x,t) = -k\frac{\partial T}{\partial x}$, the "Neumann" boundary condition can be converted into

   $$\left.\frac{\partial T}{\partial x}\right|_{(0,t)} = \gamma_0, \quad \left.\frac{\partial T}{\partial x}\right|_{(L,t)} = \gamma_L$$

   But $\beta_0\gamma_0 < 0$, $\beta_L\gamma_L < 0$.

3. Perfect insulation. Then it means that no heat can enter or leave the bar. So the flux is 0.

   $$q(0,t) = q(L,t) = 0$$

   "No flux" condition.

It's difficult to find the solutions for the PDE with boundary conditions. But we can learn the system by considering equilibrium solution. Notice that we are assuming the BC are independent of time, the steady state exist and the steady state is also independent of time:

$$\lim_{t\to\infty} T(x,t) = T_\infty(x)$$
$$\frac{\partial}{\partial t} T_\infty(x) = 0$$

Example 25.1

Consider

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}, \quad 0 < x < 1, \quad T(0,t) = \alpha_0$$
$$T(1,t) = \alpha_1 \quad \text{and} \quad T(x,0) = T_0(x)$$

Find the steady state T_∞.

$$\frac{\partial T_\infty}{\partial t} = 0 \implies D\frac{\partial^2 T_\infty}{\partial x^2} = 0$$
$$\implies \frac{\partial^2 T_\infty}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T_\infty}{\partial x}\right) = 0$$
$$\implies \frac{\partial T_\infty}{\partial x} = C_1$$

C_1 is independent of t and x because T_∞ is independent of t, so $T_\infty = T_\infty(x)$.

$$\implies T_\infty(x) = C_1 x + C_2$$

Now, let's use the BC

$$T(0,t) = \alpha_0 \implies C_1 \cdot 0 + C_2 = \alpha_0 \implies C_2 = \alpha_0$$
$$T(1,t) = alph_1 \implies C_1 + C_2 = \alpha_1 \implies C_1 = \alpha_1 - \alpha_0$$
$$\implies T_\infty(x) = (\alpha_1 - \alpha_0)x + \alpha_0$$

Example 25.2

Consider

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}, \quad 0 < x < 1, \quad T(0,t) = \alpha_0$$
$$\left.\frac{\partial T}{\partial x}\right|_{x=1} = 0 \quad \text{and} \quad T(x,0) = T_0(x)$$

Find the steady state T_∞.
From the above example, we have $T_\infty(x) = C_1 x + C_2$. Next, let's apply the BC on T_∞

$$T(0,t) = \alpha_0 \implies T_\infty(0) = C_2 = \alpha_0$$
$$\left.\frac{\partial T}{\partial x}\right|_{x=1} = 0 \implies \left.\frac{\partial T_\infty}{\partial x}\right|_{x=1} = C_1 = 0$$

Thus, $T_\infty(x) = \alpha_0$.