Math 134 – Nonlinear ODE

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This is math 134 – Linear and Nonlinear System of Differential Equations taught by Professor Wink. The class lecture is prerecorded, and we have live session every Monday and Friday at 3:00 pm – 3:50 pm for Q & A. We use *Nonlinear Dynamics and Chaos* 2^{nd} by *Steven Strogatz* as our main book for the class. Other course notes can be found through my github. Any error spotted in the notes is my responsibility, and please let me know through my email at ducvu2718@ucla.edu if you notice it.

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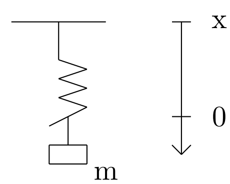
§1 Lec 1: Jan 4, 2021

§1.1 Intro to Dynamical Systems

There are two types of dynamical systems:

1. Discrete in time:
   - Difference equation
   - Iterated map: $a_{n+1} = f(a_n)$
2. Continuous in time: differential equation
   - Partial Differential Equation (PDE):
     e.g. heat equation
     $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$$
     wave equation
     $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$$
     where the derivatives w.r.t time and space.
   - Ordinary Differential Equation (ODE):
     i) Harmonic oscillator



m: mass
k: spring constant

$$m\dot{x} + kx = 0$$

If $\omega^2 = \frac{k}{m}$, then
$$x(t) = x_0 \cos(\omega t) + x_1 \sin(\omega t)$$

ii) Damped harmonic oscillator
$$m\ddot{x} + b\dot{x} + kx = 0, \qquad b\text{: damping constant}$$

iii) Forced, damped harmonic oscillator
$$m\ddot{x} + b\dot{x} + kx = F\cos(t), \qquad F\text{: force}$$

so derivatives w.r.t time only.

Definition 1.1 (Order of ODE) — Highest occurring derivative is defined as the order of the ODE.

Remark 1.2. We can always write an ODE of n^{th} order as a system of ODEs of 1^{st} order.

Trick: Consider the damped harmonic oscillator

$$m\ddot{x} + b\dot{x} + kx = 0$$

Set

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned}$$

Then,

$$\begin{aligned} \dot{x}_1 &= \dot{x} = x_2 \\ \dot{x}_2 &= \ddot{x} = -\frac{b}{m}\dot{x} - \frac{k}{m}x \\ &= -\frac{b}{m}x_2 - \frac{k}{m}x_1 \end{aligned}$$

i.e.,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{b}{m}x_2 - \frac{k}{m}x_1 \end{aligned}$$

General framework: $\dot{x} = f(t, x)$

$$f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$

i.e.,

$$\begin{aligned} \dot{x}_1 &= f_1(t_1, x_1, \dots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(t, x_1, \dots, x_n) \end{aligned} \tag{1}$$

which is 1^{st} order n-dimensional ODE.

Definition 1.3 (Linear ODE) — The ODE (1) is called linear if $f(t, x) = A(t) \cdot x$ for a time dependent matrix $A(t)$, otherwise we call it non-linear.

Example 1.4

The damped harmonic oscillator is linear.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Question 1.1. Why are linear equations special?

They satisfy the principle of superposition. If ϕ, ψ solve $\dot{x} = A(t)x$, then $y(t) = c \cdot \phi(t) + \psi(t), c \in \mathbb{R}$ also solves $\dot{x} = A(t)x$. This is valid because $\dot{y} = c\dot{\phi} + \dot{\psi} = cA\phi + A\psi = A(c\phi + \psi) = Ay$. For non-linear ODEs, the principle of superposition fails.

Definition 1.5 (Autonomous ODE) — The ODE (1) is called autonomous if f does not depend on t, i.e., $f(t,x) = f(x)$.

Example 1.6

$$m\ddot{x} + b\dot{x} + kx = F\cos(t)$$

is non-autonomous.

However, we can always consider an autonomous system instead. Set

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \\ x_3 &= t \end{aligned}$$

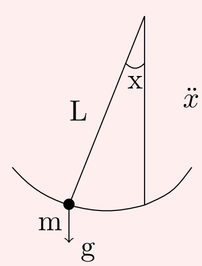
Then

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{b}{m}x_2 - \frac{k}{m}x_1 + F\cos(x_3) \\ \dot{x}_3 &= 1 \end{aligned}$$

We will primarily study autonomous 1^{st} order system in 1 or 2 variables.

Example 1.7 (Swinging Pendulum)

Consider a swinging pendulum



$$\ddot{x} + \frac{g}{L}\sin(x) = 0$$

Set

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned}$$

Then

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{L}\sin(x_1) \end{aligned}$$

1^{st} order, non-linear autonomous ODE in 2 variables.

Question 1.2. What can we say about the behavior of a solution $x_1(t), x_2(t)$ for larger time t? How does it depend on $\frac{g}{L}$?

Idea: Use geometric methods, without solving $\dot{x} = f(x)$ explicitly, to make qualitative statements about the long time behavior of the solution.

§2 Lec 2: Jan 6, 2021

§2.1 Phase Portraits

We want to study 1D autonomous dynamical systems

$$\dot{x} = f(x), \qquad f : \mathbb{R} \to \mathbb{R}$$

Remark 2.1. $x(t)$ is the solution to $\dot{x} = f(x)$ with $x(0) = x_0$. Find the solution $y(t)$ with $y(t_0) = x_0$.

Ans: $y(t) = x(t - t_0)$ because $y(t_0) = x(0) = x_0$ and $\dot{y}(t) = \dot{x}(t - t_0) = f(x(t - t_0)) = f(y(t))$.

Example 2.2

$x = \sin(x)$. Suppose $x_0 = \frac{\pi}{4}, x(t)$ solution with $x(0) = x_0$. Answer the followings

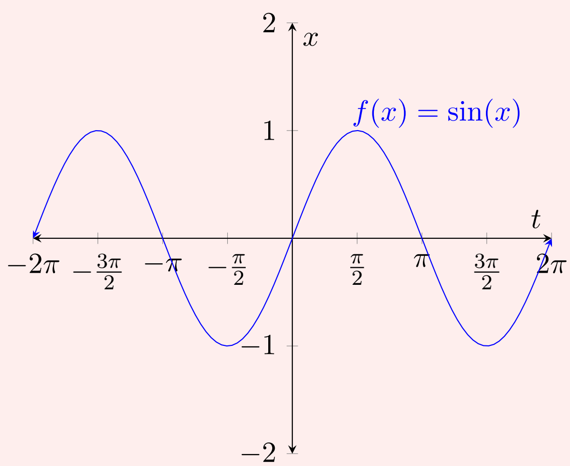
- Describe the long time behaviors of $x(t)$ as $t \to \infty$.
- How does the long time behavior depend on $x_0 \in \mathbb{R}$?

Attemp 1: Find explicit solution

$$\begin{aligned} \frac{dx}{dt} &= \sin(x) \\ dt &= \frac{dx}{\sin(x)} \\ t &= -\ln\left|\frac{1}{\sin(x)} + \frac{\cos(x)}{\sin(x)}\right| + c \end{aligned}$$

We know $x(0) = x_0$, so $c = \ln\left|\frac{1+\cos(x_0)}{\sin(x_0)}\right|$. But what is $x(t) =$? This approach fails!

Attempt 2: Draw a phase portrait/diagram. We want to interpret the velocity $\dot{x} = f(x)$ as a vector field on the real line.

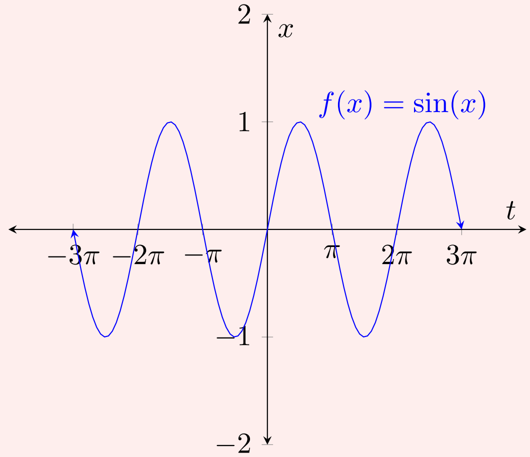


Idea:

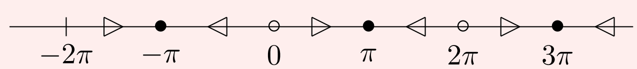
- If $f(x_0) > 0$, then the solution to $\dot{x} = f(x), x(0) = x_0$ increase near x_0.
- If $f(x_0) < 0$, then the solution to $\dot{x} = f(x), x(0) = x_0$ decrease near x_0.
- If $f(x_0) = 0$, then the solution to $\dot{x} = f(x), x(0) = x_0$ is $x(t) = x_0$ for all $t \in \mathbb{R}$, i.e., we have a fixed point/equilibrium point.

Example 2.3

$\dot{x} = f(x) = \sin(x)$



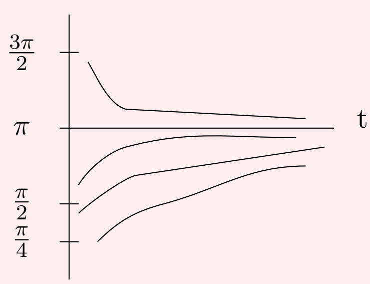
Phase portrait:



stable fixed point
(sink, attractors)

unstable fixed point
(source,)

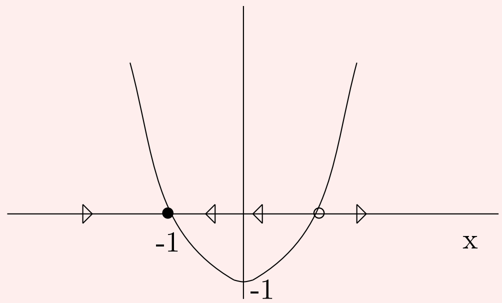
Qualitative plot of solution:



$x(t)$ is concave up
if $f(x(t))$ is increasing
$x(t)$ is concave down
if $f(x(t))$ is decreasing

Example 2.4

$\dot{x} = x^2 - 1$. Fixed points: $f(x) = x^2 - 1 = 0 \implies x = \pm 1$



Note: If $x_0 > 1$, then solution $x(t)$ with $x(0) = x_0 > 1$ is unbounded. In fact, $x(t) \to \infty$ in finite time.

§3 Lec 3: Jan 8, 2021

§3.1 Stability Types of Fixed Points

Definition 3.1 (Stability Types) — Consider the ODE $\dot{x} = f(x)$ and suppose that $f(x_*) = 0$. The fixed point x_* is called

1. Lyapunov stable if every solution $x(t)$ with $x(0) = x_0$ closed to x_* remain close to x_* for all $t \geq 0$, otherwise unstable.
2. Attracting if every solution $x(t)$ with $x(0) = x_0$ close to x_* satisfies $x(t) \to x_*$ as $t \to \infty$.
3. (asymptotically) stable if x_* is both Lyapunov stable and attracting.

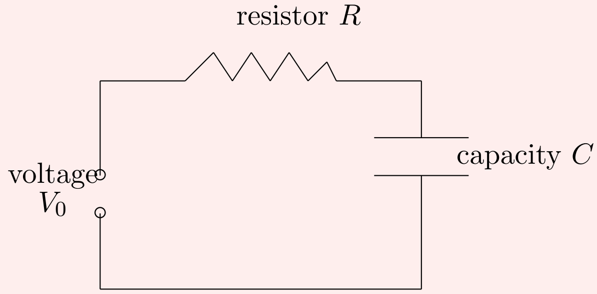
Example 3.2

Let $\alpha \in \mathbb{R}, \dot{x} = \alpha x$. General solution $x(t) = x_0 e^{\alpha t}$.

- $x_* = 0$ is always an equilibrium solution.
- $x_* = 0$ is
  1. attracting if $\alpha < 0$
  2. Lyapunov stable if $\alpha \leq 0$
  3. unstable if $\alpha > 0$

Example 3.3 (RC circuit)

We have the following circuit



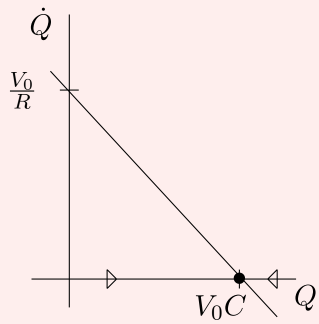
$$V_0 = RI + \frac{Q}{C}$$

$$I: \text{ current}, Q: \text{ change}$$

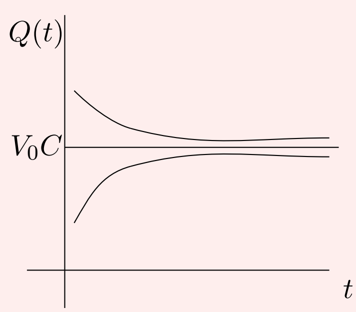
$$I = \dot{Q}$$

$$\dot{Q} = \frac{V_0}{R} - \frac{Q}{RC}$$

Phase portrait



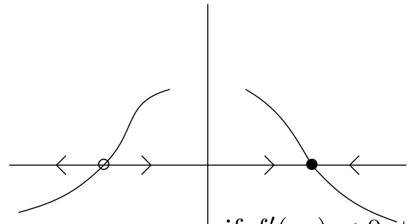
$Q_* = V_0C$ globally stable because every $Q(t)$ approaches Q_* as $t \to \infty$.



§3.2 Linear Stability Analysis

We have $\dot{x} = f(x), f(x_*) = 0$. Our task is to find an analytic criterion to decide if a fixed point x_* is stable/unstable.

Picture:



If $f'(x*) > 0$, then x_* is unstable. On the other hand, if $f'(x_*) < 0$, then x_* is a stable fixed point.
<u>The linearization</u>:
Consider: $\eta(t) = x(t) - x_*$ where $x(t)$ is the solution of $\dot{x} = f(x)$ with $x(0)$ close to x_*, $f(x_*) = 0$.
<u>Note</u>: $\dot{\eta}(t) = \dot{x}(t) = f\left(x(t)\right) = f\left(x(t) - x_* + x_*\right) = f\left(\eta(t) + x_*\right)$.
Taylor's Theorem:

$$f(x_* + \eta) = \underbrace{f(x_*)}_{=0} + f'(x_*)\eta + \underbrace{\mathcal{O}(\eta^2)}_{\text{error term and negligible if} f'(x_*)\neq 0 \text{ and } \eta \text{ is small}}$$

$\implies \quad \dot{\eta}(t) \approx f'(x_*)\eta(t)$ (as long as $\eta(t)$ is small) which is called the linearization of $\dot{x} = f(x)$ about x_*. The general solution is

$$\eta(t) = \eta_0 e^{f'(x_*)\cdot t}$$

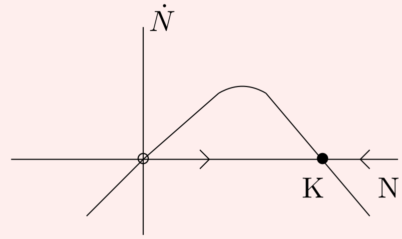
In particular, η grows exponentially if $f'(x_*) > 0$ or decreases exponentially if $f'(x_*) < 0$.

Definition 3.4 (Characteristics Time Scale) — $\frac{1}{|f'(x_*)|}$ is called the characteristics time scale.

Example 3.5 (Logstics Equation)
$N \geq 0$ population size, $r > 0$ growth rate, $K > 0$ carrying capacity

$$\dot{N} = rN\left(1 - \frac{N}{K}\right)$$

Fixed points: $\dot{N} = 0 \implies N_* = 0$ or $N_* = K$.
Let $f(N) = rN\left(1 - \frac{N}{K}\right) \implies f'(N) = r - 2\frac{r}{K}N$. In particular, $f'(0) = r > 0 \implies N_* = 0$ is an unstable fixed point and $f'(K) = r - 2r = -r < 0 \implies N_* = K$ is stable.
Phase portrait:

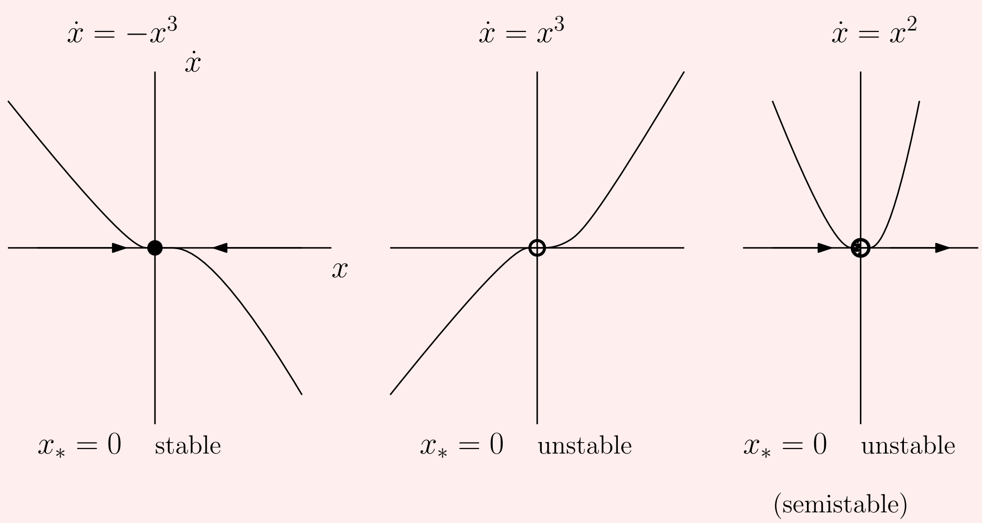


Thus, if $N(t)$ is the population with

$$N(0) = N_0 > 0 \implies N(t) \to K \text{ as } t \to \infty$$
$$N(0) = 0 \to N(t) = 0 \quad \forall t \text{ (no spontaneous outbreak)}$$

Characteristics time scale: $\frac{1}{|f'(N_*)|} = \frac{1}{r}$ for both $N_* = 0, K$.

Example 3.6
What if $f'(x_*) = 0$? Then we can't tell.

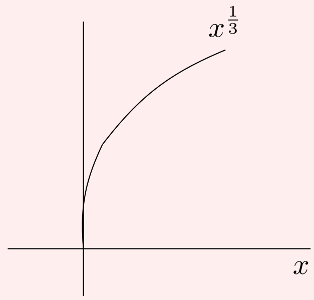


§4 Lec 4: Jan 11, 2021

§4.1 Existence and Uniqueness

Example 4.1 (Non-uniqueness)

$\dot{x} = x^{\frac{1}{3}} \implies x_1(t) \equiv 0$ (for all t) is a solution with $x_1(0) = 0$ but $x_2(t) = \left(\frac{2}{3}t\right)^{\frac{3}{2}}$ is also a solution with $x_2(0) = 0$



Is $x_0 = 0$ really a fixed point? No, it's unclear how it would behave (according to $x(t) = 0$ or $x(t) = \left(\frac{2}{3}t\right)^{\frac{3}{2}}$).

Theorem 4.2 (Picard's)

Let $I = (a, b) \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ differentiable and f' continuous. Let $x_0 \in I$. Then there is $\tau > 0$ s.t. the initial value problem

$$\dot{x} = f(x), x(0) = x_0$$

has a unique solution $x : (-\tau, \tau) \to \mathbb{R}$.

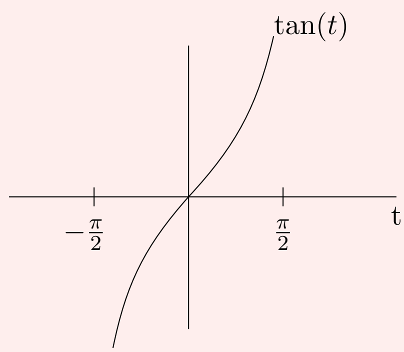
Example 4.3

(The solution might not exist for all times) Consider

$$\frac{dx}{dt} = \dot{x} = 1 + x^2, \quad x(0) = 0$$

So,

$$\begin{aligned} dt &= \frac{dx}{1+x^2} \\ t &= \int \frac{dx}{1+x^2} = \arctan x + C \\ 0 &= 0 + C \implies C = 0 \\ x(t) &= \tan(t) \end{aligned}$$



In particular,

$$\begin{aligned} x(t) &\to +\infty \text{ as } t \to \frac{\pi}{2} \\ x(t) &\to -\infty \text{ as } t \to \frac{-\pi}{2} \end{aligned}$$

i.e., $x(t)$ reaches infinity in finite time, i.e., the solution $x(t)$ blows up in finite time.

Remark 4.4. (Hw 1) If $x_0 > 0$, then the solution to $\dot{x} = x^2, x(0) = x_0 > 0$ blows up in finite time. In fact, if $\alpha > 1$, then the solution to $\dot{x} = x^\alpha$, $x(0) = x_0 > 0$ blows up in finite time.

Theorem 4.5 (ODE Comparison)

If $x_1(t)$ solves $\dot{x} = f(x)$, $x_2(t)$ solves $\dot{x} = q(x)$ and $x_1(0) \le x_2(0)$, $f(x) < q(x)$, then $x_1(t) \le x_2(t)$ for all $t > 0$.
In particular, if $x_1(t) \to \infty$ in finite time, then $x_2(t) \to \infty$ in finite time.

Example 4.6

The solution to $\dot{x} = 1 + x^2 + x^3, x(0) = 0$ blows up in finite time.
Note: For $x \geq 0$:

$$1 + x^2 \leq 1 + x^2 + x^3$$

Recall: $\tan(t)$ solves $\dot{x} = 1 + x^2, x(0) = 0$. By comparison: the solution $x(t)$ to $\dot{x} = 1 + x^2 + x^3, x(0) = 0$ satisfies $x(t) \geq \tan(t)$. Thus, $x(t)$ blows up in finite time. We may indeed assume that $x(t) > 0$. Since $\dot{x}(0) = 1$, it follows that $x(t) > 0$ for $t > 0$ small. In fact, $\dot{x} = 1 + x^2 + x^3 > 0$ for $x(t)$ small, i.e., whenever $x(t)$ is close to zero, it must increase $\implies x(t) > 0$ for $t > 0$.

Example 4.7 (No Oscillating Solution in 1D)

Let $f \in C^1(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} | f \text{ differentiable}, f' \text{ continuous}\}$. Suppose $f(x_*) = 0, x(t)$ solution of $\dot{x} = f(x)$. If $x(t_0) = x_*$ for some t_0. Then $x(t) = x_*$ for all time t. Geometrically this says that a solution can never reach/cross a fixed point (unless it is a fixed point).

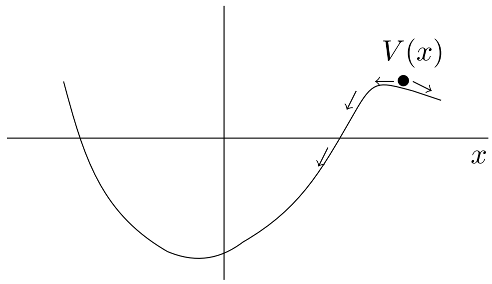
- $f(x(t)) > 0$ and $\dot{x}(t) > 0$, i.e., $x(t)$ increases.
- $f(x(t)) = 0$ and $x(t) =$ constant for all t.
- $f(x(t)) < 0$ and $\dot{x}(t) < 0$ i.e., $x(t)$ decreases.

In particular, there is no oscillating solution.

§5 Lec 5: Jan 13, 2021

§5.1 Potential

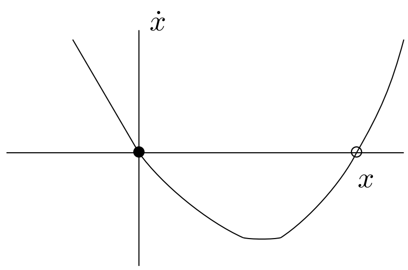
Consider the movement of a particle (with lots of friction) in a potential.



Notice:

- Particle approaches the local minimum of $V(x)$ (minimum energy level) no fixed point.
- Local minima of $V(x)$ are stable fixed points.
- Local maxima of $V(x)$ are unstable fixed points.

$\implies \dot{x} = f(x) = -\frac{dV}{dx} = -V'(x)$.



Expect $t \to V(x(t))$ is non-increasing for a solution $x(t)$ of $\dot{x} = -V'(x)$.
Indeed:

$$\begin{aligned} \frac{d}{dt} V(x(t)) &= V'(x(t)) \frac{d}{dt} x(t) \\ &= V'(x(t)) \left(-V'(x(t)) \right) \\ &= -\left(V'(x(t)) \right)^2 \leq 0 \end{aligned}$$

$\implies$ particle always moves towards a lower energy level.

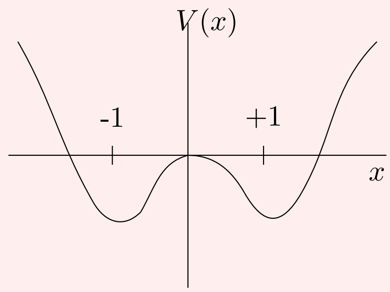
Definition 5.1 (Potential) — A function $V(x)$ s.t. $\dot{x} = f(x) = -\frac{dV}{dx}$ is called a potential.

Example 5.2

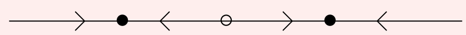
Graph potential for $\dot{x} = x - x^3$. Find/characterize equilibria (fixed points).

$$\dot{x} = f(x) = x - x^3 = -\frac{dV}{dx} \overset{\int}{\Longrightarrow} V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + C$$

$\Longrightarrow$ V is only defined up to a constant, we may choose any $C \in \mathbb{R}$, e.g., choose $C = 0$.



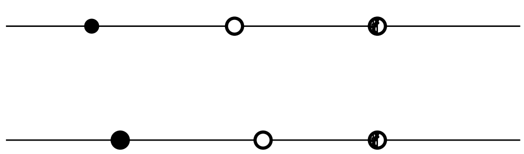
Local minima of V correspond to stable fixed points $\Longrightarrow 0 = -\frac{dV}{dx} = f(x) = x - x^3$, i.e., $x = \pm 1$.
Local maximum of V corresponds to an unstable fixed point at $x = 0$.
Phase portrait:



Remark 5.3. This system is often called <u>bistable</u> because it has two stable fixed points.

§5.2 Bifurcations

The qualitative behavior of 1D dynamical systems $\dot{x} = f(x)$ is determined by fixed points.

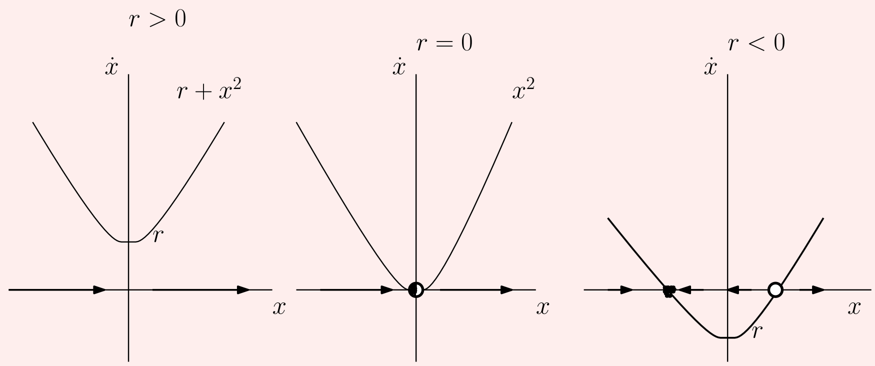


qualitative the same

(dynamics)

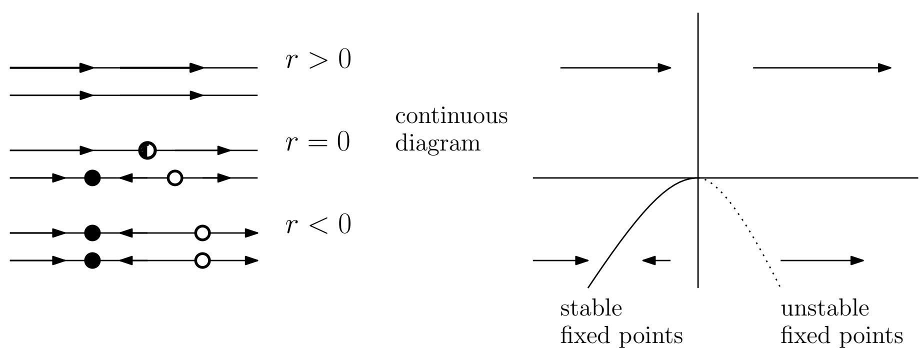
If $\dot{x} = f(r, x)$ depends on a parameter r, then the numbers of fixed points and their stability may change as r varies. This is called bifurcation.

Example 5.4 (Saddle-node, blue sky bifurcation)
$\dot{x} = r + x^2, \quad r \in \mathbb{R}.$

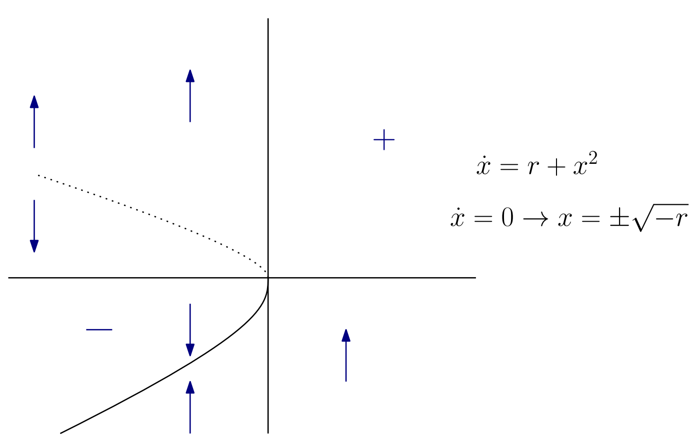


Hence, the qualitative behavior changes at $r_* = 0$, i.e., $r_* = 0$ is called a bifurcation point.

Ways to plot the dependence on the parameter:



Most common: bifurcation diagram



§6 Lec 6: Jan 15, 2021

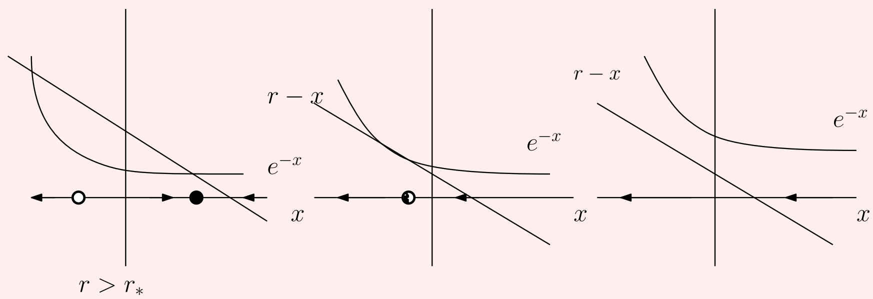
§6.1 Saddle-Node Example

Example 6.1

Argue geometrically that the ODE

$$\dot{x} = r - x - e^{-x}$$

undergoes a saddle-node bifurcation. Furthermore, find the bifurcation point.
Note: Fixed points of $\dot{x} = r - x - e^{-x}$ correspond to intersection points of the functions $r - x, e^{-x}$ because $r - x - e^{-x} = 0 \iff r - x = e^{-x}$.



Indeed we have a saddle-node bifurcation.
Note: At $r = r_*$, the graph of $r - x$ and e^{-x} intersect tangentially. Thus, for the bifurcation point we require:

$$0 = \dot{x} = r - x - e^{-x} \implies r - x = e^{-x}$$
$$0 = \frac{d}{dx}(r - x - e^{-x}) \implies \frac{d}{dx}(r - x) = \frac{d}{dx}e^{-x}$$

So,

$$\begin{aligned} -1 &= -e^{-x} \\ e^{-x} &= 1 \\ x &= 0 \\ r_* &= x_* + e^{-x_*} = 0 + 1 = 1 \end{aligned}$$

Thus the bifurcation point is $(r_*, x_*) = (1, 0)$.

Note:

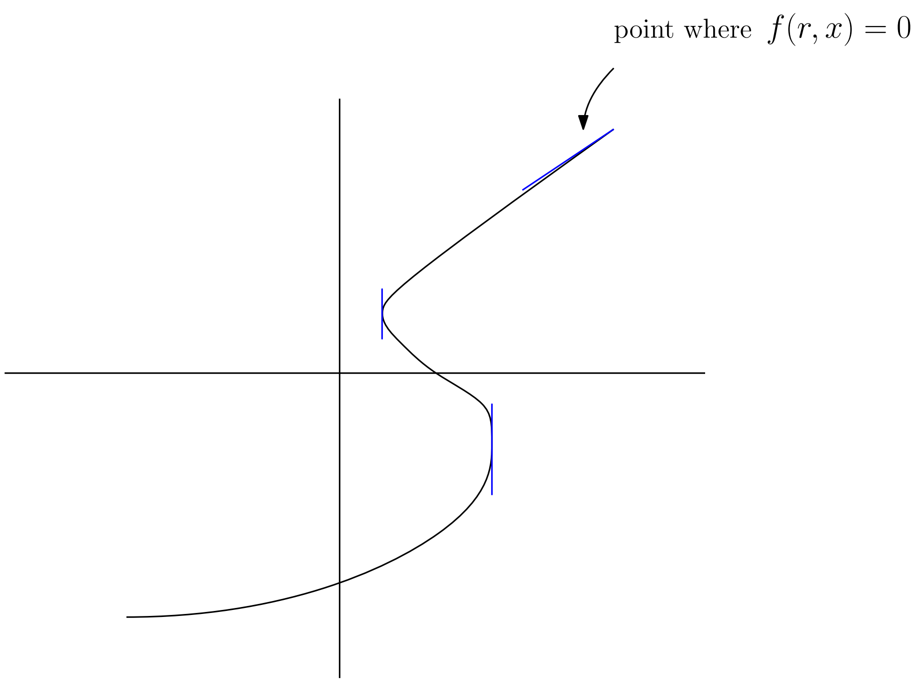
$$\begin{aligned} \dot{x} &= r - x - e^{-x} = r - x - \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \dots\right) \\ &= r - 1 - \frac{1}{2}x^2 + \frac{x^3}{6} - \dots \\ &\approx (r - 1) - \frac{1}{2}x^2 \text{ for x near } x_* = 0 \end{aligned}$$

Set $R = r - 1$, then $\dot{x} \approx R - \frac{1}{2}x^2$.
Upshot: Up to appropriate rescalings/coordinate changes, every saddle-node bifurcation looks like its normal form

$$\dot{x} = r - x^2 \qquad (\text{or } \dot{x} = r + x^2)$$

close to the bifurcation point $(r_*, x_*) = (0, 0)$.

§6.2 Normal Forms



Recall:

- Normal vector: $\begin{pmatrix} \partial_r f \\ \partial_x f \end{pmatrix}$
- Tangent vector: $\begin{pmatrix} -\partial_x f \\ \partial_r f \end{pmatrix}$

Note: Bifurcation points have vertical tangent vectors, i.e., $\partial_x f = 0, \partial_r f \neq 0$.

Theorem 6.2 (Taylor's)
Suppose $f(r_*, x_*) = 0$.

$$f(r,x) = f(r_*,x_*) + \underbrace{\frac{\partial f}{\partial r}(r_*,x_*)}_{p_1}(r - r_*) + \underbrace{\frac{\partial f}{\partial x}(r_*,x_*)}_{q_1}(x - x_*)$$
$$+\frac{1}{2}\underbrace{\frac{\partial^2 f}{\partial r^2}(r_*,x_*)}_{p_2}(r-r_*)^2 + \underbrace{\frac{\partial^2 f}{\partial r \partial x}(r_*,x_*)}_{R}(r-r_*)(x-x_*) + \frac{1}{2}\underbrace{\frac{\partial^2 f}{\partial x^2}(r_*,x_*)}_{q_2}(x-x_*)^2 + \dots$$

Remark 6.3. If $q_1 \neq 0$, then there is no bifurcation at (r_*, x_*), linear stability (sign of q_1) determines if (r_*, x_*) is (un)stable.

Theorem 6.4

Suppose that $f(r_*, x_*) = 0, q_1 = 0, p_1 \neq 0, q_2 \neq 0$, then $\dot{x} = f(r, x)$ undergoes a saddle node bifurcation at (r_*, x_*) and

$$\dot{x} = \frac{\partial f}{\partial r}(r^*, x^*)(r - r^*) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x - x_*)^2 + \mathcal{O}(\epsilon^3)$$

for $|r - r_*| < \epsilon^2, \quad |x - x_*| < \epsilon$.

Remark 6.5.

i) Note that the constant $(r - r_*)(x - x_*)$ is $\mathcal{O}(\epsilon^3)$

ii) With a coordinate change $(t, x, r) \mapsto (s, y, R)$ we can arrange that ODE looks like

$$\frac{d}{ds} y = R + y^2$$

near $(0, 0) = (R(r_*), y(x_*))$

Example 6.6

$\dot{x} = e^r - x - e^{-x}$ undergoes a saddle-node bifurcation near $(r_*, x_*) = (0,0)$. Apply the theorem 6.4,

$$\begin{aligned} f(r,x) &= e^r - x - e^{-x} \\ f(0,0) &= 1 - 0 - 1 = 0 \\ \frac{\partial f}{\partial x}(r,x) &= -1 + e^{-x} \implies \frac{\partial f}{\partial x}(0,0) = 0 \\ \frac{\partial f}{\partial r}(r,x) &= e^r \implies \frac{\partial f}{\partial r}(0,0) = 1 \neq 0 \\ \frac{\partial^2 f}{\partial x^2}(r,x) &= -e^{-x} \implies \frac{\partial^2 f}{\partial x^2}(0,0) = -1 \neq 0 \end{aligned}$$

Therefore, by theorem 6.4, $(r_*, x_*) = (0,0)$ is a bifurcation point of a saddle-node bifurcation.
Normal form near $(r_*, x_*) = (0,0)$:

$$\begin{aligned} \dot{x} &= e^r - x - e^{-x} \\ &= 1 + r + \frac{r^2}{2} + \mathcal{O}(r^3) - x - \left(1 - x + \frac{x^2}{2} + \mathcal{O}(x^3)\right) \\ &= r + \underbrace{\frac{r^2}{2}}_{\mathcal{O}(\epsilon^4)} - \frac{x^2}{2} + \mathcal{O}(r^3) + \mathcal{O}(x^3) \\ &= \underbrace{r - \frac{x^2}{2}}_{\mathcal{O}(\epsilon^2)} + \mathcal{O}(\epsilon^3) \text{ if } |r - r_*| = |r| < \epsilon^2 \\ &\qquad\qquad \text{if } |x - x_*| = |x| < \epsilon \end{aligned}$$

Set $y = \frac{x}{2}$, then

$$\dot{y} = \frac{1}{2}\dot{x} = \frac{r}{2} - \frac{x^2}{4} + \mathcal{O}(\epsilon^3) = \frac{r}{2} - y^2 + \mathcal{O}(\epsilon^3)$$

Set $s = -t$, then

$$\frac{d}{ds}y = -\frac{d}{dt}y = -\frac{r}{2} + y^2 + \mathcal{O}(\epsilon^3)$$

Set $R = -\frac{r}{2}$, then

$$\underbrace{\frac{d}{ds}y = R + y^2}_{\text{normal form of a saddle-node bifurcation}} \qquad + \mathcal{O}(\epsilon^3)$$

§7 Lec 7: Jan 20, 2021

§7.1 Classification of Bifurcations

Let's rewrite $\dot{x}$ in theorem 6.4 as

$$\dot{x} = p(r - r_*) + \frac{c}{2}(x - x_*)^2 + \mathcal{O}(\epsilon^3)$$

if $|r - r_*| < \epsilon^2, |x - x_*| < \epsilon$. After a coordinate change $(t, x, r) \mapsto (s, y, R)$ such that

$$\begin{aligned} s &= t \\ y &= \frac{c}{2}(x - x_*) \\ R &= p\frac{c}{2}(r - r_*) \end{aligned}$$

the ODE is represented by the normal form.

$$\frac{d}{ds}y = \dot{y} = R + y^2 + \mathcal{O}(\epsilon^3)$$

for $|R| < \epsilon^2, |y| < \epsilon$.
If $f(x_*, r_*) = 0$, and also $\frac{\partial f}{\partial x}(x_*, r_*) = 0 = \frac{\partial f}{\partial r}(x_*, r_*)$, then the second derivatives determines the bifurcation type.

$$\text{Hessian} \quad \text{Hess}f = \begin{pmatrix} \frac{\partial^2 f}{\partial r^2} & \frac{\partial^2 f}{\partial r \partial x} \\ \frac{\partial^2 f}{\partial r \partial x} & \frac{\partial^2 f}{\partial x^2} \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

Second test: if $AC - B^2 > 0$, (r_*, x_*) is a local maximum/minimum. In particular, (r_*, x_*) is an isolated fixed point. (irrelevant case)
Practically relevant case: If $AC - B^2 < 0$: (r_*, x_*) is a saddle. If also $C \neq 0$: transcritical bifurcation.

$$\dot{y} = Ry - y^2 + \mathcal{O}(\epsilon^2)$$

for $|R| < \epsilon, |y| < \epsilon$ (after an appropriate coordinate change)

$$\mathcal{O}(r - r_*) = \mathcal{O}(R), \quad \mathcal{O}(x - x_*) = \mathcal{O}(y)$$

If also $C = 0$: Pitchfork bifurcation

- Supercritical Pitchfork bifurcation:

  $$y' = Ry - y^3 + \mathcal{O}(\epsilon^3)$$

- Subcritical Pitchfork bifurcation

  $$y' = Ry + y^3 + \mathcal{O}(\epsilon^3)$$

  for $|R| < \epsilon^2, |y| < \epsilon$

Again,

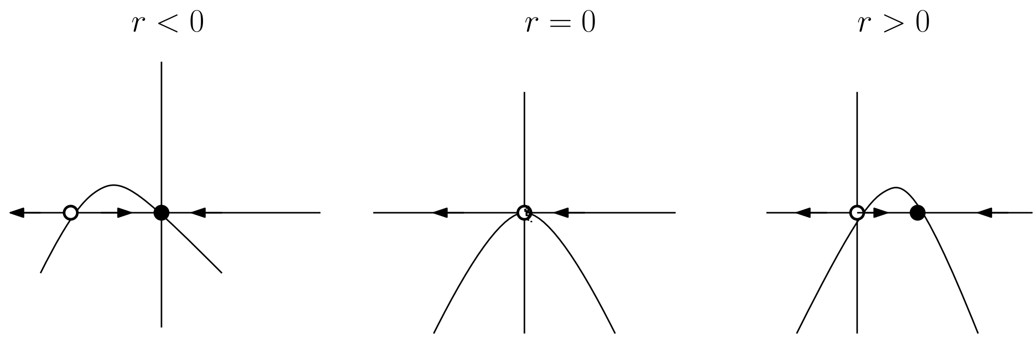
$$\mathcal{O}(r - r_*) = \mathcal{O}(R), \quad \mathcal{O}(x - x_*) = \mathcal{O}(y)$$

§7.2 Transcritical Bifurcation

Normal form:

$$\dot{x} = rx - x^2 = x(r - x)$$

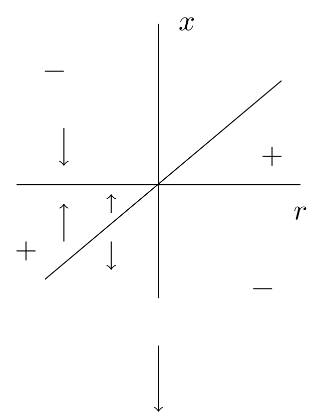
In particular, $x_* = 0$ is always a fixed point but it changes stability.



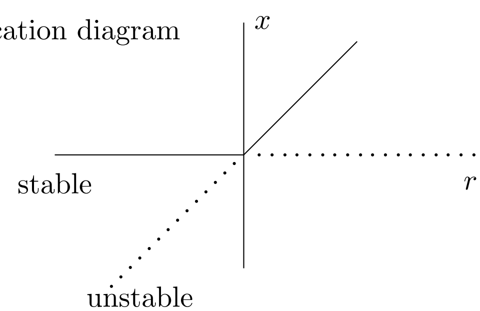
Bifurcation diagram: $\dot{x} = x(r - x) = rx - x^2 = f(x)$. Fixed points:

$$x_* = 0, \quad x_* = r \quad r \in \mathbb{R}$$

intermediate step:
draw fixed points
(without stability)



bifurcation diagram



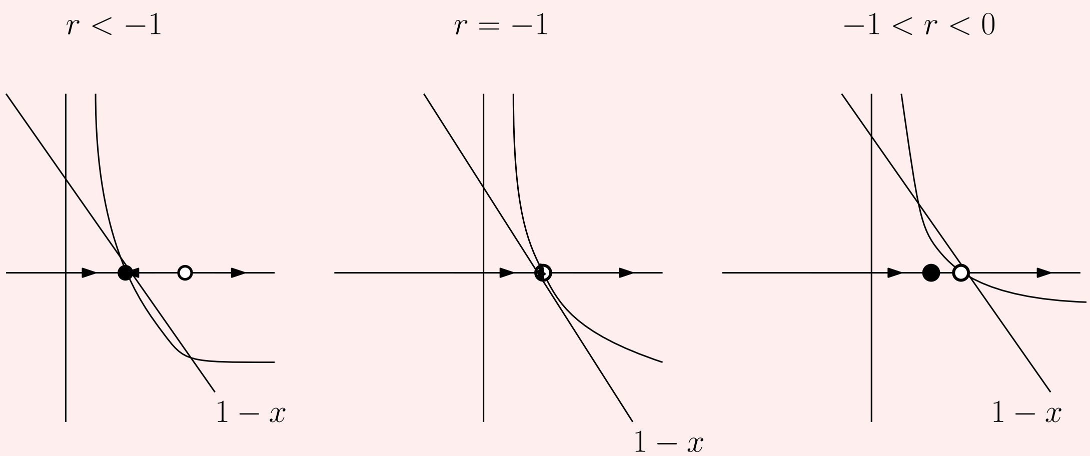
§8 Lec 8: Jan 22, 2021

§8.1 Example of Transcritical Bifurcation

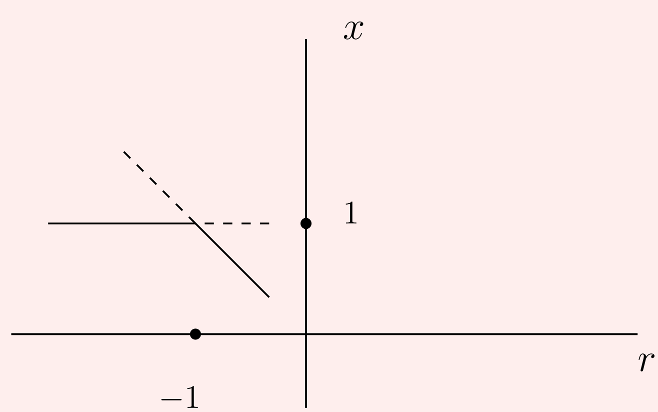
Example 8.1

$\dot{x} = r\ln(x) + x - 1$ has a transcritical bifurcation at $(r_*, x_*) = (-1, 1)$.
Geometric approach:

$$\dot{x} = 0 \iff r\ln(x) = 1 - x$$



Bifurcation near $(r_*, x_*) = (-1, 1)$



Normal form: $\dot{x} = r\ln(x) + x - 1$.

Remark 8.2. $\ln(1 + x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k, \quad |x| < 1$

So,

$$\begin{aligned}
\dot{x} &= r\ln(x) + x - 1 \\
&= r(x - 1 - \frac{1}{2}(x-1)^2 + \mathcal{O}((x-1)^3) + x - 1 \\
&= (r+1)(x-1) - \frac{1}{2}\left((r+1)-1\right)(x-1)^2 + \mathcal{O}\left(r(x-1)^3\right) \\
&= (r+1)(x-1) + \frac{1}{2}(x-1)^2 + \mathcal{O}(\epsilon^3)
\end{aligned}$$

if $|r - (-1)| < \epsilon$ and $|x - 1 < \epsilon|$.
Now, set $R = r+1, y = c\cdot(x-1)$. Then,

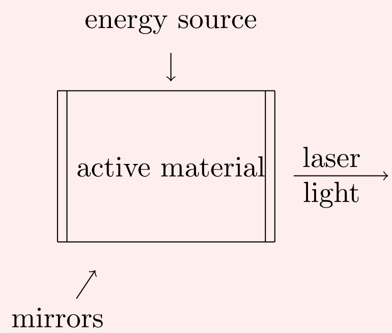
$$\begin{aligned}
\dot{y} &= c\dot{x} \\
&= (r+1)c(x-1) + \frac{1}{2}c(x-1)^2 + \mathcal{O}(\epsilon^3) \\
&= Ry + \frac{1}{2c}\left(c(x-1)\right)^2 + \mathcal{O}(\epsilon^3) \\
&= Ry + \underbrace{\frac{1}{2c}}_{=1} y^2 = Ry + y^2
\end{aligned}$$

for $c = \frac{1}{2}$.

§8.2 Application of Transcritical Bifurcations

Example 8.3 (Laser Threshold)

Consider



Simple model:

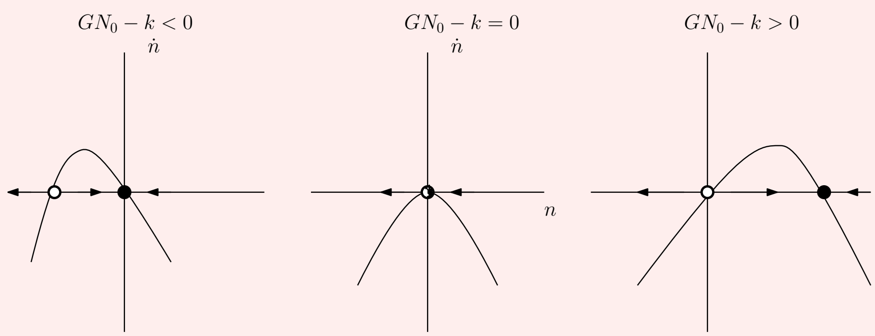
$$n = n(t) = \#\text{ photons in the laser}$$

Then

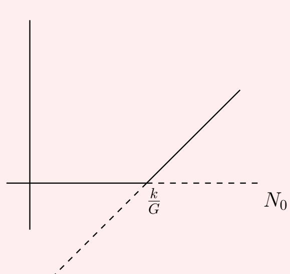
$$\begin{aligned}\dot{n} &= G \cdot \underbrace{N}_{\#\text{ excited atoms}} \cdot n - kn \\ &= N_0 - \alpha \cdot n \\ &= G(N_0 - \alpha n)n - kn \\ &= (GN_0 - k)n - \alpha G n^2\end{aligned}$$

where $G, k, \alpha > 0$. Fixed points:

$$\dot{n} = 0 \iff n = 0 \text{ or } n = \frac{GN_0 - k}{\alpha G}$$



Bifurcation diagram



transcritical bifurcation at

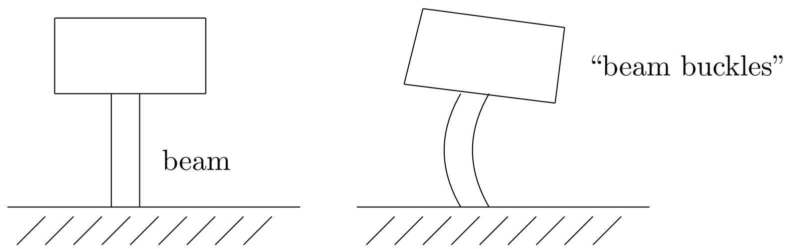
$(N, n) = (\frac{k}{G}, 0)$

$\frac{k}{G} =$ laser threshold

§9 Lec 9: Jan 25, 2021

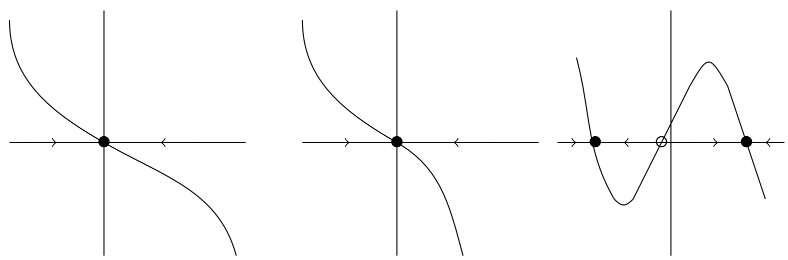
§9.1 Supercritical Pitchfork Bifurcation

Fixed points appear/disappear in symmetric pairs



Supercritical Pitchfork Bifurcation:

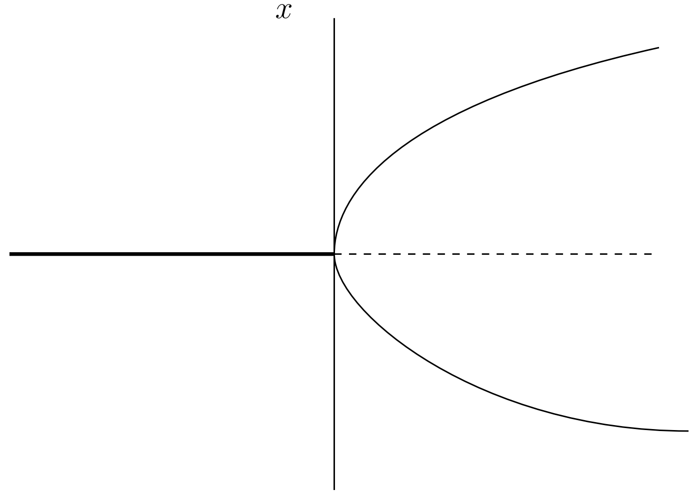
$$\dot{x} = rx - x^3$$



> **Remark 9.1.** Decay towards $x_* = 0$ is not exponential in time for $r = 0$.

Bifurcation diagram:

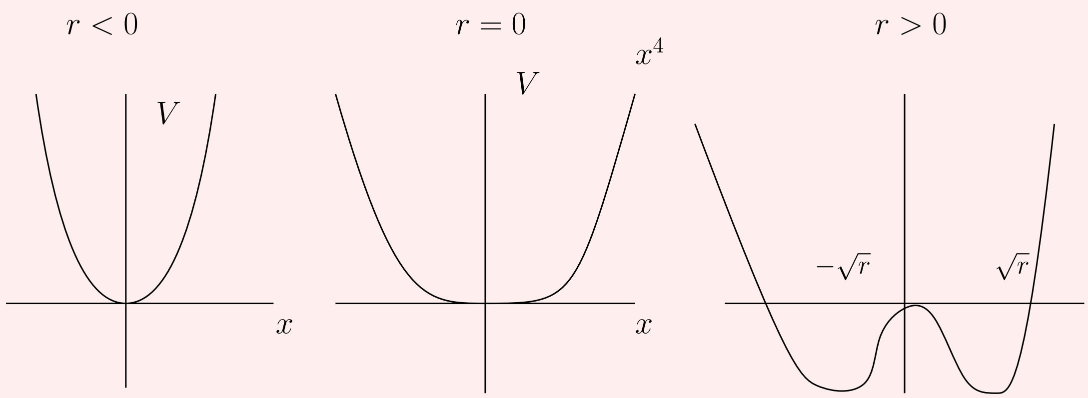
$$\dot{x} = rx - x^3 = 0$$
$$\implies x = 0, \qquad x = \pm\sqrt{r}, \quad r > 0$$



Example 9.2

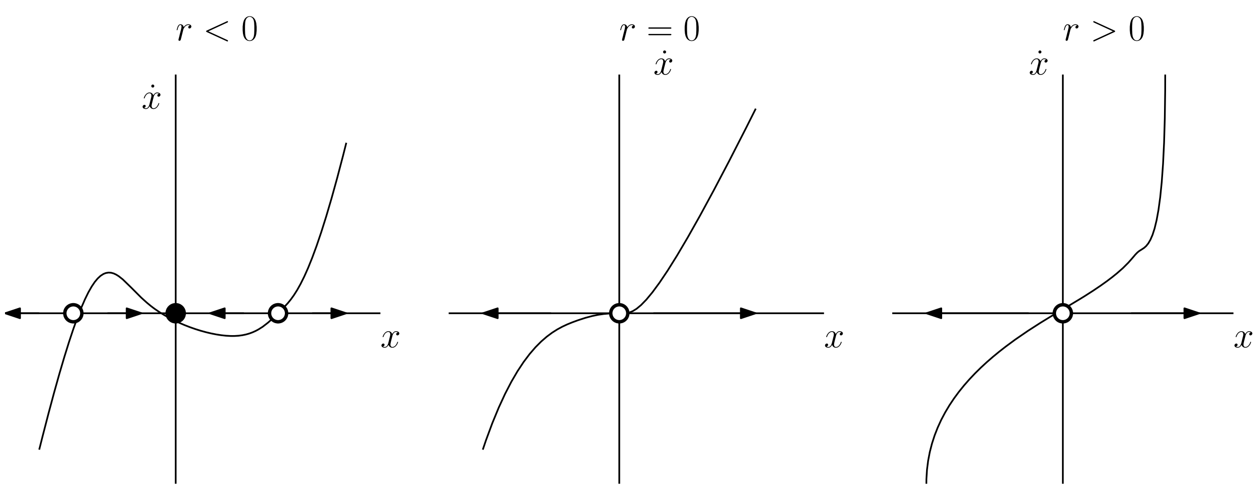
Potential for $\dot{x} = rx - x^3 = -\frac{dV}{dx}$

$$\implies V(x) = -\frac{1}{2}rx^2 + \frac{1}{4}x^4 + \underbrace{C}_{=0}$$



§9.2 Subcritical Pitchfork Bifurcation

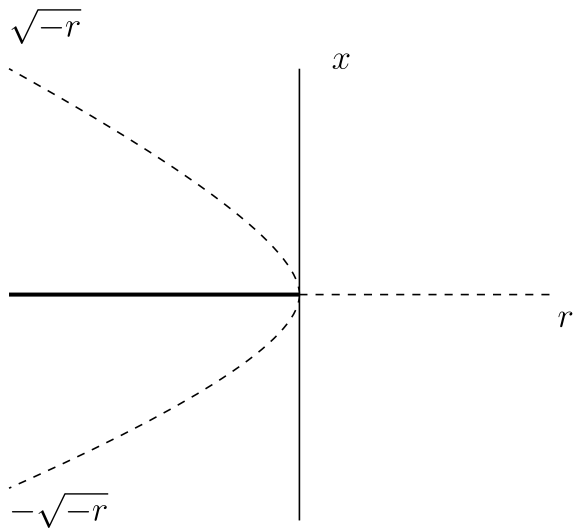
$$\dot{x} = rx + x^3$$



Fixed points:

$$\dot{x} = rx + x^3 = 0$$
$$\implies x = 0, \qquad x = \pm\sqrt{-r}, \quad r < 0$$

Bifurcation Diagram:



> **Remark 9.3.** If $r > 0, x_0 > 0$, then the solution $x(t)$ with $x(0) = x_0 > 0$ blows up in finite time (cf. homework). Interpretation: $+x^3$ is destabilizing.

Physically more realistic scenario:

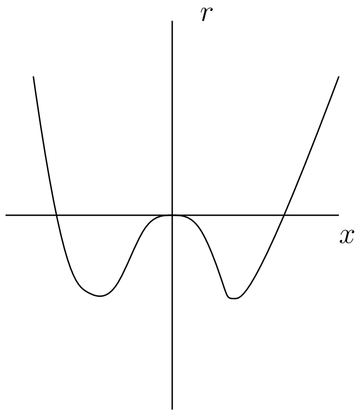
$$\dot{x} = rx + x^3 - x^5$$

where x^5 is the stabilizing higher order term.
Fixed points:

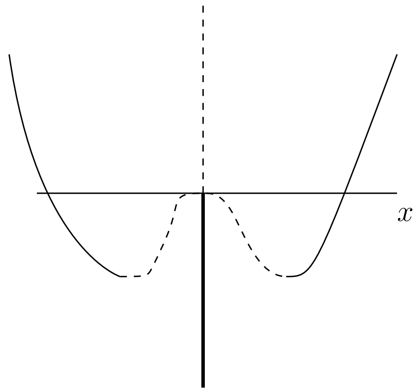
$$\dot{x} = 0 \iff x = 0, \quad r = -x^2 + x^4$$

Bifurcation diagram:

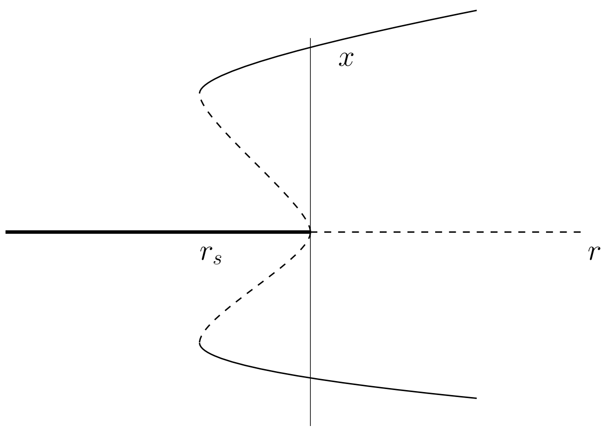
1. Intermediate step



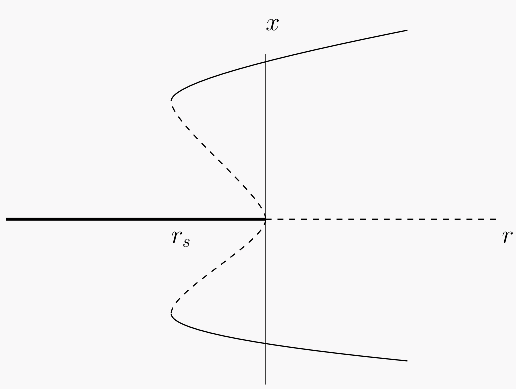
2. Stability Types



3. Change axes: bifurcation diagram



$$\dot{x} = rx + x^3 - x^5$$

> **Remark 9.4.** i) Subcritical pitchfork bifurcation at $(r_*, x_*) = (0, 0)$ and saddle node bifurcation at $(r_s, x_*) = \left(-\frac{1}{4}, \pm\sqrt{2}\right)$.
>
> 
>
>
> $$\dot{x} = r(t)x + x^3 - x^5$$
>
> ii) jump at $r_* = 0$: A small perturbation of a stable fixed point at $(0, r)$ with $r < 0$ jumps to the stable large amplitude branch as r becomes positive, but does not jump back until $r < r_s$.
>
> This non-reversibility is called hysteresis.

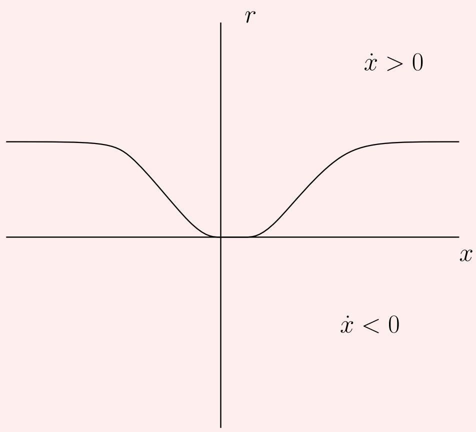
§10 Lec 10: Jan 27, 2021

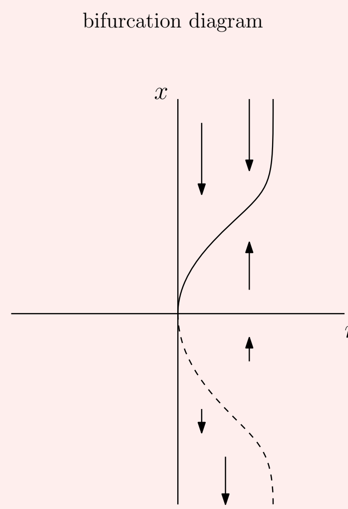
§10.1 Bifurcation at Infinity

Example 10.1

$\dot{x} = r - \frac{x^2}{1+x^2}$

Fixed points: $\dot{x} = 0 \iff r = \frac{x^2}{1+x^2}$





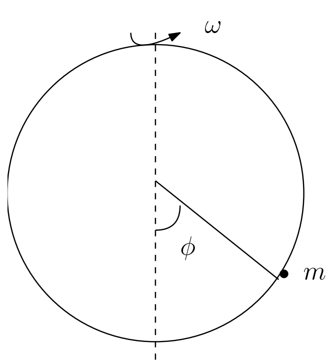
Note:

- At (r_*, x_*) we have a saddle node bifurcation.
- If $r \in (0, 1)$ we have two fixed points.
- For $r \geq 1$ we have no fixed points.

Thus, we have a bifurcation at (spatial) infinity.

§10.2 Dimensional Analysis and Scaling

Over-damped bead over a hoop:



forces: gravitation: $-mg\vec{e_2}$

centrifugal: $mr\sin\phi\omega^2\vec{e_x}$

damping: $-b\phi\vec{e_\phi}$

Physics: $mr\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mr\omega^2 \sin\phi\cos\phi$
Experiment: Provided ω large enough, bead slides slowly towards a fixed angle, after an initial acceleration phase.

Question 10.1. When we can neglect second order term $\ddot{\phi}$?

Problem 10.1. We're working with different dimensions, e.g.

$$[m] = kg$$
$$[b] = \frac{kg \cdot m}{s}$$

What is small – what quantity is actually small so we can neglect the second order term?

Idea: Non-dimensionalize

- small means $\ll 1$
- reduce the numbers of parameters
- no general algorithm

Quantity ω large, time scale T.
Set $\tau = \frac{t}{T} \implies d\tau = \frac{1}{T}dt$, where T is the characteristics time scale.
$\dot{\phi} = \frac{d\phi}{dt} = \frac{d\phi}{d\tau}\frac{d\tau}{dt} = \frac{1}{T}\frac{d\phi}{d\tau}$
Similarly, $\ddot{\phi} = \frac{1}{T^2}\frac{d^2\phi}{d\tau^2}$

$$mr\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mr\omega^2\sin\phi\cos\phi \tag{1}$$

So

$$\implies \frac{mr}{T^2}\frac{d^2\phi}{d\tau^2} = -\frac{b}{T}\frac{d\phi}{d\tau} - mg\sin\phi + mr\omega^2\sin\phi\cos\phi \qquad \text{(unit force)}$$
$$\implies \frac{r}{gT^2}\frac{d^2\phi}{d\tau^2} = -\frac{b}{mgT}\frac{d\phi}{d\tau} - \sin\phi + \frac{r\omega^2}{g}\sin\phi\cos\phi \qquad \text{(dimensionless)}$$

Thus 1st order term $\frac{d\phi}{d\tau}$ dominates $\frac{d^2\phi}{d\tau^2}$ if $\frac{r}{gT^2} \ll 1$ and $\frac{b}{mgT} \approx \mathcal{O}(1)$, i.e., $\frac{b}{mgT} = 1$ and $\epsilon = \frac{r}{gT^2}$

$$\implies T = \frac{b}{mg}$$
$$\implies \epsilon = \frac{rgm^2}{b^2} \ll 1$$

Set $\gamma = \frac{r\omega^2}{g}$. Then the non-dimensionalize equation becomes

$$\epsilon\frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin\phi + \gamma\sin\phi\cos\phi$$

Overdamped limit: $\epsilon \to 0$

$$\begin{aligned}\frac{d\phi}{d\tau} &= -\sin\phi + \gamma\sin\phi\cos\phi \\ &= \sin\phi(\gamma\cos\phi - 1)\end{aligned}$$

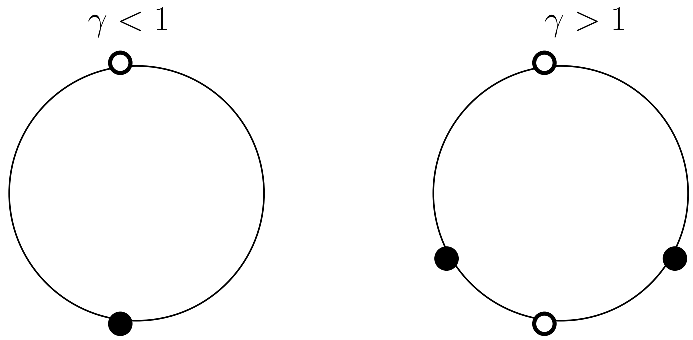
Dynamics: $\frac{d\phi}{d\tau} = 0$ (fixed points)

$$\implies \sin\phi = 0 \iff \phi = 0, \pi \text{ (bottom/top of hoop)}$$

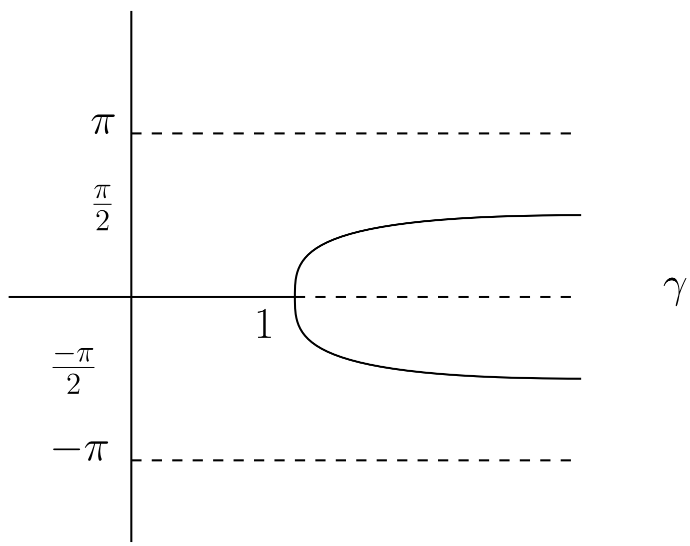
or

$$\cos\phi = \frac{1}{\gamma} \in (0, 1] \implies \gamma \geq 1$$

Fixed points:



Bifurcation Diagram:



In particular, we have a supercritical pitchfork bifurcation at $\gamma = 1$.

§11 Lec 11: Jan 29, 2021

§11.1 Imperfect Bifurcation and Catastrophes

$\dot{x} = h + rx - x^3$

- If $h = 0$: symmetry, if $x(t)$ is a solution then $-x(t)$ is also a solution (supercritical pitchfork bifurcation).
- If $h \neq 0$: imperfect parameter, breaks symmetry.

Aim: Study qualitative behavior of ODE as parameters vary.
Strategy: keep h fixed and vary r

- $h = 0$: supercritical pitchfork bifurcation

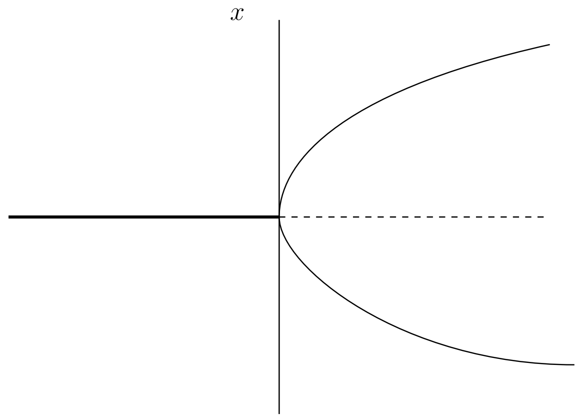


Figure 1: Bifurcation Diagram

- $h > 0$: fixed points: $\dot{x} = 0 \iff x^3 = h + rx$

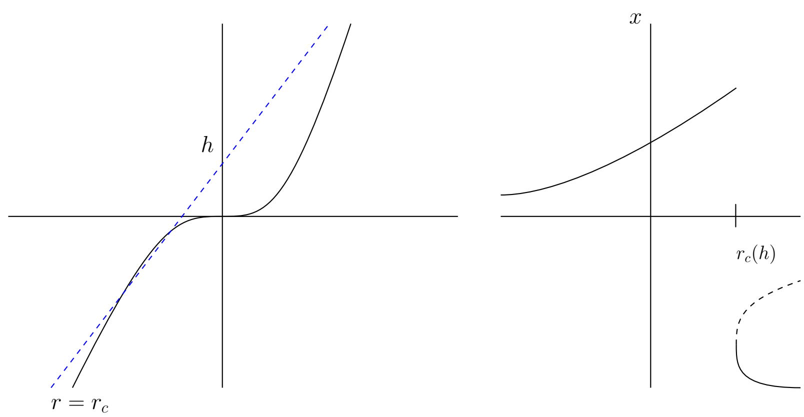


Figure 2: Bifurcation Diagram

- $h < 0$: Fixed points: $x^3 = h + rx$

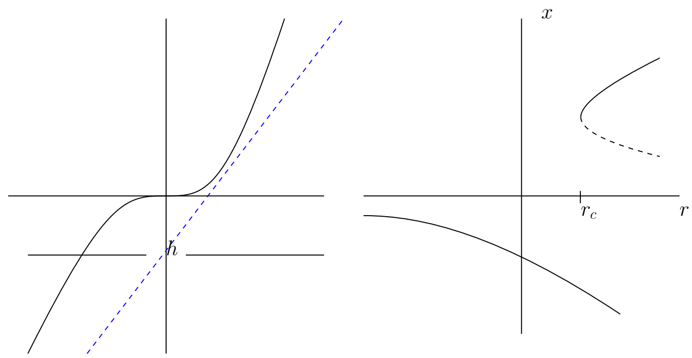


Figure 3: Bifurcation Diagram

Note:We have saddle node bifurcation at $r_c = r(h)$

Bifurcation Curves

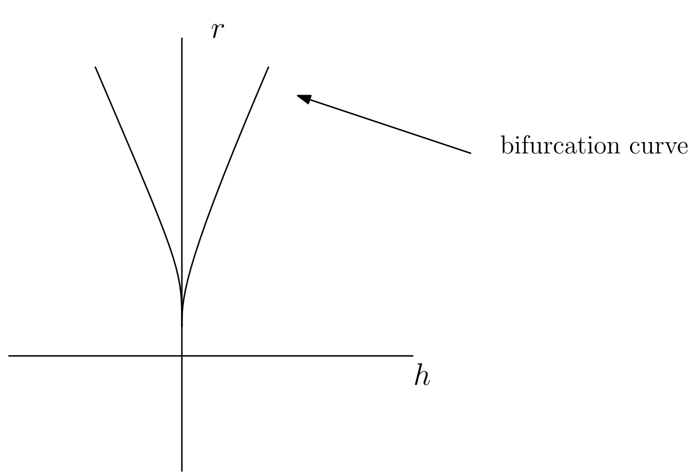
$$\left\{(h,r)|(h,r,x) \text{ solves } f=0, \frac{\partial f}{\partial x}=0\right\}$$

in our example $\dot{x} = h + rx - x^3$

$$\begin{aligned} 0 = \frac{\partial f}{\partial x} = r - 3x^2 &\implies x = \pm\sqrt{\frac{r}{3}} \\ 0 = f = h + rx - x^3 &\implies h = x^3 - rx \\ &\implies h = x^3 - rx = \pm\frac{2\sqrt{3}}{9}r^3 \\ h = h_c(r) &= \pm\frac{2\sqrt{3}}{9}r^{\frac{3}{2}} \\ &\implies r = r_c(h) = \left(\frac{9}{2\sqrt{3}}|h|\right)^{\frac{2}{3}} \end{aligned}$$

Stability Diagram:

Plot the bifurcation curves in the parameters space (= (h,r) plane).



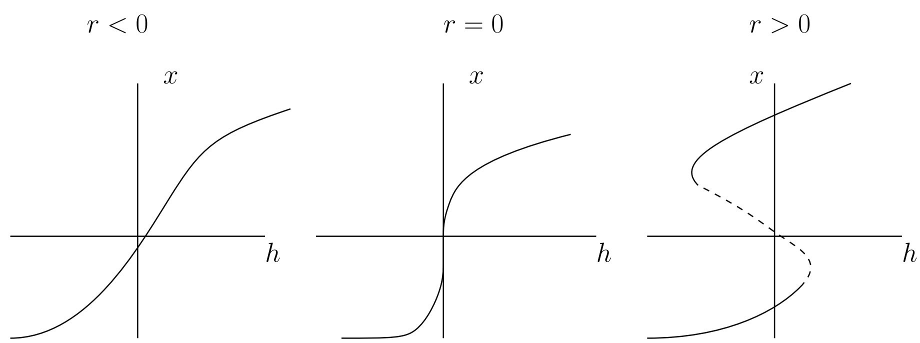
Note: qualitative behavior of ode changes as (h, r) cross bifurcation curve.
In example:

- "below" bifurcation curve: ODE has one (stable) fixed point.
- "on" bifurcation curve: two fixed points.
- "above" bifurcation curve: three fixed points.

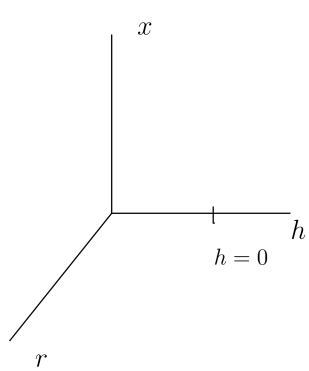
Remark 11.1.
- Saddle-node bifurcation occurs along bifurcation curve for $(h, r) \neq (0, 0)$
- At $(h, r) = (0, 0)$, the branches $r_c(h) = \left(\frac{9}{2\sqrt{3}}|h|\right)^{\frac{2}{3}}$ for $h > 0$ and $h < 0$ meet tangentially, and we have a cusp point at $(h, r) = (0, 0)$. This is an example of a codimension 2 bifurcation (i.e., we need two parameters to model this type of bifurcation).

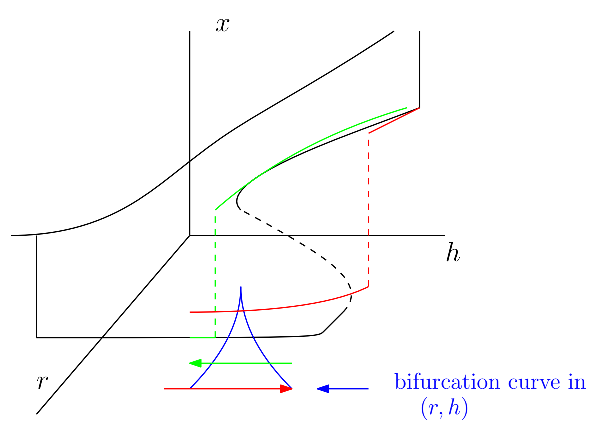
Bifurcation diagrams for fixed $r \in \mathbb{R}$.

$$\dot{x} = h + rx - x^3 = 0 \iff h = x^3 - rx$$



3D plot$(h, r$, fixed points $x)$

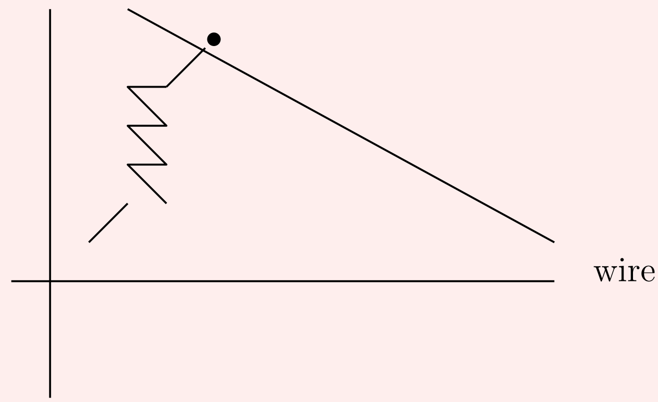




Picture/surface of cusp catastrophe solutions close to "upper" stable fixed points drop to "lower" stable fixed points as (r, h) vary (and vice versa).

Example 11.2 (practical)

Details in the book, page 74

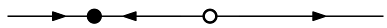


§12 Midterm 1: Feb 1, 2021

§13 Lec 12: Feb 3, 2021

§13.1 Flows on the Circle

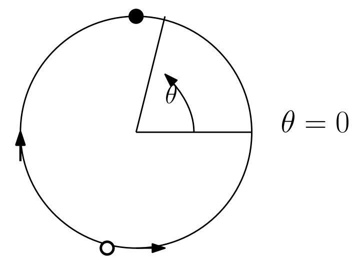
real line R



$\dot{x} = f(x)$

x = position

circle (with radius 1)



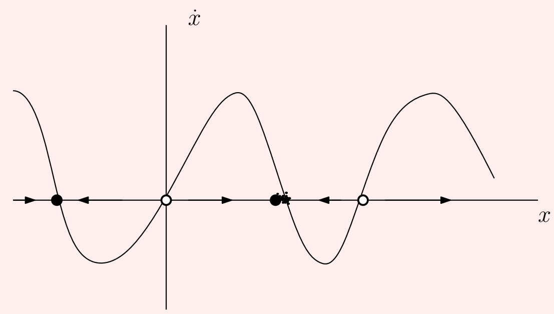
$\theta =$ angle

$\dot{\theta} = f(\theta)$

Example 13.1 i) $\dot{x} = \sin(x)$. Fixed points: $\dot{x} = 0$

$$\iff x = \dots, -\pi, 0, \pi, 2\pi, \dots$$

i.e., $x = k\pi, k \in \mathbb{Z}$.

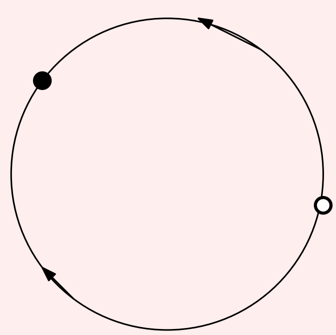


$$\dot{\theta} = \sin\theta$$
$$\dot{\theta} = 0$$
$$\iff \theta = 0 \text{ or } \theta = \pi$$
$$\underbrace{\theta = 2\pi}_{\text{same position on circle}}$$

i.e., θ is defined up to multiples of 2π.

Note: If $f(\theta) > 0$: flow is counterclockwise, and if $f(\theta) < 0$: flow is clockwise.

$$\dot{\theta} = \sin(\theta)$$



ii) $\dot{x} = x$ where $f(x) = x$ is not periodic.

Thus $\dot{\theta} = \theta$ does not work, because $\theta = 0, \theta = 2\pi$ describe the same position on the circle but $f(\theta) = \theta$ yields different values at $\theta = 0, 2\pi$, i.e. $f(\theta)$ is not a vector field on the circle.

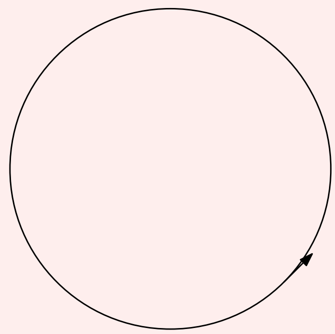
Correspondence:

$f(x)$ is 2π-periodic, i.e. $f(x + 2\pi) = f(x)$, and f is continuously differentiable $\iff f(\theta)$ defines a vector field on the circle.

Example 13.2 iii) $\dot{x} = c > 0$

$$x(t) = ct + x_0$$

$\dot{\theta} = \omega > 0$ – uniform oscillator



$\theta(t) = \omega t + \theta_0$

Period T:

$$\theta(T) = \theta(0) + 2\pi$$
$$\omega T + \theta_0 = \theta_0 + 2\pi$$
$$T = \frac{2\pi}{\omega}$$

In particular, periodic solutions are possible.

Example 13.3

Two runners are on a circular track, running in the same direction, with constant speed:

- Runner 1: period $T_1 = \frac{2\pi}{\omega_1}$, angle θ_1
- Runner 2: period $T_2 = \frac{2\pi}{\omega_2}$, angle θ_2

Runner 1, 2 start at the same position. Suppose $T_1 < T_2$, i.e. Runner 1 is faster than runner 2.

Question 13.1. How long does it take runner 1 to lap runner 2?

Ans: T_{lap} = time when phase difference

$$\phi = \theta_1 - \theta_2 \text{ is } 2\pi$$
$$\dot{\phi} = \dot{\theta}_1 - \dot{\theta}_2 = \omega_1 - \omega_2,\ \phi(0) = 0$$
$$\implies \phi(t) = (\omega_1 - \omega_2)t$$
$$\implies T_{\text{lap}} = \frac{2\pi}{\omega_1 - \omega_2} = \frac{1}{\frac{1}{T_1} - \frac{1}{T_2}} = \left(\frac{1}{T_1} - \frac{1}{T_2}\right)^{-1}$$

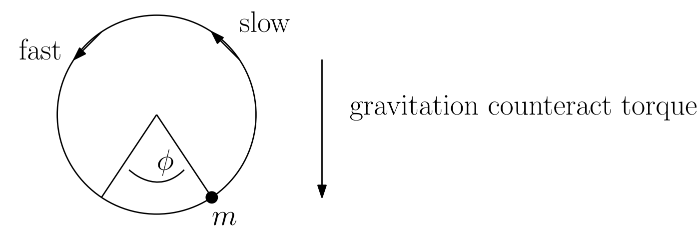
i.e. Runner 1,2 are in phase after T_{lap} again. This is called beat phenomenon.

§14 Lec 13: Feb 5, 2021

§14.1 Non-uniform Oscillator

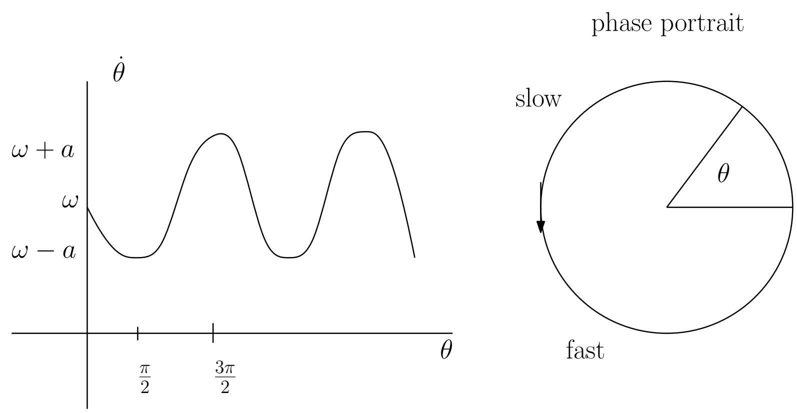
$$\dot{\theta} = \omega - a\sin\theta, \qquad \omega > 0,\ a > 0$$

Practical example: overdamped limit of pendulum driven by constant torque.



$$\dot{\phi} = \omega - a\sin\phi$$

Consider: $\dot{\theta} = \omega - a\sin\theta$
For $0 < a < \omega$:



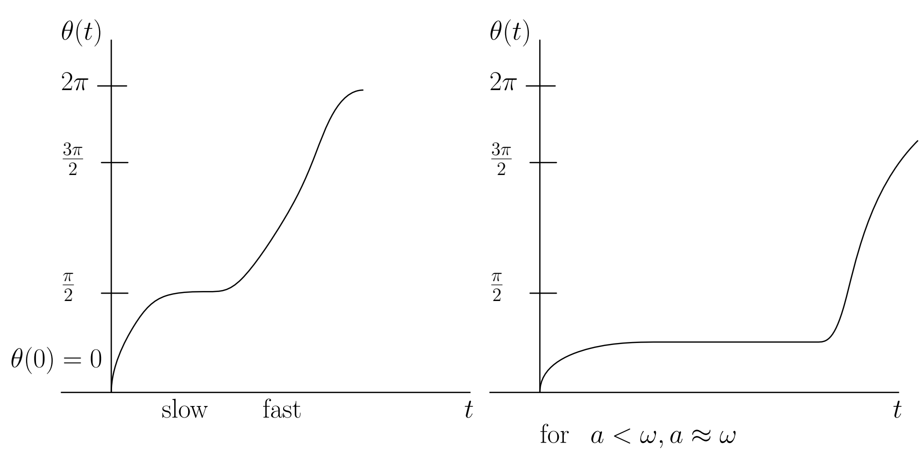
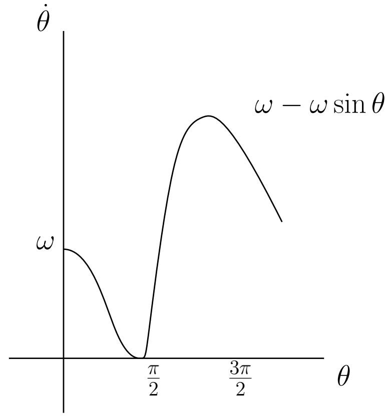
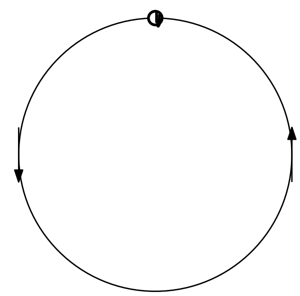


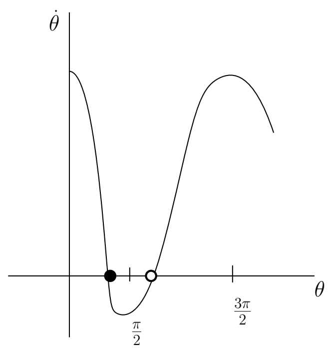
Figure 4: bottle neck remnants or "ghost" of a saddle-node bifurcation

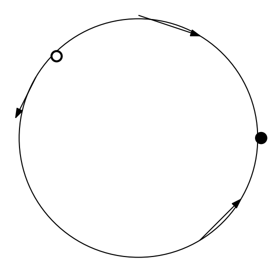
For $a = \omega$





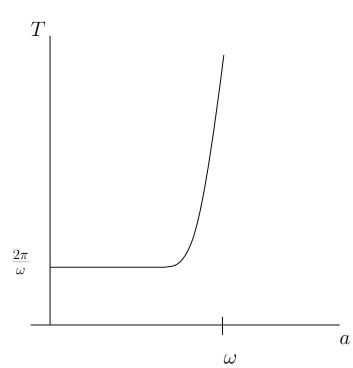
For $a > \omega$:





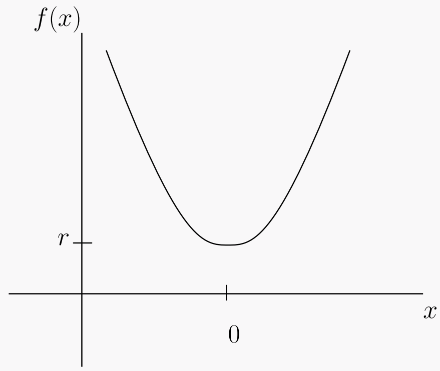
Oscillation period for $a < \omega$:

$$
\begin{aligned}
T &= \int dt = \int_0^{2\pi} \frac{dt}{d\theta} d\theta = \int_0^{2\pi} \frac{d\theta}{\omega - a\sin\theta} \\
&= \ldots = \frac{2\pi}{\sqrt{\omega^2 - a^2}} = \frac{2\pi}{\sqrt{\omega + a}} \cdot \frac{1}{\sqrt{\omega - a}} \\
&\approx \frac{2\pi}{\sqrt{2\omega}} \cdot \underbrace{\frac{1}{\sqrt{\omega - a}}}_{\text{blow up as } a \to \omega}
\end{aligned}
$$



Remark 14.1. Bottlenecks/this scaling law are a general feature of saddle-node bifurcations:

$$\text{Normal form: } \frac{dx}{dt} = \dot{x} = r + x^2$$



$$\begin{aligned} T_{\text{bottleneck}} &\approx \int dt \\ &= \int_{-\infty}^{\infty} \frac{dt}{dx} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{r + x^2}\, dx \\ T_{\text{bottleneck}} &= \frac{\pi}{\sqrt{r}} \end{aligned}$$

blows up like $\sim r^{-\frac{1}{2}} = \frac{1}{\sqrt{r}}$ as $r \to 0$ and $r > 0$.

Example 14.2

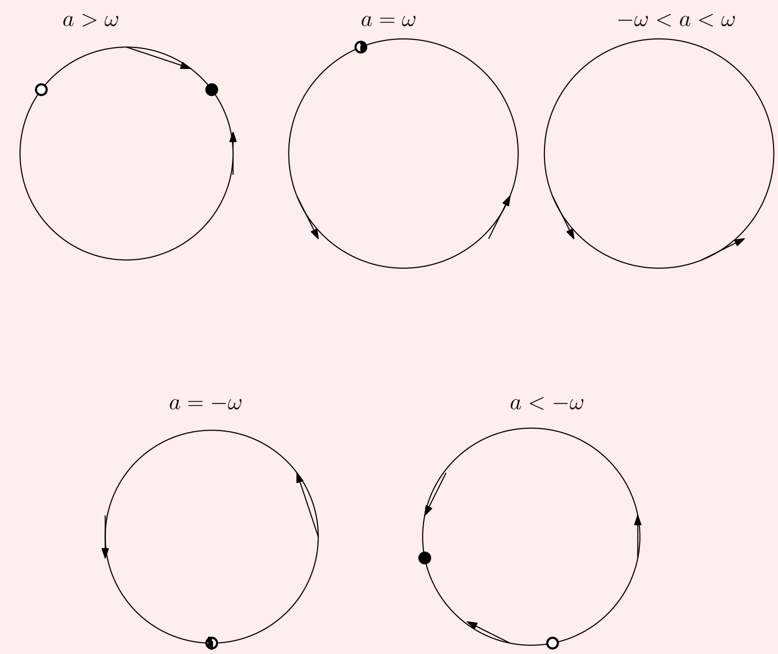
Draw all qualitatively different phase portraits of

$$\dot{\theta} = \omega - a\sin\theta \quad (\text{where } \omega > 0 \text{ fixed })$$

Bifurcation points: $\dot{\theta} = f(\theta) = 0$, $\frac{\partial f}{\partial \theta} = 0$. Thus, $0 = -a\cos\theta \implies a = 0$ or $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

$$\text{If } a = 0 : \dot{\theta} = \omega > 0 \text{ (no bifurcation)}$$
$$\text{If } \theta = \frac{\pi}{2} : 0 = \dot{\theta} = \omega - a \implies a = \omega$$
$$\text{If } \theta = \frac{3\pi}{2} : 0 = \dot{\theta} = \omega + a \implies a = -\omega$$

Bifurcation points $(a_*, \theta_*) = \left(\omega, \frac{\pi}{2}\right), \left(-\omega, \frac{3\pi}{2}\right)$.
$\dot{\theta} = \omega - a\sin\theta$



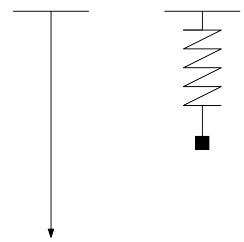
§14.2 2D Dynamical Systems

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}$$

<u>Introduction & Linear Systems</u>:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ i.e. } \dot{x} = Ax$$

<u>Harmonic Oscillator</u>: $m\ddot{x} + kx = 0$

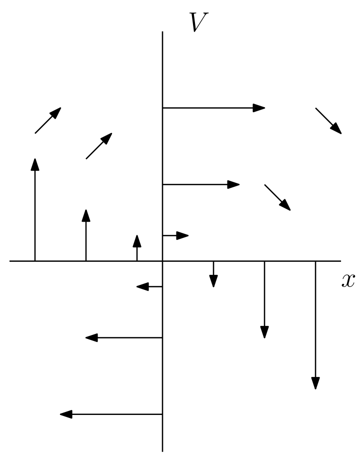


$$\ddot{x} + \omega^2 x = 0,\ \omega^2 = \frac{k}{m}$$

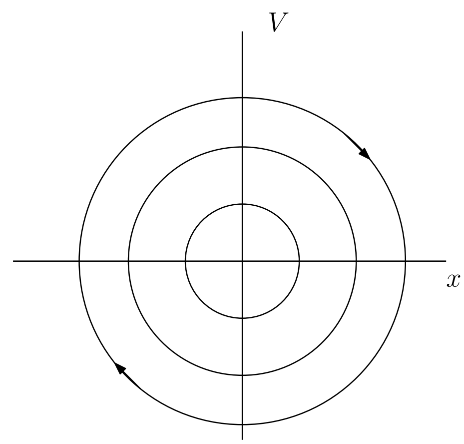
where k : spring constant and m : mass, x : position, v : velocity.

$$\begin{aligned}\dot{x} &= v \\ \dot{v} &= \ddot{x} = -\omega^2 x \\ \frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\omega^2 x \end{pmatrix}\end{aligned}$$

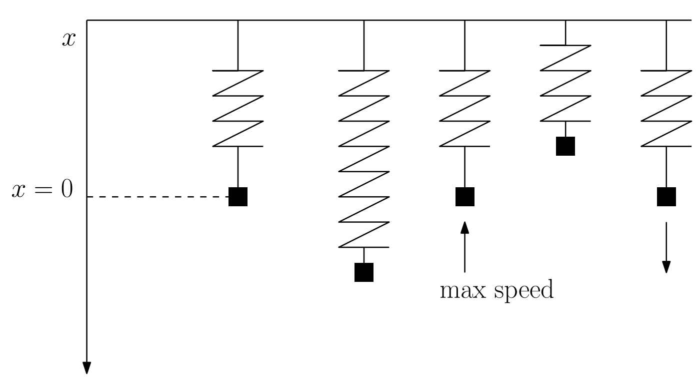
Note: the last matrix defines vector field on phase plane.



phase portrait



Harmonic oscillator:



Remark 14.3. Have:

$$\begin{aligned}\frac{d}{dt}(\omega^2 x^2 + v^2) &= 2\omega^2 x\dot{x} + 2v\dot{v} \\ &= 2\omega^2 xv - 2\omega^2 vx = 0 \\ &\implies \omega^2 x^2 + v^2 = \text{ const}\end{aligned}$$

$\implies$ trajectories $\begin{pmatrix} x(t) \\ v(t) \end{pmatrix}$ describe ellipses, in particular, they are closed orbits i.e. correspond to periodic solutions.

§15 Lec 14: Feb 8, 2021

§15.1 Classification of Linear Systems

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{i.e. } \dot{x} = Ax$$

Question 15.1. What is the stability type of $x_* = 0$?

Definition 15.1 (Eigenvector) — $v \neq 0$ is an eigenvector of A if

$$Av = \lambda v$$

for some $\lambda \in \mathbb{C}$

$\lambda \in \mathbb{C}$ is an eigenvalue

$$\begin{aligned} \iff \Lambda_\lambda(A) &= \det(A - \lambda I) = 0 \\ &= \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \\ &= (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 - \operatorname{tr}(A)\lambda + \det(A) \\ &= 0 \\ \iff \lambda_{1,2} &= \frac{1}{2}\left(\operatorname{tr}(A) \pm \sqrt{\operatorname{tr}(A)^2 - 4\det(A)}\right) \end{aligned}$$

3 cases:

i) $\lambda_1 \neq \lambda_2$ real valued $\iff \operatorname{tr}(A)^2 > 4\det(A)$

ii) $\lambda_1 = \lambda_2$ real valued $\iff \operatorname{tr}(A)^2 = 4\det(A)$

iii) $\lambda_1 = \overline{\lambda_2}$ complex conjugate $\iff \operatorname{tr}(A)^2 < 4\det(A)$

1. $\lambda_1 \neq \lambda_2 \implies$ there are linearly independent eigenvectors v_i :

   $$Av_i = \lambda_i v_i \quad \text{for } i = 1, 2$$

   A is diagonalizable.

   Coordinate change:

   $$\begin{aligned} C &= (v_1|v_2) \\ B &= C^{-1}AC = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \\ y &= C^{-1}x \end{aligned}$$

Then $\dot{y} = C^{-1}\dot{x} = C^{-1}Ax = C^{-1}ACy = By$ i.e. $\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 y_1 \\ \lambda_2 y_2 \end{pmatrix}$ i.e. the ODE decouples

$$\dot{y}_i = \lambda_i y_i \quad \text{for } i = 1, 2$$

So

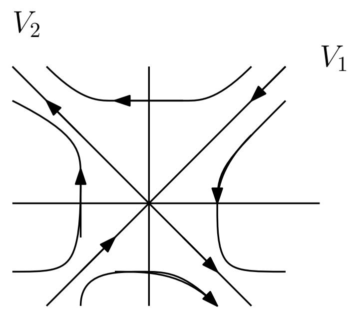
$$\begin{aligned} &\implies y(t) = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix} \\ &\implies x(t) = Cy(t) = c_1 e^{\lambda_1 t} C \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{\lambda_2 t} C \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

If $\lambda_1 \neq \lambda_2$:

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$

Phase portraits:

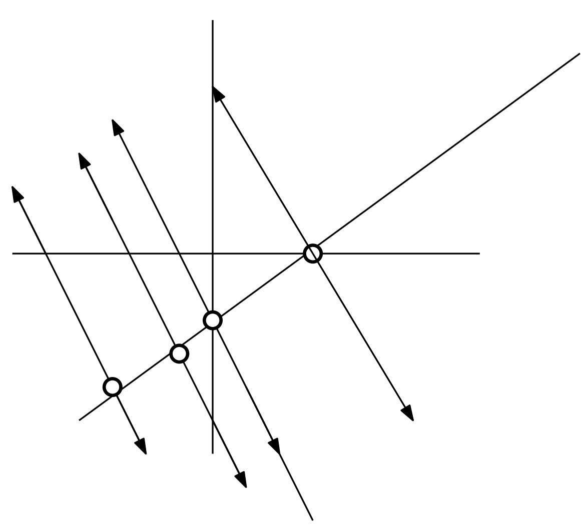
$$\lambda_1 < 0 < \lambda_2 \text{ (saddle)}$$



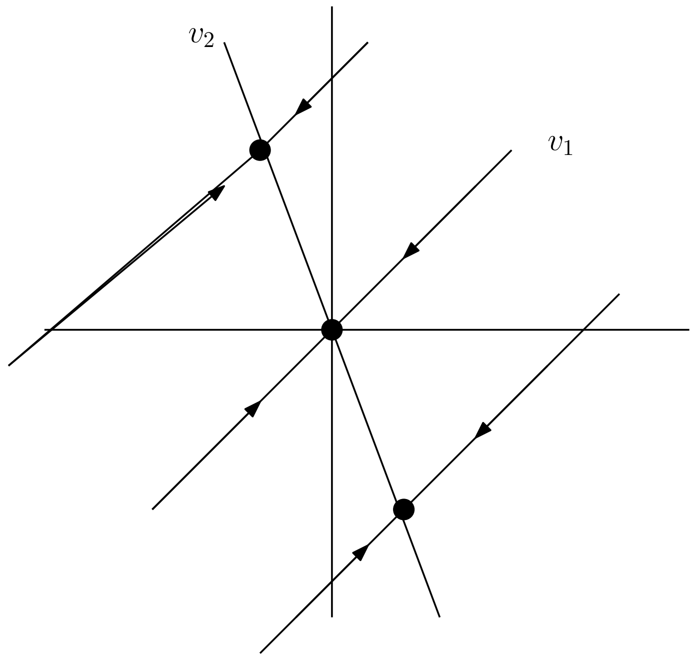
x_* is unstable

Definition 15.2 (Hyperbolic Fixed Point) — x_* is a hyperbolic fixed point if $\text{Re}(\lambda_i) \neq 0$ for $i = 1, 2$ otherwise non-hyperbolic.

$\lambda_1 = 0 < \lambda_2 : x(t) = c_1 v_1 + c_2 e^{\lambda_2 t} v_2$



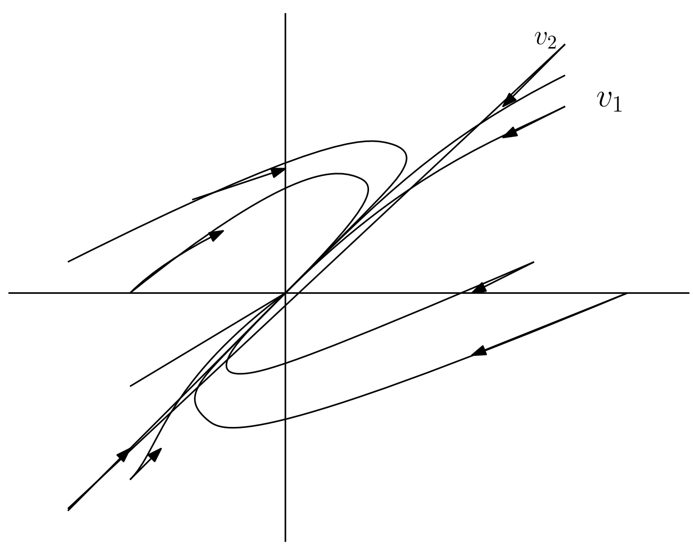
x_* is unstable and v_1 axis consists of fixed points $x_* = 0$ is a non-isolated fixed point.
$\lambda_1 < 0 = \lambda_2$: $x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 v_2$



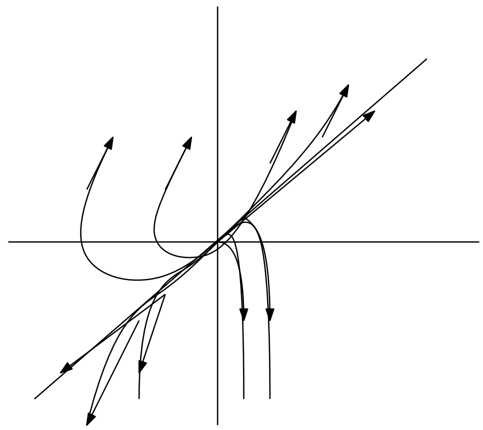
v_2 axis consists of fixed points.

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 v_2$$

$x_* = 0$ is Lypunov stable but not attracting (neutrally stable)
$\lambda_1 < \lambda_2 < 0 : \quad x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$



Trajectories approach x_* tangent to "slower" v_2 direction (note $|\lambda_1| > |\lambda_2| > 0$) – stable node.
$0 < \lambda_1 < \lambda_2$: trajectories quickly appear parallel to "faster" v_2 direction.



unstable node

Case ii) $\lambda = \lambda_1 = \lambda_2$, real valued

1. There are v_1, v_2 linearly independent eigenvectors $Av_i = \lambda v_i$ for $i = 1, 2$

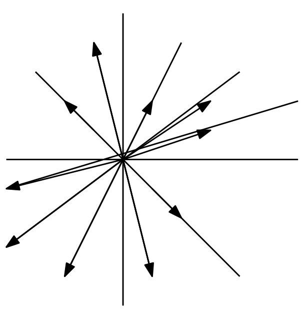
$$\implies \text{For } v \in \mathbb{R}^2 : Av = \lambda v \implies A = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \lambda I$$

So, $\dot{x} = Ax$ is solved by

$$x(t) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\lambda t}$$

Phase portraits:

$\lambda > 0$



x_* is unstable

unstable star

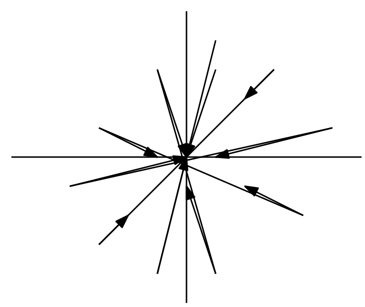
$\lambda = 0 : A = 0$

every point is a fixed point

x(t) = x(0)

($x_* = 0$
is stable non-hyperbolic,
non-isolated)

$\lambda < 0$



x_* is stable (stable star)

§16 Lec 15: Feb 10, 2021

§16.1 Classification (Cont'd)

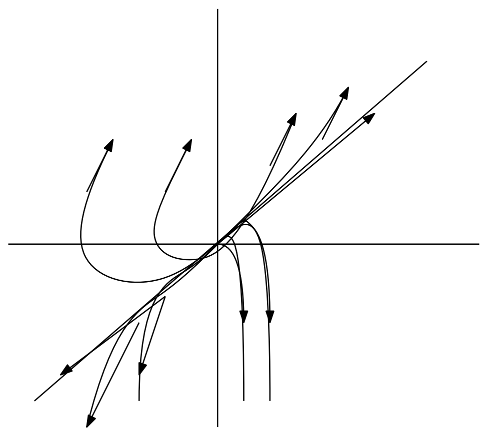
Case ii) $\lambda = \lambda_1 = \lambda_2$

2. Eigenspace $\text{Eig}_\lambda(A) = \text{span}(v)$, $v \neq 0$ A is not diagonalizable.

$$\implies x(t) = \left[(c_1 + c_2 t)v + c_2\omega\right] e^{\lambda t}$$

where λ s.t. $(A - \lambda I)\,\omega = v$. Note $\frac{x(t)}{|x(t)|} \to \frac{v}{|v|}$ as $t \to \pm\infty$ i.e. $x(t)$ tangent/parallel to v-direction as $t \to \pm\infty$.

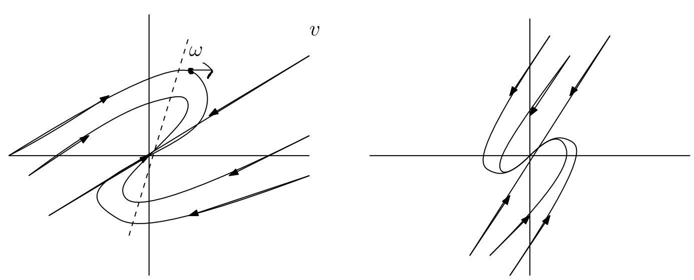
Recall: $\lambda_1 < \lambda_2 < 0$:



unstable node

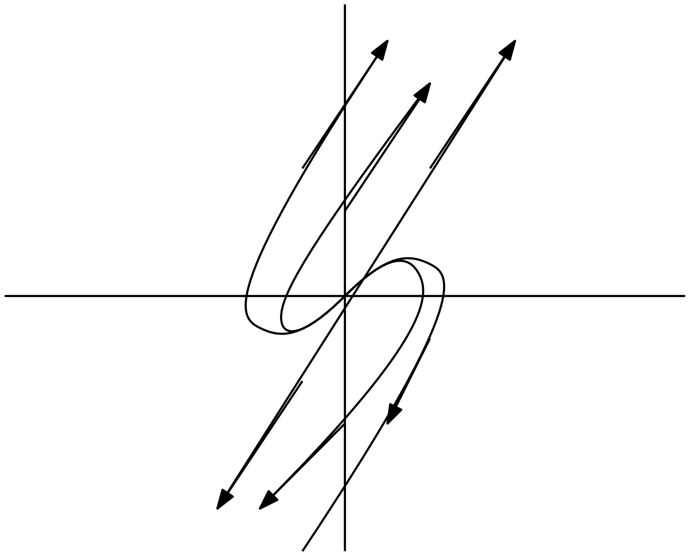
intuitively as $\lambda_1 \to \lambda_2$ and $v_1 \to v_2$.

$\lambda < 0$: stable degenerate node

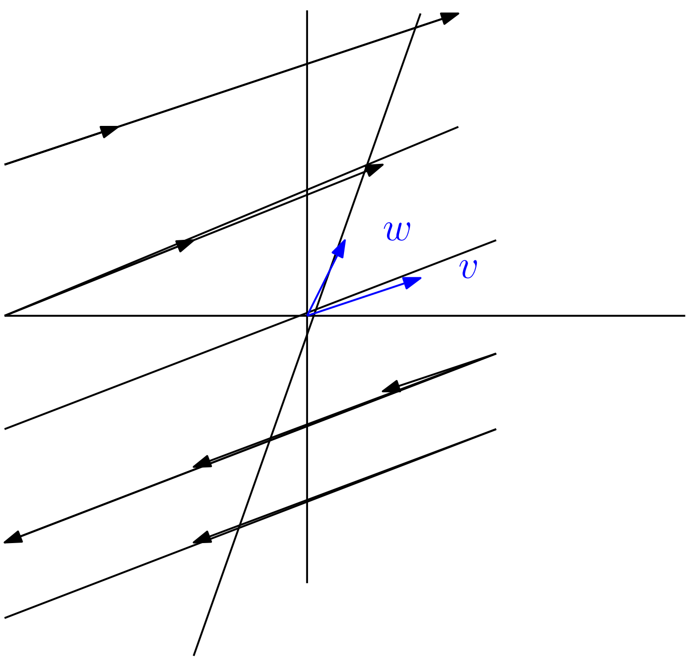


> **Remark 16.1.** Instead of solving for ω explicitly, calculate Az for some vector z to determine which way the solution "curls".

$\lambda > 0$



$\lambda = 0 : x(t) = (c_1 + c_2 t)v + c_2\omega$



Note: $x(0) = c_1 v \implies x(t) = c_1 v$ for all t i.e. the $v-$axis consists of fixed points (non-isolated fixed points, $x_* = 0$ unstable).

Remark 16.2. If $\lambda = \lambda_1 = \lambda_2$, $\text{Eig}_\lambda(A) = \text{span}(v)$. Then there is ω s.t.

$$\begin{aligned} &(A - \lambda I)\omega = v \\ &\implies v_1\omega \text{ lin. indep} \\ &\implies v_1\omega \text{ form a basis of } \mathbb{R}^2 \end{aligned}$$

Coordinate change:

Set

$$\begin{aligned} C &= (v|w) \\ B = C^{-1}AC &= \underbrace{\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}}_{\text{Jordan normal form}} \\ y = C^{-1}x: &\quad \dot{y} = By \end{aligned}$$

So

$$\begin{aligned} \dot{y}_2 = \lambda y_2 &\implies y_2(t) = c_2 e^{\lambda t} \\ \dot{y}_1 = \lambda y_1 + y_2 &\implies y_1(t) = (c_1 + c_2 t)\, e^{\lambda t} \\ \implies x = Cy &= [(c_1 + c_2 t) + c_2 \omega]\, e^{\lambda t} \end{aligned}$$

Case iii)

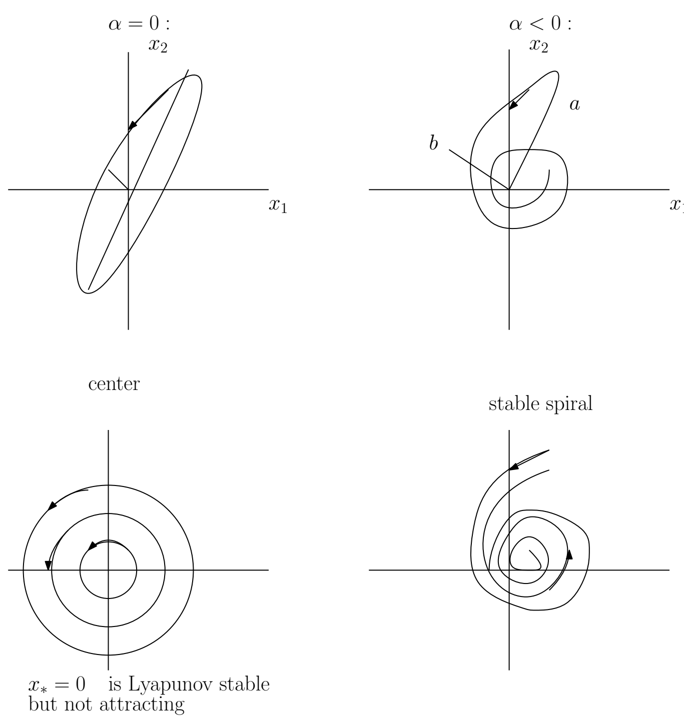
$$\begin{cases} \lambda_1 = \lambda = \alpha + i\beta \\ \lambda_2 = \overline{\lambda} = \alpha - i\beta \end{cases} \quad (\beta > 0)$$

$\implies$ A is diagonalizable over $\mathbb{C}$, in particular there is $v \in \mathbb{C}^2, v \neq 0$, s.t. $Av = \lambda v$. Let $v = a - ib$, $a, b \in \mathbb{R}^2$. Assume $a \perp b$. General solution:

$$x(t) = (a|b) \underbrace{\begin{pmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{pmatrix}}_{\text{rotation} R(\beta t) \text{ period } \frac{2\pi}{\beta}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \underbrace{e^{\lambda t}}_{\text{stretching factor}}$$

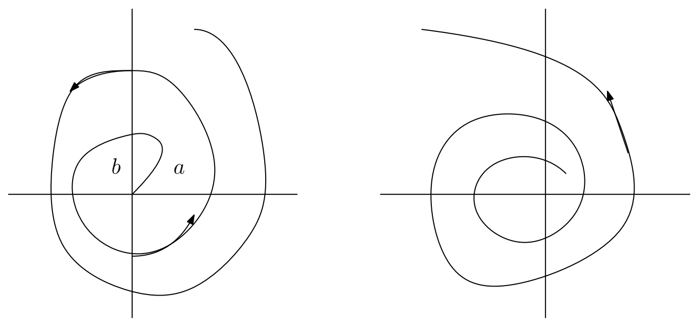
In particular, $x(t) = [a\cos(\beta t) + b\sin(\beta t)]\, e^{\lambda t}$ is the solution with $x(0) = a$ and $x\left(\frac{\pi}{2\beta} = be^{\alpha t}\right)$ $\left[\text{set } \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$.

Phase portraits:



$\alpha > 0$: unstable spiral

$\alpha > 0$: unstable spiral



Remark 16.3. i) If $\alpha = 0$, $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies x(t) = \cos(\beta t) \cdot a + \sin(\beta t) \cdot b$. Then since $a \perp b$:

$$\frac{1}{|a|^2} \langle x(t), \frac{a}{|a|} \rangle^2 + \frac{1}{|b|^2} \langle x(t), \frac{b}{|b|} \rangle^2 = \frac{1}{|a|^2} \left(\frac{a \cdot a}{|a|} \cdot \cos(\beta t) \right)^2 + \frac{1}{|b|^2} \left(\frac{b \cdot b}{|b|} \cdot \sin(\beta t) \right)^2$$
$$= (\cos(\beta t))^2 + (\sin(\beta t))^2 = 1$$

$\implies x(t)$ is on an ellipse with axes $\frac{a}{|a|}, \frac{b}{|b|}$.

ii) $\lambda = \alpha + i\beta$, $v = a - ib$. If a is not orthogonal to b, then replace v by

$$w = (\gamma + i\delta)\, v$$

with $\gamma = -2ab$

$$\delta = \left(|a|^2 - |b|^2\right) \pm \sqrt{\left(|a|^2 - |b|^2\right)^2 + 4(ab)^2}$$

Then $A\omega = \lambda\omega$ and Re $\omega \perp$ Im ω.
Assume $Av = \lambda v$, $v = a - ib$, $a \perp b$.

$$Aa - iAb = A(a - ib) = Av = \lambda v = (\alpha + i\beta)(a - ib)$$
$$= (\alpha a + \beta b) + i(\beta a - \alpha b)$$

So

$$Aa = \alpha a + \beta b$$
$$Ab = -\beta a + \alpha b$$

Set $C = (a|b)$. Then

$$AC = C \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$
$$B = C^{-1}AC = \underbrace{\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}}_{\text{normal form}}$$

Set $y = C^{-1}x$, $\dot{y} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} y$ with solution:

$$y(t) = \begin{pmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\alpha t}$$
$$\implies x(t) = C \cdot y(t)$$

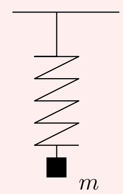
§17 Lec 16: Feb 12, 2021

§17.1 Linear Systems – Harmonic Oscillator

Example 17.1 (Harmoinc oscillator)

$$m\ddot{x} + kx = 0$$

where k : spring constant.



$\implies \ddot{x} + \omega^2 x = 0$ where $\omega^2 = \frac{k}{m}$. Set

$$\begin{cases} x_1 = x \\ x_2 = \dot{x} \end{cases} \implies \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\omega^2 x_1 \end{cases}$$

i.e.

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

eigenvalues:

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \det\begin{pmatrix} -\lambda & 1 \\ -\omega^2 & -\lambda \end{pmatrix} \\ &= \lambda^2 + \omega^2 \end{aligned}$$

$\implies \lambda_{1,2} = \pm i\omega \implies$ center
Phase portrait:

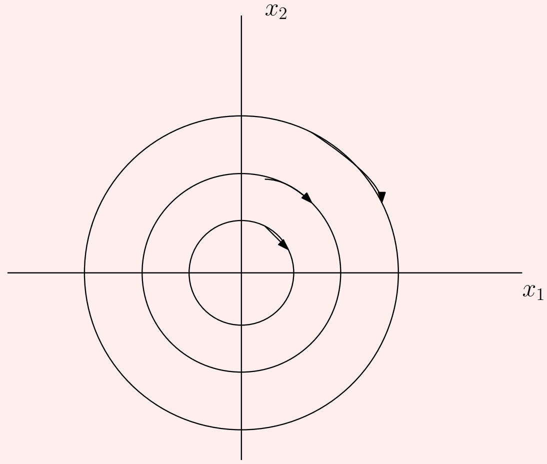
i) in practice: compute $\dot{x} = Ax$ for a specific vector to determine which way solutions turn

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} x$$

e.g. $\begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\omega^2 \end{pmatrix}$.

Example 17.2 (Cont'd of example 17.1)

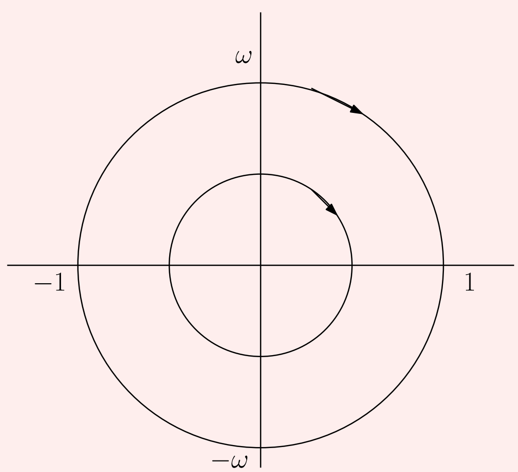
Then,



ii) more precise quantitative analysis, eigenvectors solutions of $(A - \lambda I)v = 0$

$$A - i\omega I = \begin{pmatrix} -i\omega & 1 \\ -\omega^2 & -i\omega \end{pmatrix} \to \begin{pmatrix} -i\omega & 1 \\ 0 & 0 \end{pmatrix}$$

eigenvector $v = \begin{pmatrix} -i \\ \omega \end{pmatrix} = \begin{pmatrix} 0 \\ \omega \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} i$



Recall:

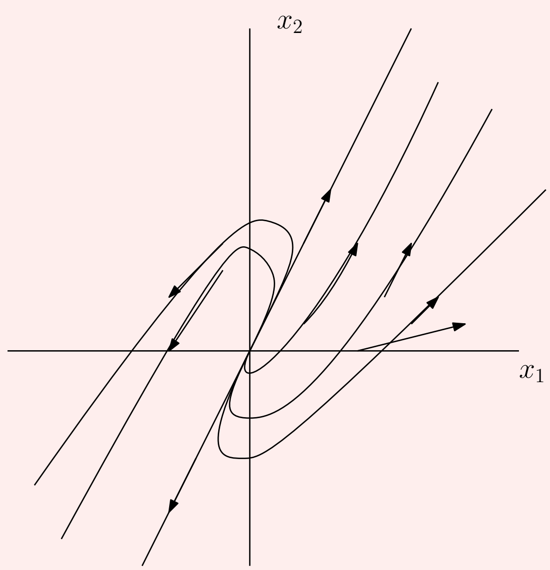
$$x(t) = C \cdot \left[\begin{pmatrix} 0 \\ \omega \end{pmatrix} \cos(\omega t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(\omega t) \right]$$

Example 17.3

$\dot{x} = Ax \quad A = \begin{pmatrix} 8 & -1 \\ 4 & 4 \end{pmatrix}$. Eigenvalues:

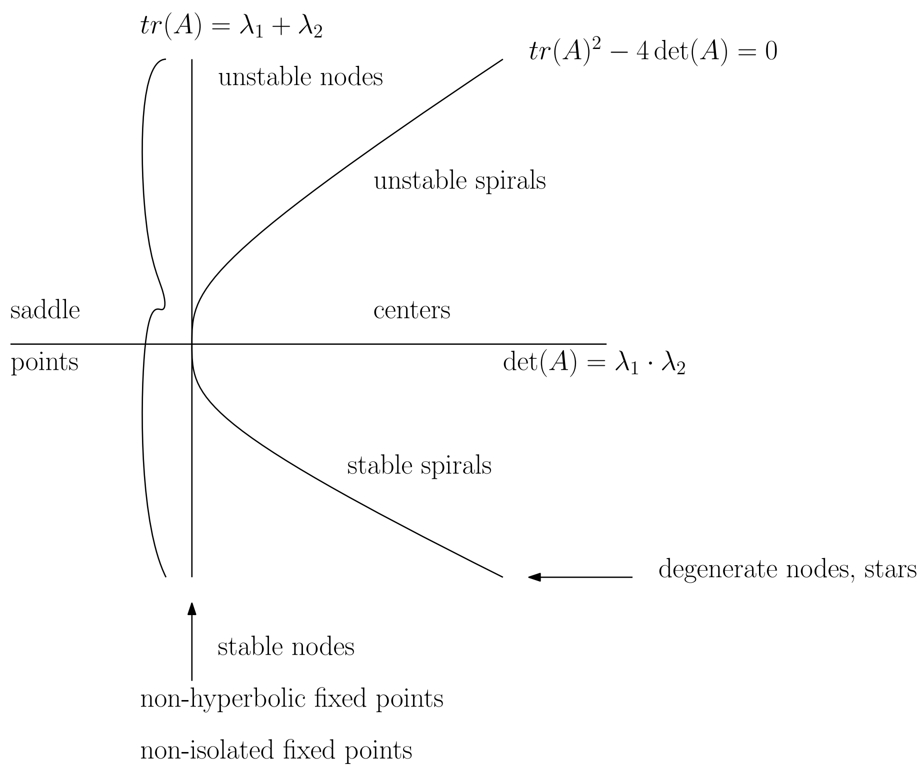
$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \det \begin{pmatrix} 8-\lambda & -1 \\ 4 & 4-\lambda \end{pmatrix} \\ &= (8-\lambda)(4-\lambda) - 4(-1) \\ &= \lambda^2 - 12\lambda + 36 = 0 \\ \implies \lambda &= 6 \end{aligned}$$

$A \neq \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$, we have an unstable degenerate node. Eigenvector: $A - \lambda I = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \to \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}$, so $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector. Note $A \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$.
Phase portrait:



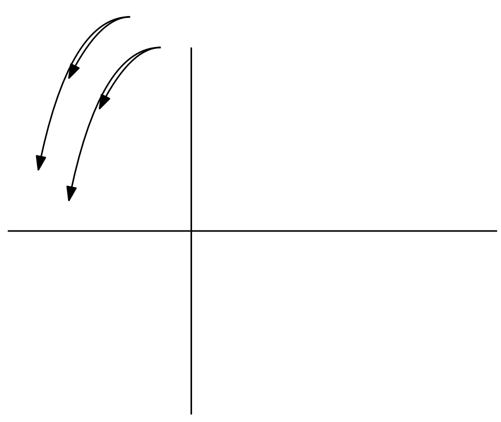
Summary:

Recall $\lambda_{1,2} = \frac{1}{2}\left(\operatorname{tr}(A) \pm \sqrt{\operatorname{tr}(A)^2 - 4\det(A)}\right)$



§17.2 Nonlinear Systems – Existence and Uniqueness

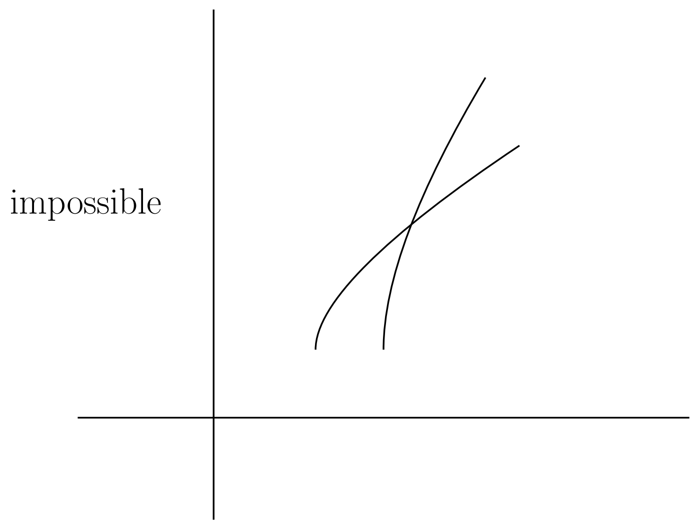
$$\dot{x} = f(x) \qquad \text{i.e.} \qquad \begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned}$$



Theorem 17.4 (Existence & Uniqueness of Systems)
Let $D \subseteq \mathbb{R}^n$ be open, $f : D \to \mathbb{R}^n$ s.t. $\frac{\partial f_i}{\partial x_j}$ exist and are continuous, that is $f \in C^1(D)$. Then for every $x_0 \in D$ there $\tau > 0$ s.t. $\dot{x} = f(x)$, $x(t_0) = x_0$ has a unique solution $\phi : (t_0 - \tau, t_0 + \tau) \to \mathbb{R}^n$ i.e. $\dot{\phi}(t) = f(\phi(t))$, $\phi(t_0) = x_0$.

> **Remark 17.5.** $f \in C^2(D)$ if $\frac{\partial^2 f_i}{\partial x_k \partial x_l}$ exist and continuous.

Consequence: Different trajectories in the phase portrait cannot intersect



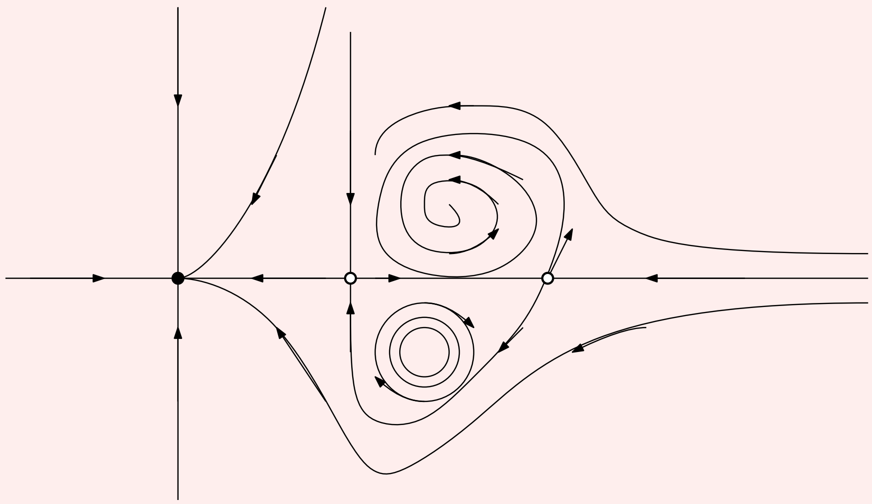
§18 Lec 17: Feb 15, 2021

§18.1 Nonlinear Systems – Nullclines

$\dot{x} = f(x)$ and $\dot{x}_1 = f_1(x_1, x_2)$, $\dot{x}_2 = f_2(x_1, x_2)$

Example 18.1

Consider



Definition 18.2 (Isocline and Nullcline) — Let $c \in \mathbb{R}$. The curves $\{(x_1, x_2) | f_i(x_1, x_2) = c\}$ $i = 1, 2$ are called isoclines. Specifically, if $c = 0$

- $f_1(x_1, x_2) = 0$ is called vertical nullcline.
- $f_2(x_1, x_2) = 0$ is called horizontal nullcline.

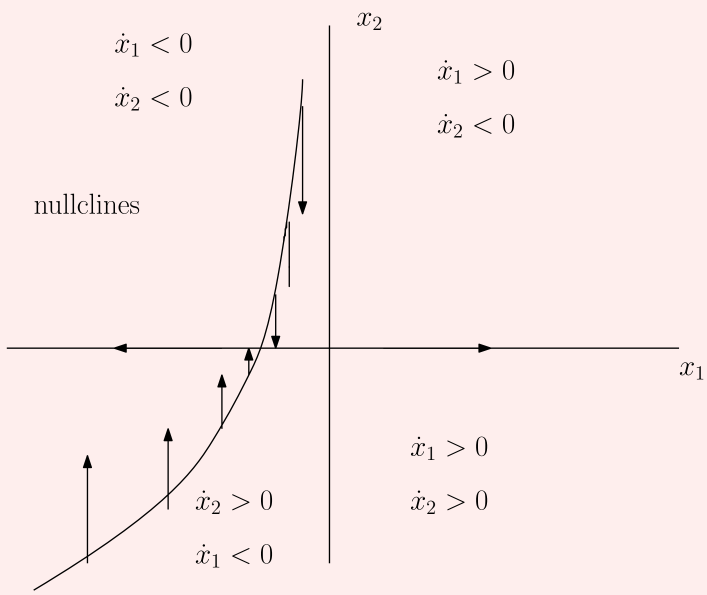
Example 18.3

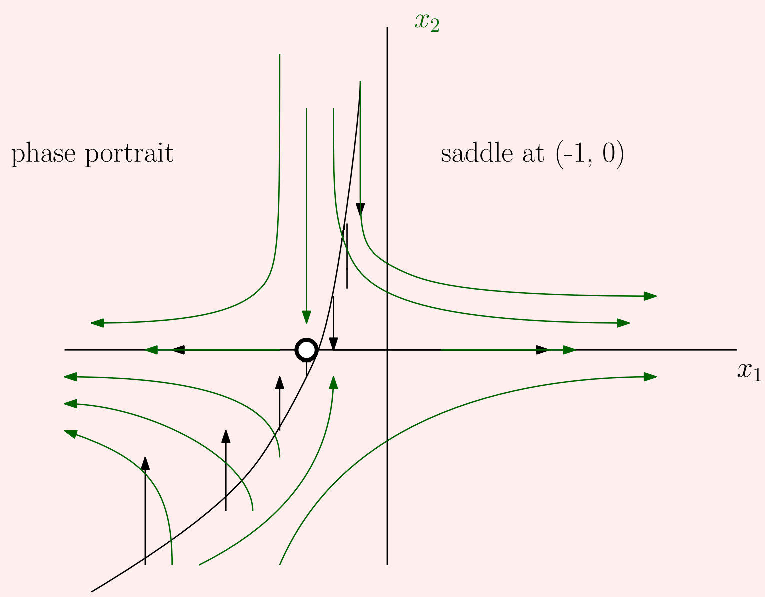
Consider:

$$\dot{x}_1 = x_1 + e^{-x_2}$$
$$\dot{x}_2 = -x_2$$

Fixed points: $\dot{x} = f(x) = 0 \iff (x_1, x_2) = (-1, 0)$.
Nullclines:

$$\dot{x}_1 = 0 : \; x_1 = -e^{-x_2} \text{ (vertical nullcline)}$$
$$\dot{x}_2 = 0 : \; x_2 = 0 \text{ (horizontal nullcline)}$$





Remark 18.4. A nullclines typically are not/do not consist of trajectories. Vertical(horizontal) nullclines consist of trajectories if it is exactly vertical(horizontal.

§18.2 Principle of Linear Stability

$\dot{x} = f(x)$, $f \in C^1(D)$, $f(x_*) = 0$. We want to approximate the nonlinear DE near the fixed point.

$$\begin{aligned}\frac{d}{dt}(x - x_*) &= \dot{x} = f(x) = f(x - x_* + x_*) \\ &\overset{\text{Taylor}}{=} \underbrace{f(x_*)}_{=0} + Df(x_*)(x - x_*) + \mathcal{O}(|x - x_*|^2)\end{aligned}$$

i.e. $y = x - x_*$ approximately solves the linear ODE

$$\dot{y} = Df(x_*)y$$

where

$$Df(x_*)\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}$$

Let λ_1, λ_2 denote the eigenvalues of $Df(x_*)$.

Theorem 18.5 (Linear Stability)

Similar to the linear systems,

i) If $\text{Re}(\lambda_1) < 0$, $\text{Re}(\lambda_2) < 0$ then x_* is asymptotically stable, i.e. x_* is Lyapunov stable and attracting.

ii) If $\text{Re}(\lambda_i) > 0$ for $i = 1$ or $i = 2$ then x_* is unstable.

§19 Lec 18: Feb 19, 2021

§19.1 The Stable/Unstable Manifold Theorem

$f \in C^1$, $\dot{x} = f(x)$, $f(x_*) = 0$ i.e. x_* fixed point, λ_1, λ_2 eigenvalues of $Df(x_*)$.
Let x_* be a hyperbolic fixed point and $x(t, x_0)$ be the solution of

$$\dot{x} = f(x), \quad x(0) = x_0$$

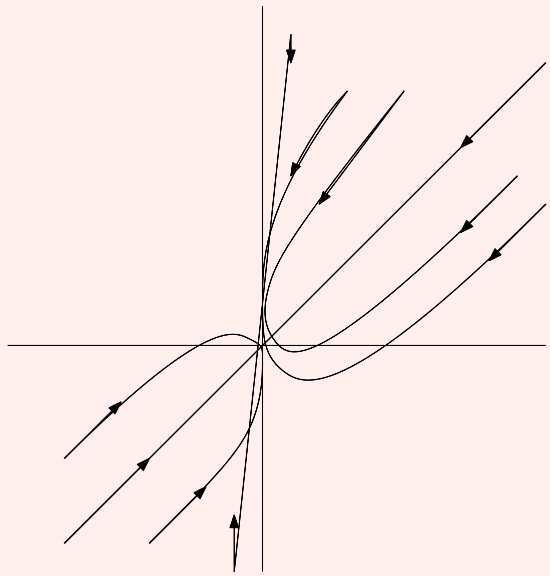
Set

$$\mathcal{M}_s \coloneqq \left\{ x_0 \in D | x(t, x_0) \text{ defined for all } t \geq 0 \text{ and } \lim_{t \to \infty} x(t, x_0) = x_* \right\} \text{ (stable manifold)}$$

$$\mathcal{M}_u \coloneqq \left\{ x_0 \in D | x(t, x_0) \text{ defined for all } t \leq 0 \text{ and } \lim_{t \to -\infty} x(t, x_0) = x_* \right\} \text{ (unstable manifold)}$$

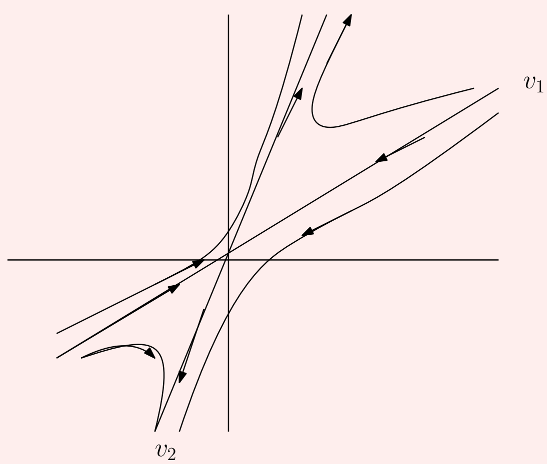
Example 19.1

Linear stable node



$$\mathcal{M}_s = \mathbb{R}^2$$
$$\mathcal{M}_u = \{x_*\} = \{0\}$$

Linear saddle



$$\begin{aligned}\mathcal{M}_s &= \text{span}(v_1) \\ &= \text{ line through } v_1(\lambda_1 < 0) \ \ \text{(trajectories that approach } x_*) \\ \mathcal{M}_u &= \text{ span}(v_2) \\ &= \text{line through } v_2(\lambda_2 > 0) \ \ \text{(trajectories that emanate from } x_*)\end{aligned}$$

Theorem 19.2 (Stable/Unstable Manifold)
Let $f \in C^1, x_*$ is a hyperbolic fixed point.

i) If $\text{Re}(\lambda_i) < 0$ for $i = 1, 2$, then $\mathcal{M}_s$ contains an open neighborhood of x_* and $\mathcal{M}_u = \{x_*\}$.

ii) If $\text{Re}(\lambda_i) > 0$ for $i = 1, 2$, then $\mathcal{M}_s = \{x_*\}$ and $\mathcal{M}_u$ contains an open neighborhood of x_*.

iii) If $\text{Re}(\lambda_1) < 0 < \text{Re}(\lambda_2)$, then $\mathcal{M}_s, \mathcal{M}_u$ are C^1-curves through x_*. $\mathcal{M}_s$ tangent to v_1 at x_*, $Df(x_*)v_1 = \lambda_1 v_1$, and $\mathcal{M}_u$ tangent to v_2 at x_*, $Df(x_*)v_2 = \lambda_2 v_2$

Theorem 19.3
Suppose x_* is a hyperbolic fixed points of $\dot{x} = f(x)$. Then the phase portrait of $\dot{y} = Df(x_*)y$ near $y_* = 0$ gives a qualitatively accurate picture of the phase portrait of $\dot{x} = f(x)$ near x_* if

a) $f \in C^2$ i.e. $\frac{\partial^2 f}{\partial x_i \partial x_j}$ exists and are continuous.

or

b) $f \in C^1$ and $\lambda_1 \neq \lambda_2$.

Example 19.4

Consider:

$$\begin{aligned}\dot{x}_1 &= x_1 + e^{-x_2}\\ \dot{x}_2 &= -x_2\end{aligned}$$

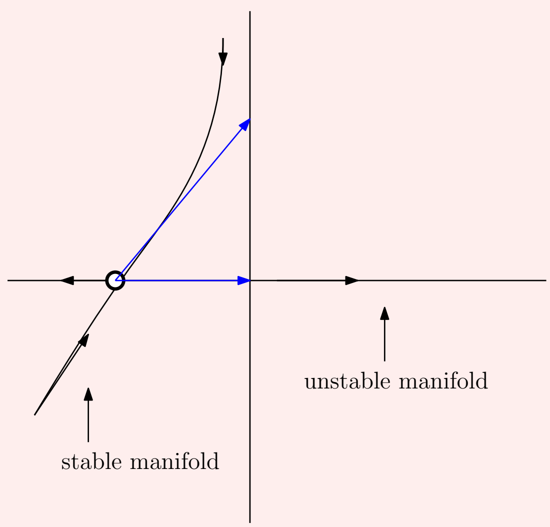
only fixed point: $(x_1, x_2) = (-1, 0)$ and note that $f(x_1, x_2) = \begin{pmatrix} x_1 + e^{-x_2} \\ -x_2 \end{pmatrix}$.

$$\begin{aligned}Df &= \begin{pmatrix} 1 & -e^{-x_2} \\ 0 & -1 \end{pmatrix}\\ Df(x_*) &= \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}\end{aligned}$$

Eigenvalues: $\lambda_1 = -1, \lambda_2 = 1 \implies (-1, 0)$ is unstable (by Theorem 18.5)
Eigenvectors:

$$\begin{aligned}A - (-1)I &= \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \implies v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\\ A - (1)I &= \begin{pmatrix} 0 & -1 \\ 0 & -2 \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \implies v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\end{aligned}$$

where v_1 is the tangent direction of stable manifold at $x_* = (-1, 0)$ and v_2 is the tangent direction of unstable manifold at $x_* = (-1, 0)$.

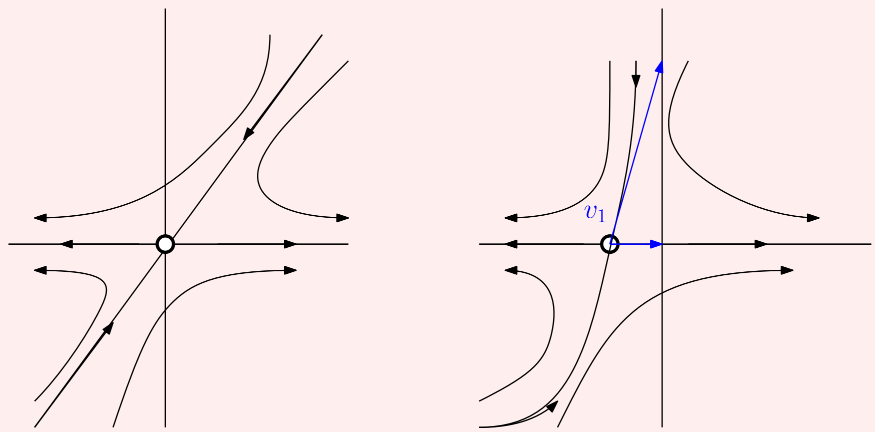


Example 19.5 (Cont'd from above)

<u>*Note*</u>: $f(x_1, x_2) = \begin{pmatrix} x_1 + e^{-x_2} \\ -x_2 \end{pmatrix}$ is infinitely often differentiable, in particular, $f \in C^2$ (or $f \in C^1$), thus the phase portrait of

$$\dot{y} = Df(x_*) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \text{ near } y_* = 0$$

is an accurate picture of the phase portrait of $\dot{x} = f(x)$ near x_*.



$\dot{x} = f(x)$ near $x_* = (-1, 0)$

where the left figure denote the approximation $\dot{y}$.

Theorem 19.6 (Hartman – Grobman)
Let $f \in C^1$, x_* a hyperbolic fixed point of $\dot{x} = f(x)$. Then the phase portrait of $\dot{x} = f(x)$ near x_* and $\dot{y} = Df(x_*)y$ near $y_* = 0$ are topologically equivalent i.e. the same up to continuous deformation (homeomorphisms).

Morally: hyperbolic fixed points are <u>structurally stable</u>.

§19.2 Lotka Volterra Model

Example 19.7 (Lotka Volterra model for competition of two species for limited resources)

Recall: logistic model

$$\dot{x} = rx\left(1 - \frac{x}{k}\right)$$

Consider:

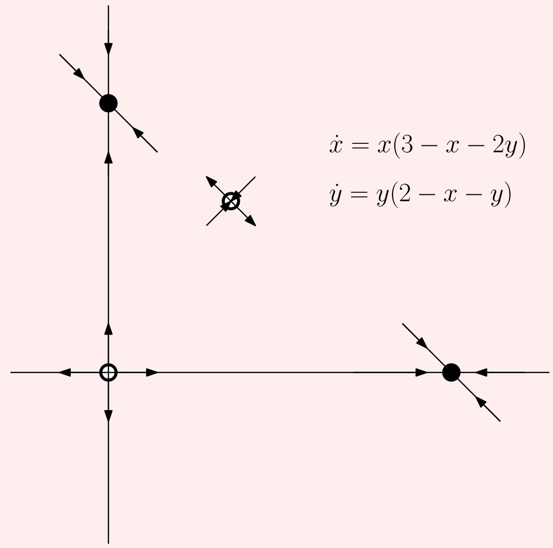
$$\dot{x} = x(3 - x - 2y)$$
$$\dot{y} = y(2 - x - y)$$

fixed points (x_*, y_*)	eigenvalues/eigendirections of $Df(x_*, y_*)$			
	λ_1	v_1	λ_2	v_2
$(0, 0)$	3	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	2	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$(0, 2)$	-1	$\begin{pmatrix} 1 \\ -2 \end{pmatrix}$	-2	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$(3, 0)$	-3	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	-1	$\begin{pmatrix} 3 \\ -1 \end{pmatrix}$
$(1, 1)$	$-1 + \sqrt{2}$	$\begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}$	$-1 - \sqrt{2}$	$\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$

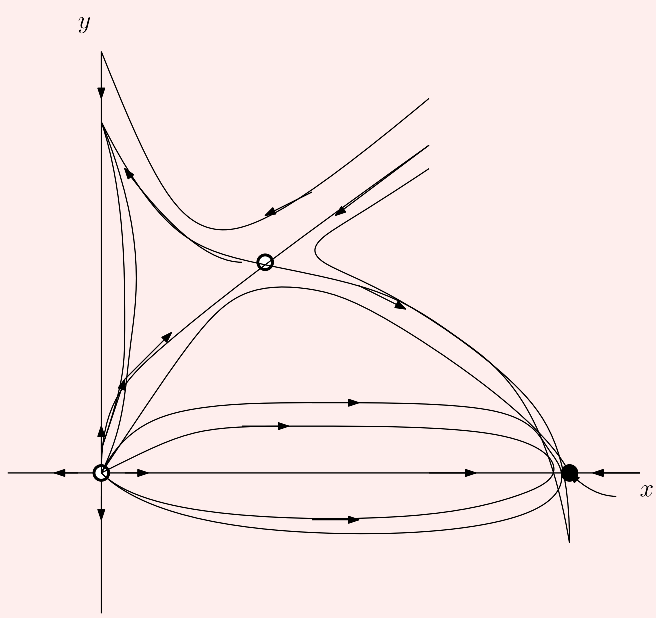
where all the fixed points above are hyperbolic fixed points.

Example 19.8 (Cont'd from above)

Phase portrait: tangent directions of stable/unstable manifolds



Phase portrait:



<u>Conclusion</u>: Only one species survives.

§20 Lec 19: Feb 22, 2021

§20.1 Non-Hyperbolic Fixed Points

Example 20.1 (Sheet 7, Ex A)

The phase portrait of a non-linear ODE near a non-hyperbolic fixed point can be very different from the phase portrait of the linearization at the fixed point.

Example 20.2 (Centers)

For $a \in \mathbb{R}$, consider

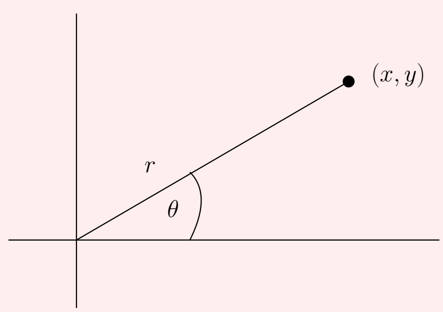
$$\dot{x} = -y + ax(x^2 + y^2)$$
$$\dot{y} = x + ay(x^2 + y^2)$$

$(0, 0)$ is the only fixed point.

$$Df(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \implies \text{eigenvalues: } \lambda = \pm i$$
$$\implies \text{phase portrait of linearization is center around origin}$$

In polar coordinates, (r, θ)

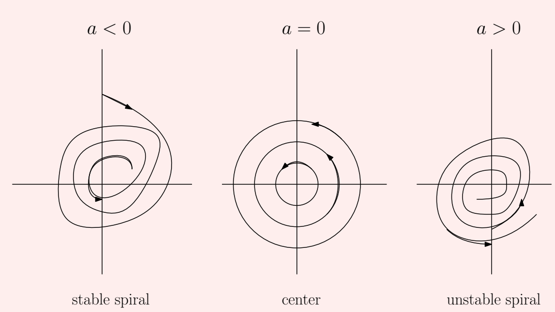
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$



Have

$$\dot{r} = \frac{1}{r}(x\dot{x} + y\dot{y}) = ar^3$$
$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = 1$$

Thus phase portrait of non-linear ODE:



i.e. we have qualitatively different phase portraits (linearization compared to non-linear ODE) for $a \neq 0$.

§20.2 Conservative Systems

Consider Newton's Law: $m\ddot{x} = F(x)$. The force F is called conservative if there is $V(x)$ s.t. $F(x) = -\frac{dV}{dx}$. V is called potential energy. In this case,

$$m\ddot{x} + \frac{dV}{dx} = 0 \tag{*}$$

Proposition 20.3

The total energy $E = \frac{1}{2}m\dot{x}^2 + V(x)$ is preserved, i.e. if $x(t)$ solves $(*)$ then $E(x(t)) =$ const.

Proof. Observe

$$\begin{aligned}\frac{d}{dt}E(x(t)) &= \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + +V(x)\right) \\ &= \frac{1}{2}m \cdot 2 \cdot \dot{x}\ddot{x} + V'(x(t))\,\dot{x} \\ &= \dot{x}\left(m\ddot{x} + V'(x)\right) = 0\end{aligned}$$

□

Definition 20.4 (Conserved Quantity/First Integral) — Suppose $f : D \to \mathbb{R}^2, D \subseteq \mathbb{R}^2$. A conserved quantity/first integral for $\dot{x} = f(x)$ is a function $E : D \to \mathbb{R}$ s.t.

i) $\frac{d}{dt}E(x(t)) = 0$ for every solution $x(t)$ of $\dot{x} = f(x)$.

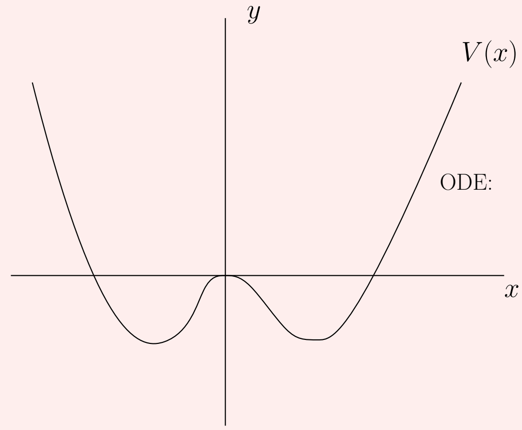
ii) E is non-constant on every ball $B_r(x_0) \subset D$.

Remark 20.5. If E is a first integral of $\dot{x} = f(x)$ then $\dot{x} = f(x)$ cannot have attracting fixed points.

Example 20.6 (Particle of mass $m = 1$ in a double-well potential)

Consider the following:

$$V(x) = \frac{-1}{2}x^2 + \frac{1}{4}x^4$$



The ODE is equivalent to

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -V'(x_1) = x - x^3 = x_1(1 - x_1^2) \end{aligned}$$

Fixed points: $(-1, 0), (0, 0), (1, 0)$

$$Df = \begin{pmatrix} 0 & 1 \\ 1 - 3x_1^2 & 0 \end{pmatrix}$$

$$Df(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \implies \text{eigenvalues } \lambda = \pm 1$$

$$\implies (0,0) \text{ is saddle for both linear and nonlinear ODE}$$

$$Df(\pm 1, 0) = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \implies \text{eigenvalues: } \lambda^2 + 2 = 0 \implies \lambda = \pm i\sqrt{2}$$

$$\implies (-1,0), (1,0) \text{ are linear centers}$$

Theorem 20.7

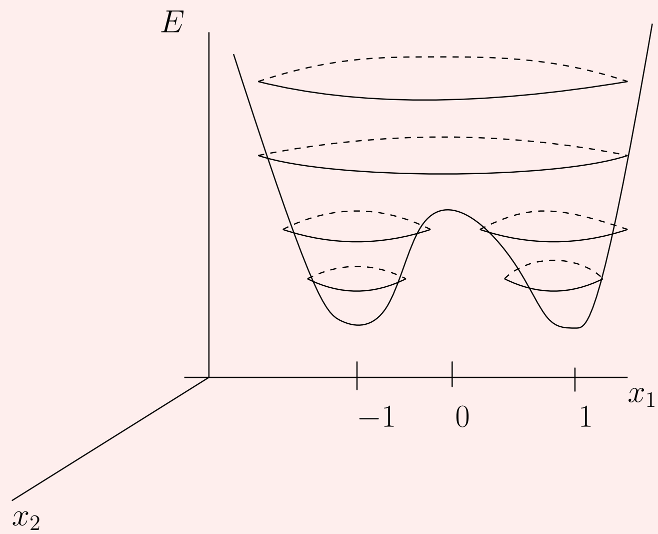
$f \in C^1(D)$. Suppose E is a preserved quantity for $\dot{x} = f(x)$. Suppose x_* is an isolated fixed point. If x_* is a local minimum (or maximum) of E, then all trajectories sufficiently close to x_* are closed trajectories. In particular, x_* is a center for the ODE $\dot{x} = f(x)$.

Example 20.8

Recall from the previous example, $\ddot{x} + V'(x) = 0$, $V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$ i.e. equivalently for $x_1 = x$ and $x_2 = \dot{x}$:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1 - x_1^3$$

By example, $E = \frac{1}{2}\dot{x}^2 + V(x) = \frac{1}{2}x_2^2 - \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4$ is a preserved quantity.



Look at level sets: $E = \text{const}$

$$x_1 \text{ large} : E \approx \frac{x_2^2}{2} + \frac{x_1^4}{4} = \text{const}$$
$$x_1 \text{ small} : E \approx \frac{x_2^2}{2} - \frac{x_1^2}{2} = \text{const}$$

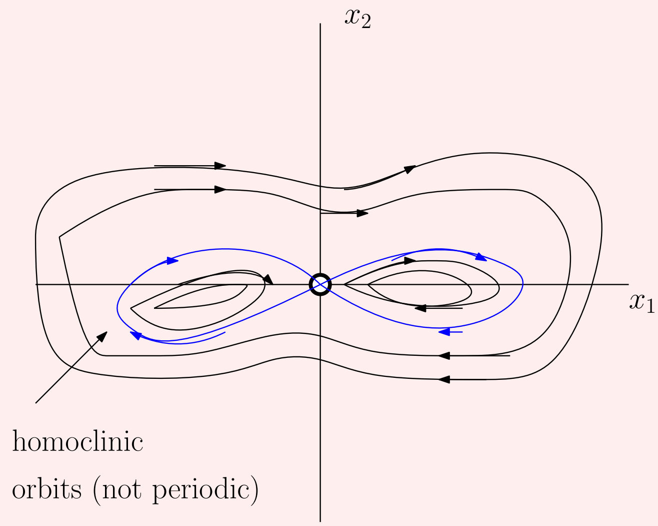
Recall if $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ solves $\dot{x} = f(x)$, then $E\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \text{const}$ i.e. $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ is on level set.

§21 Lec 20: Feb 24, 2021

§21.1 Conservative System (Cont'd)

Example 21.1 (Cont'd from the last example in Lec 19)

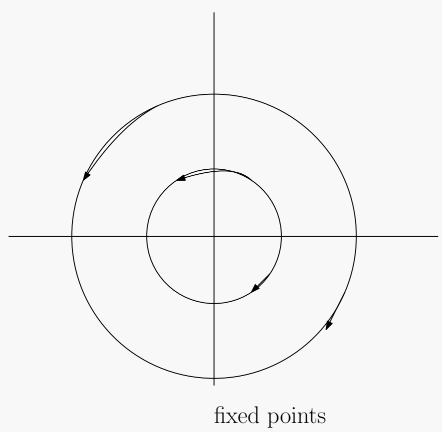
Phase portrait:



Remark 21.2. The assumption that x_* is isolated is necessary:

$$\dot{x} = xy$$
$$\dot{y} = -x^2$$

has the preserved quantity $E = x^2 + y^2$ ($\frac{d}{dt}E = 2x\dot{x} + 2y\dot{y} = 2x^2y - 2yx^2 = 0$), E has a minimum at $(x, y) = (0, 0)$, but $\{(0, y) | y \in \mathbb{R}\} = y$-axis is a line of fixed points.



and in particular, the ODE has no closed orbit (around $(0, 0)$).

Recall: Suppose $E : \mathbb{R}^2 \to \mathbb{R}$,

$$DE = \left(\frac{\partial E}{\partial x_1}, \frac{\partial E}{\partial x_2}\right) = 0 \text{ at } x_*$$

If

$$\text{Hess } E = \begin{pmatrix} \frac{\partial^2 E}{\partial x_1^2} & \frac{\partial^2 E}{\partial x_1 \partial x_2} \\ \frac{\partial^2 E}{\partial x_1 \partial x_2} & \frac{\partial^2 E}{\partial x_2^2} \end{pmatrix}$$

has only negative (positive) eigenvalues, then x_* is a local maximum (minimum) of E (alternatively, if $\det \text{Hess } E > 0$, then E has either a local minimum or local maximum at x_*).
If Hess E has eigenvalues $\lambda_1 < 0 < \lambda_2$ (i.e. $\det \text{Hess } E < 0$), these x_* is a saddle.

Example 21.3

Consider:

$$\begin{aligned} E &= \frac{1}{2}m\dot{x}^2 + V(x) \\ &= \frac{1}{2}x_2^2 - \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4 \\ DE &= \left(-x_1 + x_1^3, x_2\right) = 0 \\ &\iff (x_1, x_2) = (-1,0), (0,0), (1,0) \\ \text{Hess } E &= \begin{pmatrix} -1 + 3x_1^2 & 0 \\ 0 & 1 \end{pmatrix} \\ \text{Hess E}(\pm 1, 0) &= \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \implies (\pm 1, 0) \text{ are local minima} \\ \text{Hess } E(0,0) &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \implies (0,0) \text{ is a saddle} \end{aligned}$$

Remark 21.4. If E is a preserved quantity, then the trajectories are on the level sets, $a, b > 0$.

$$\begin{aligned} &\text{If } E \approx ax_1^2 + bx_2^2 = 1 \leftrightarrow \text{ ellipse} \\ &\text{If } E \approx ax_1^2 - bx_2^2 = 1 \leftrightarrow \text{ saddle} \end{aligned}$$

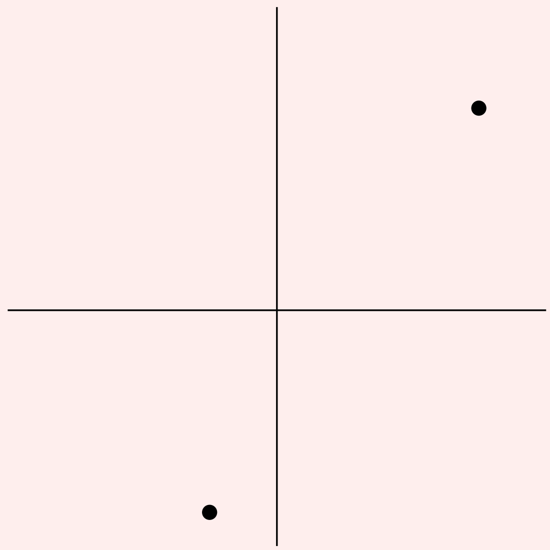
§21.2 Reversible Systems

Definition 21.5 (Involution) — A map $R : \mathbb{R}^2 \to \mathbb{R}^2$ is an involution if $R^2(x) = R\left(R(x)\right) = x$.

Example 21.6 i) $R(x, y) = (x, y)$ identity

ii) R is a reflection ,e.g. $R(x, y) = (x, -y)$ reflection along x-axis.

iii) $R(x, y) = (-x, -y)$ antipodal map



Definition 21.7 (Time-Reversible) — Let R be an involution. The ODE $\dot{x} = f(x)$ is <u>time − reversible</u> with respect to R if for every solution $x(t)$ of $\dot{x} = f(x)$, $R\left(x(-t)\right)$ is also a solution.

Example 21.8

$m\ddot{x} = F(x)$ i.e.

$$(*) \begin{cases} \dot{x} = v \\ \dot{v} = \frac{1}{m}F(x) \end{cases}$$

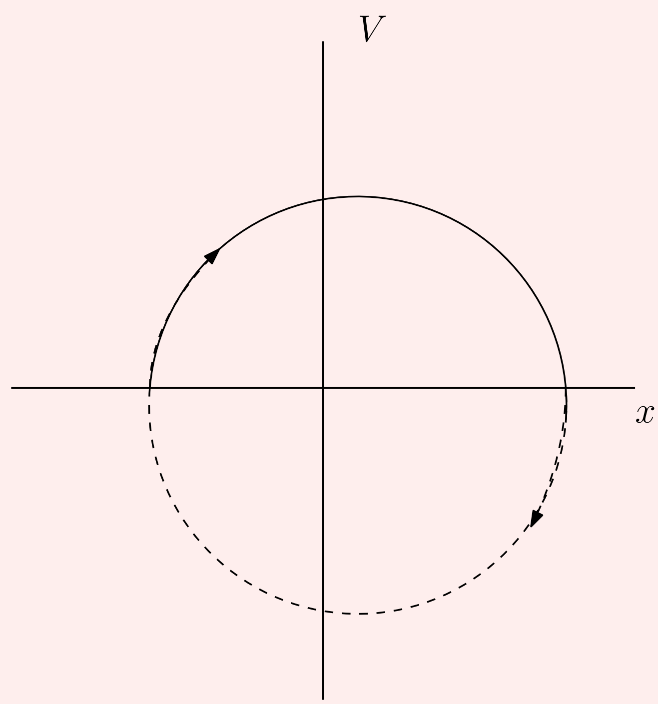
Consider: $R(x, v) = (x, -v)$. Let

$$(X, V)(t) = R\left(x(-t), v(-t)\right) = \left(x(-t), -v(-t)\right)$$

Then

$$\begin{aligned} \frac{d}{dt}(X, V)(t) &= \left(-\dot{x}(-t), \dot{v}(-t)\right) \\ &= \left(-v(-t), \frac{1}{m}F\left(x(-t)\right)\right) \\ &= \left(V(t), \frac{1}{m}F\left(X(t)\right)\right) \end{aligned}$$

i.e. $(X, V)(t)$ indeed solves the ODE (*) geometrically:



harmonic oscillator: $F(x) = -kx$ with spring constant k. Recall: conservation of energy

$$\left(\frac{k}{m}\right)^2 x^2 + v^2 = \text{const}$$

Remark 21.9. Reversible systems may not be conservative, e.g.

$$\dot{x} = -2\cos(x) - \cos(y)$$
$$\dot{y} = -2\cos(y) - \cos(x)$$

has a sink at $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$. On the other hand, the ODE is time-reversible with respect to $R(x, y) = (-x, -y)$ – more details: Strogatz example 6.6.

§22 Midterm 2: Feb 26, 2021

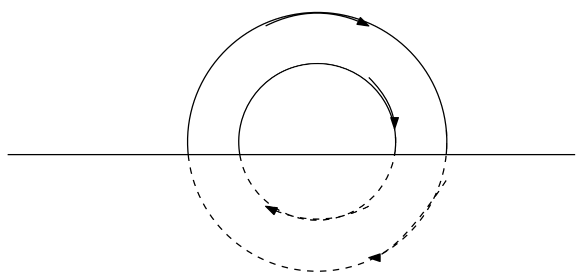
§23 Lec 21: Mar 1, 2021

§23.1 Reversible Systems (Cont'd)

Theorem 23.1

Let $f \in C^1(\mathbb{R}^2)$, $f(x_*) = 0$ and suppose that x_* is a center for the linearization $\dot{y} = Df(x_*) = y$. If $\dot{x} = f(x)$ is time-reversible with respect to a reflection through x_*, then x_* is a center for $\dot{x} = f(x)$, i.e. all trajectories close to x_* are closed orbits.

Idea:



linear centers induces rotational behavior, hence yields intersections with reflection axis, thus closed trajectory.

Example 23.2

Consider:

$$\dot{x} = y$$
$$\dot{y} = x - x^2 = x(1-x)$$

Fixed points: $y = 0$, $x = 0$ or $x = 1$.

$$Df = \begin{pmatrix} 0 & 1 \\ 1-2x & 0 \end{pmatrix}$$
$$\implies \text{Eigenvectors} : \lambda = \pm 1 : \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$$
$$Df(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\text{Eigenvalues} : \lambda = \pm 1$$

$\implies$ $(0,0)$ is a saddle for both the linear and non-linear ODE.

$$Df(1,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ eigenvalues} : \lambda = \pm i$$

$\implies$ $(1,0)$ is a linear center. ODE time-reversible wrt the reflection $R(x,y) = (x,-y)$. Check: Suppose $(x(t), y(t))$ is a solution. Then

$$(X(t), Y(t)) = R\left(x(-t), y(-t)\right) = (x(-t), -y(-t))$$

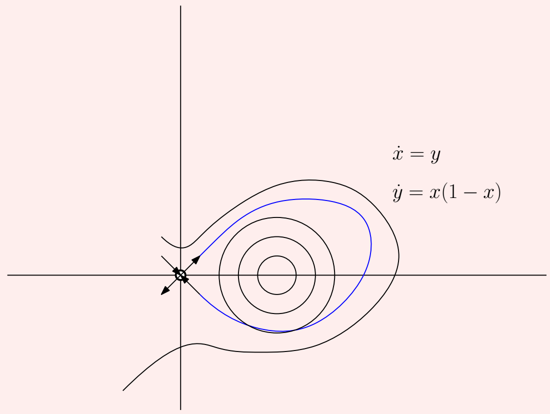
satisfies

$$\begin{aligned} \frac{d}{dt}\left(X(t), Y(t)\right) &= (-\dot{x}(-t), \dot{y}(-t)) \\ &= (-y(-t), x(-t)(1-x(-t))) \\ &= (Y(t), X(t)(1-X(t))) \end{aligned}$$

i.e. $(X(t), Y(t))$ is a solution $\overset{\text{theorem}}{\implies}$ $(1,0)$ is also a non-linear center.

Example 23.3 (Cont'd from above)

Phase portrait:



Note:

$$\dot{x} > 0 \iff y > 0$$
$$\dot{y} > 0 \iff 0 < x < 1$$

$\implies$ solution (x, y) with $x > 0$, $y > 0$ in the unstable manifold of $(0, 0)$ satisfies $x = 1$ for TBA, then $\dot{y} < 0$ as long as $x > 1$, hence it must cross the x-axis; time reversibility yields a homoclinic orbit.

Remark 23.4. The ODE

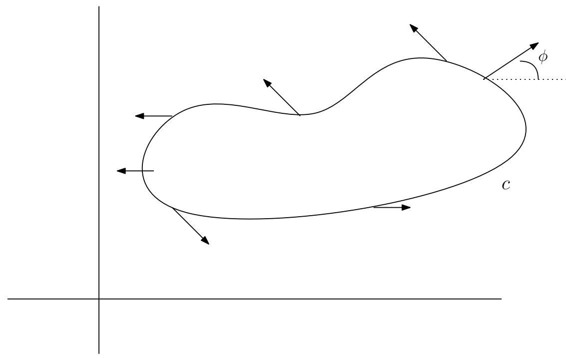
$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

is time reversible wrt $R(x, y) = (x, -y)$

$$\iff \begin{cases} f \text{ is odd in } y,\ f(x, -y) = -f(x, y) \\ g \text{ is even in } y,\ g(x, -y) = g(x, y) \end{cases}$$

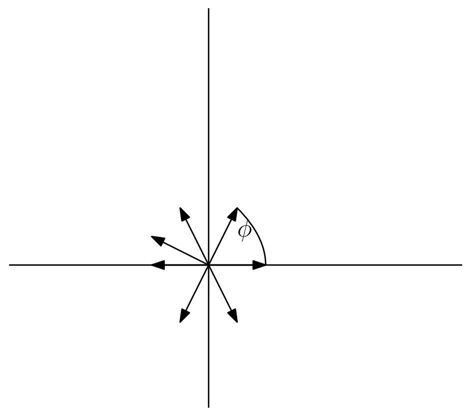
§23.2 Index Theory

$\dot{x} = f(x)$
Phase plane:



unless stated explicitly otherwise C is a simple (=no self-intersections) closed curve, no fixed points on C, oriented counterclockwise.

Remark 23.5. Usually C is not a trajectory.



Definition 23.6 (Index of a Curve) — Index of C : $I_C(f) = I_C =$ net numbers of counter-clockwise rotations of the vector field f along $C = \frac{1}{2\pi}$ (change of angle).

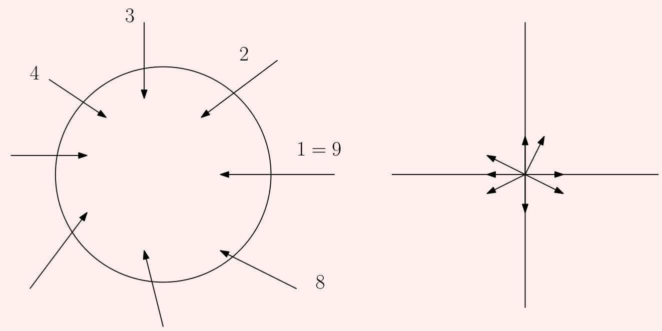
Theorem 23.7

If C can be continuously deformed into C' without passing through fixed points, then $I_C = I_{C'}$

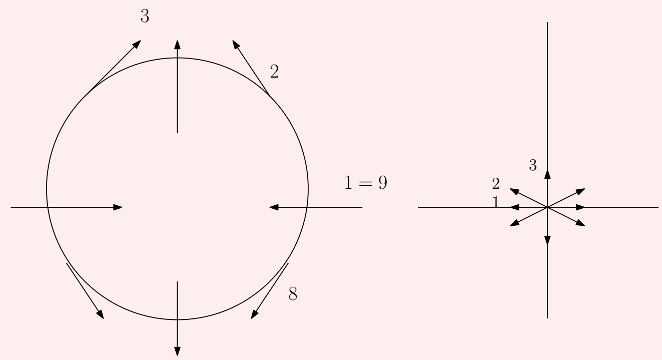
Idea: C changes continuously, I_C is an integer, hence it cannot jump.

Example 23.8

Consider:



i) $\implies I_C = 1$. In particular, if C encloses a stable node (and no other fixed points), then $I_C = 1$.

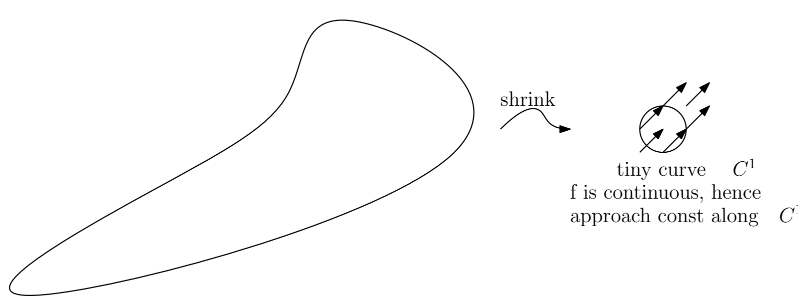


ii) $\implies I_C = -1$. In particular, if C encloses a saddle (and no other fixed points), then $I_C = -1$.

Proposition 23.9

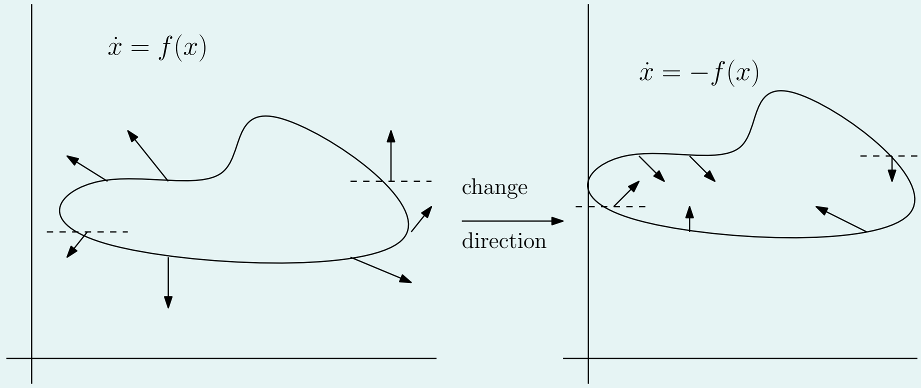
If C does not enclose a fixed point, then $I_C = 0$.

Idea:



Proposition 23.10

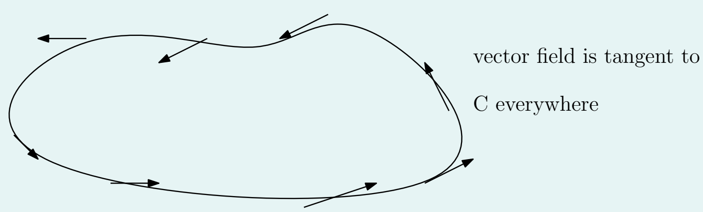
$I_C(f) = I_C(-f)$ i.e. the index does not change when reversing all arrows.



Idea: angle changes from ϕ to $\phi + \pi$, hence the difference stays the same.

Proposition 23.11

If C is a trajectory, i.e. a closed orbit of $\dot{x} = f(x)$, then $I_C = 1$. Intuition:



precise result: Hopfscher Umlaufsatz.

Note: Closed orbits precisely correspond to periodic solutions.

§24 Lec 22: Mar 3, 2021

§24.1 Index Theory (Cont'd)

Definition 24.1 (Index of a Fixed Point) — Let x_* be a fixed point of $\dot{x} = f(x)$, $f(x_*) = 0$. The index of x_* is $I_{x_*} = I_C$ where C encloses x_* and no other fixed point.

Proposition 24.2

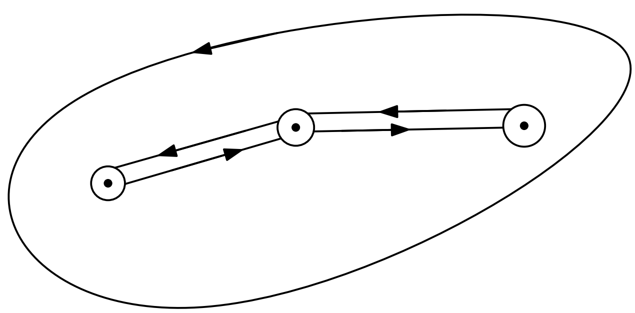
If x_* is a hyperbolic fixed point, then $I_* = \text{sign} \det Df(x_*)$. In particular, $I_* = -1 \iff x_*$ saddle.

Proposition 24.3

If C encloses the fixed points $x_1^*, \ldots, x_n^*$ then

$$I_C = \sum_{i=1}^{n} I_{x_1^*}$$

<u>Idea</u>:



Theorem 24.4

Any closed orbit in $\mathbb{R}^2$ must enclose fixed point(s) whose indices sum up to $+1$. In particular, every closed orbit encloses a fixed point.

Corollary 24.5

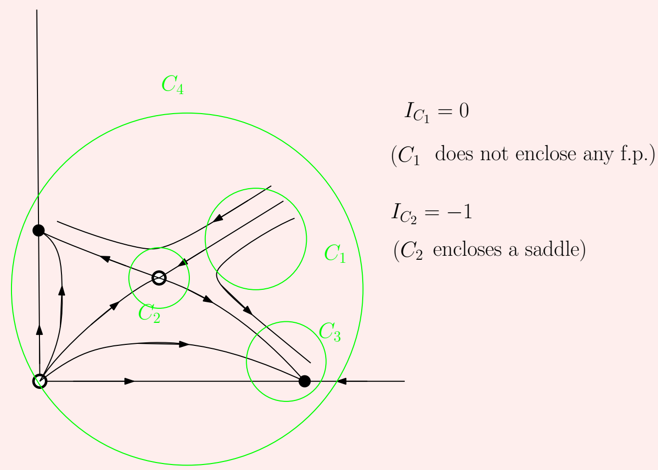
If $\dot{x} = f(x)$, $f \in C^1(\mathbb{R}^2)$, does not have any fixed points, then it does not have a closed orbit.

Example 24.6

The ODE:

$$\dot{x} = x(3 - x - 2y)$$
$$\dot{y} = y(2 - x - y)$$

does not have closed orbit, $(0,0)$ – unstable node, $(0,2),(3,0)$ – stable nodes, $(1,1)$ – saddle.
Phase portrait:

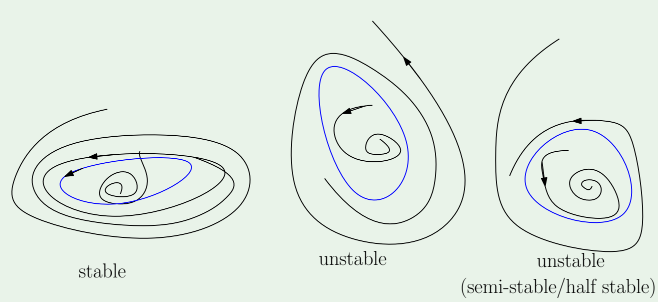


Any closed has index $= +1$, so C_1, C_2 cannot be closed orbit. $I_{C_3} = I_{C_4} = 1$ but C_3, C_4 intersect the x- or y-axis. However, the x-axis, y-axis consist of trajectories. By uniqueness, trajectories cannot intersect, hence C_3, C_4 cannot be trajectories.

The same argument applies to any other curve with index $+1$ since all f.p. with index $+1$ are on the x- or y-axis.

§24.2 Limit Cycles

Definition 24.7 (Limit Cycles) — Limit cycles are isolated closed trajectories.



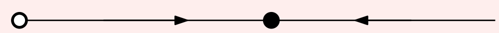
Remark 24.8. Limit cycles are a non-linear phenomenon.

Example 24.9

$\dot{r} = r(1 - r^2)$, $\dot{\theta} = 1$ (counter-clockwise rotation with speed +1)

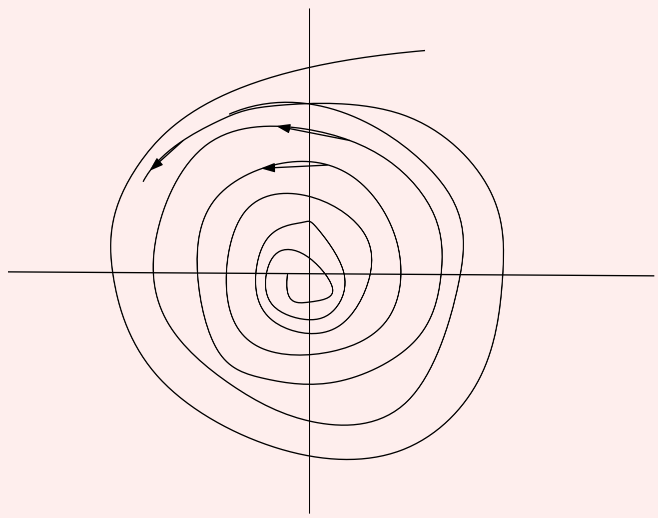
$$\dot{r} = 0: \; r = 0, \, r = 1$$

Phase portrait for radius:



$r = 1$

Phase portrait of ODE:



at $r = 1$ we have a stable limit cycle.

§24.3 Gradient Systems

Definition 24.10 (Gradient) — $\dot{x} = f(x)$ is gradient if $f(x) = -\nabla V = -\begin{pmatrix} \partial_{x_1} V \\ \partial_{x_2} V \end{pmatrix}$ for a scalar function $V(x_1, x_2)$. V is called potential.

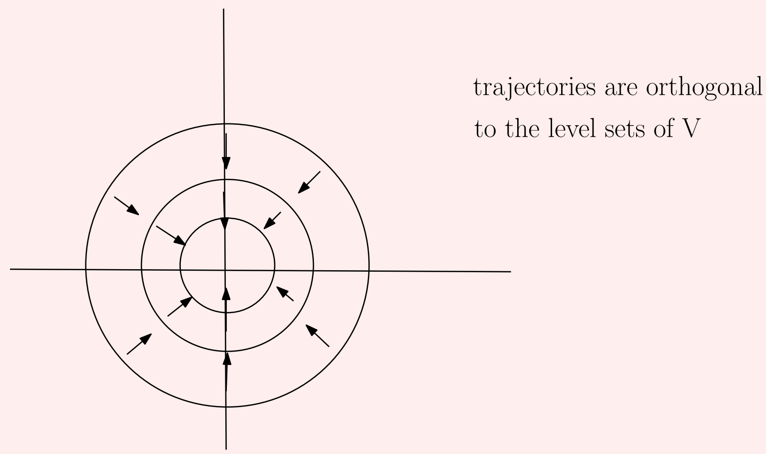
Example 24.11

Consider:

i) $V = x^2 + y^2$ where

$$\dot{x} = -\partial_x V = -2x$$
$$\dot{y} = -\partial_y V = -2y$$

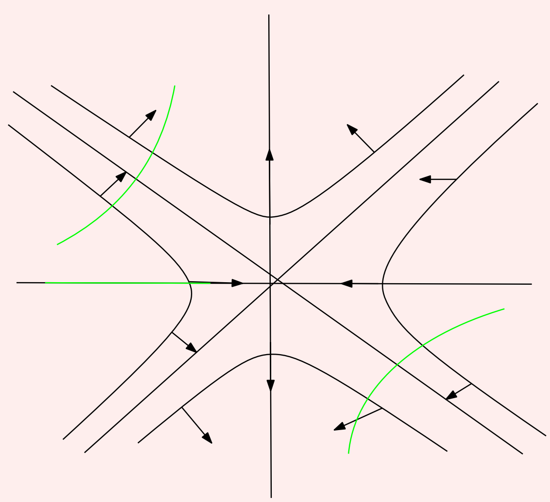
i.e. $(0,0)$ is a stable star. Level sets of $V = x^2 + y^2 = r^2$



ii) $V = x^2 - y^2$ where

$$\dot{x} = -\partial_x V = -2x$$
$$\dot{y} = -\partial_y V = 2y$$

i.e. $(0,0)$ is a saddle.



Theorem 24.12

Gradient systems cannot have closed orbits.

Proof. Otherwise, let $x(t)$, $t \in [0, t]$ be a closed orbit. Then

$$\begin{aligned} 0 &= V\left(x(T)\right) - V\left(x(x(0))\right) \\ &= \int_0^T \frac{d}{dt} V\left(x(t)\right)\, dt \\ &= \int_0^T \left\langle \nabla V\left(x(t)\right), \dot{x}(t) \right\rangle dt \\ &= -\int_0^T \left\| \dot{x}(t) \right\|^2 dt < 0 \end{aligned}$$

unless $\dot{x} = 0$ i.e. $x(t) = \text{const}$ is a fixed point. Contradiction. □

§25 Lec 23: Mar 5, 2021

§25.1 Gradient Systems (Cont'd)

Remark 25.1. If $\dot{x} = f(x)$ is gradient, i.e.

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = -\begin{pmatrix} \partial_{x_1} V \\ \partial_{x_2} V \end{pmatrix}$$

then $\frac{\partial f}{\partial x_2} = -\partial_{x_2}\partial_{x_1} V = -\partial_{x_1}\partial_{x_2} V = \frac{\partial f_2}{\partial x_1}$ i.e. $\frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1} = 0$ and f is curl-free.

Theorem 25.2

Suppose f is curl-free. Then $\dot{x} = f(x)$ is gradient provided that the domain of f does not contain any holes e.g

$$\mathbb{R}^2 \text{ or } B_r\left((x_0, y_0)\right) = \left\{(x,y) \middle| \sqrt{(x-x_0)^2 + (y-y_0)^2} < r\right\}$$

In this case

$$\begin{aligned} V(x_1, x_2) &= -\int_{\gamma_{x_0}} \langle f(x), dx\rangle = -\left(\text{line integral from } x_0 \text{ to } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ along a path } \gamma\right) \\ &= -\int_a^b f\left(\gamma_{x_0}(t)\right) \cdot \dot{\gamma}_{x_0}(t)\, dt \end{aligned}$$

where $\gamma_{x_0}(a) = x_0, \gamma_{x_0}(b) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and also

$$V(x_1, x_2) = -\int_0^1 \left(f_1(tx_1, tx_2)x_1 + f_2(tx_1, tx_2)x_2\right)\, dt$$

for $\gamma : [0,1] \to \mathbb{R}^2$ and $\gamma(t) = t\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Example 25.3

Consider:

$$\dot{x} = \sin(y)$$
$$\dot{y} = x\cos(y)$$

Then $\frac{\partial f_1}{\partial y} = \cos(y) = \frac{\partial f_2}{\partial x}$ i.e. f is curl-free. Thus, the ODE is gradient. Then the potential

$$\begin{aligned} V(x,y) &= -\int_0^1 \left(f_1(tx,ty)x + f_2(tx,ty)y\right)\,dt \\ &= -\int_0^1 (x\sin(ty) + tx\cos(ty)y)\,dt \\ &= -x\sin(y) \end{aligned}$$

§25.2 Lyapunov Functions

Definition 25.4 (Lyapunov Function) — Let S be a set. A function $L(x)$ is a Lyapunov function for $\dot{x} = f(x)$ if

i) $L(x) \geq 0$ and $L(x) = 0 \iff x \in S$.

ii) $\frac{d}{dt}L(x(t)) < 0$ for every solution $x(t)$ of $\dot{x} = f(x)$, $x(t) \notin S$
$\frac{d}{dt}L(x(t)) = 0 \iff$ for every solution $x(t)$ of $\dot{x} = f(x)$, $x(t) \in S$.

Theorem 25.5

If $\dot{x} = f(x)$ has a Lyapunov function $L(x)$ with $L(x) = 0 \iff x = x_*$, then x_* is a globally stable fixed point. In particular, there is no closed orbit.

Example 25.6

The ODE:

$$\dot{x} = -x + 4y$$
$$\dot{y} = -x - y^3$$

does not have closed orbits, moreover $(0,0)$ is a globally stable fixed point.

Proof. The function $L(x,y) = x^2 + 4y^2$ is a Lyapunov function w.r.t $S = \{(0,0)\}$

- $L(x,y) = x^2 + 4y^2 \geq 0$, $L(x,y) = 0 \iff x = y = 0$

- Consider:

$$\begin{aligned}\frac{d}{dt}L(x,y) &= 2x\dot{x} + 8y\dot{y} \\ &= -2x^2 + 8xy - 8xy - 8y^4 \\ &= -2(x^2 + 4y^4) \le 0 \\ \frac{d}{dt}L(x,y) = 0 &\iff x = y = 0\end{aligned}$$

  Thus, the theorem applies.

□

Example 25.7

Consider:

$$\begin{aligned}\dot{x} &= x\left(1 - 4x^2 - y^2\right) - \frac{1}{2}y(1+x) \\ \dot{y} &= y\left(1 - 4x^2 - y^2\right) + 2x(1+x)\end{aligned}$$

linear stability analysis: $(0,0)$ is an unstable spiral. Consider $L(x,y) = \left(1 - 4x^2 - y^2\right)^2$

- $L(x,y) \ge 0$ and $L(x,y) = 0 \iff 4x^2 + y^2 = 1$
- Have:

$$\begin{aligned}\frac{d}{dt}L(x,y) &= 2\left(1 - 4x^2 - y^2\right)\left(-8x\dot{x} - 2y\dot{y}\right) \\ &= \dots \\ &= -4\left(1 - 4x^2 - y^2\right)^2\left(4x^2 + y^2\right) \\ &\le 0\end{aligned}$$

and $\frac{d}{dt}L(x,y) = 0 \iff x = y = 0$ or $4x^2 + y^2 = 1$.
Consequence: $4x^2 + y^2 = 1$ is a limit cycle because

- ODE does not have a f.p. on $4x^2 + y^2 = 1$. Note: if $4x^2 + y^2 = 1:$

$$\begin{aligned}\dot{x} &= \frac{1}{2}y(1+x) \\ \dot{y} &= 2x(1+x)\end{aligned}$$

  thus if $\dot{x} = 0$, then $(x,y) = \left(\pm\frac{1}{2}, 0\right)$ is the only option on $4x^2 + y^2 = 1$ and $\dot{y} \neq 0$. Similarly, if $\dot{y} = 0$, then $(x,y) = (0, \pm 1)$ and $\dot{x} \neq 0$

- Trajectories approach the minimum level set $4x^2 + y^2 = 1$ unless $(x(t), y(t)) = (0,0)$.

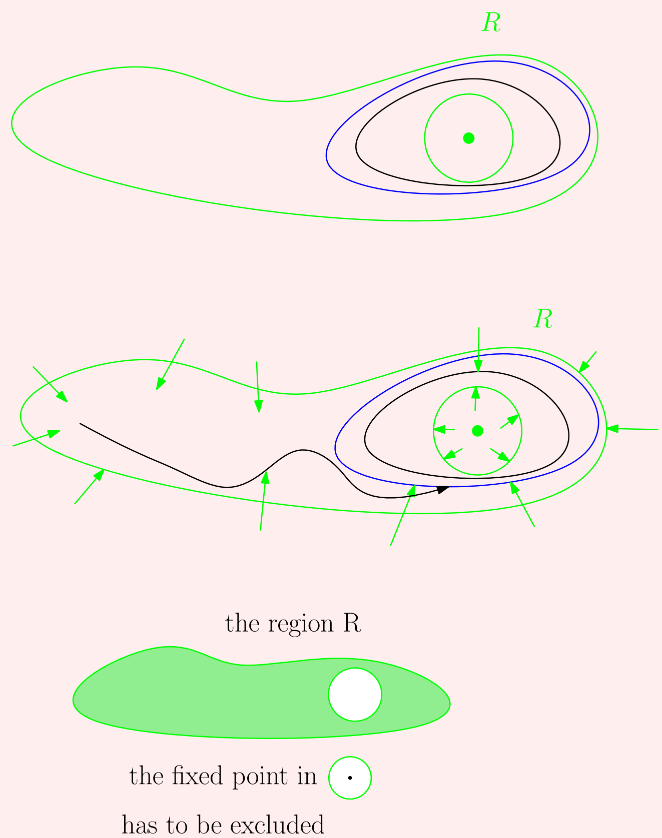
§26 Lec 24: Mar 8, 2021

§26.1 The Poincaré – Bendixson Theorem

Theorem 26.1 (Poincaré – Bendixson)
Let $D \subseteq \mathbb{R}^2$ be open, $f \in C(D)$. Let $x(t)$ be a trajectory of $\dot{x} = f(x)$ s.t. $C = \{x(t) | t \geq 0\}$ is contained in a closed, bounded region $R \subset D$.

If R does not contain any fixed points, then either C is a closed orbit or $x(t)$ spirals towards a closed orbit (in R) as $t \to \infty$. In particular, R contains a closed orbit.

Example 26.2
Consider: $D = \mathbb{R}^2$



R is a trapping region, i.e. the vector field f points inward on the boundary of R. Hence, all trajectories starting in R, remain in R. In particular, if R does not contain any fixed points, then the Poincaré – Bendixson theorem applies. In particular, R contains a closed orbit.

Remark 26.3. Poincaré – Bendixson fails in dimension 3, i.e. if $\mathbb{R}^2$ is replaced by $\mathbb{R}^3$.

P-B rules out chaotic behavior. However, in dimension 3, chaotic solutions are possible (strange attractors).

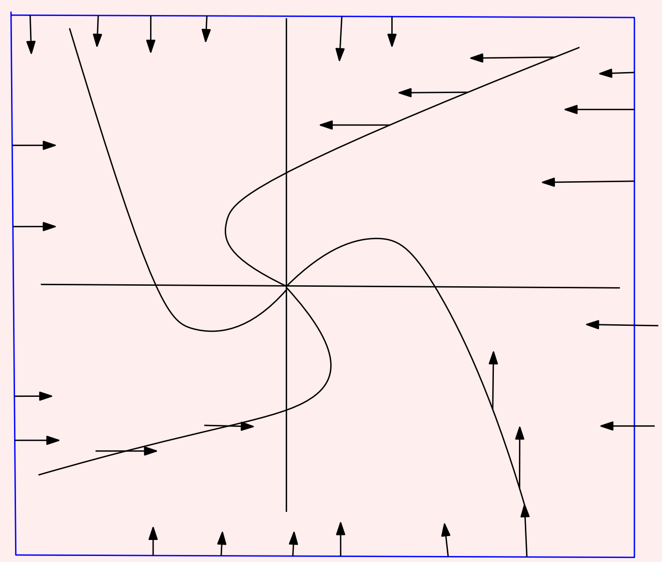
Example 26.4

Consider:

$$\dot{x} = x - y - x^3$$
$$\dot{y} = x + y - y^3$$

Claim 26.1. The ODE has a closed orbit.

$$\text{Vertical Nullclines}: \dot{x} = 0 \iff y = -x(x-1)(x+1)$$
$$\text{Horizontal Nullclines}: \dot{y} = 0 \iff x = y(y-1)(y+1)$$



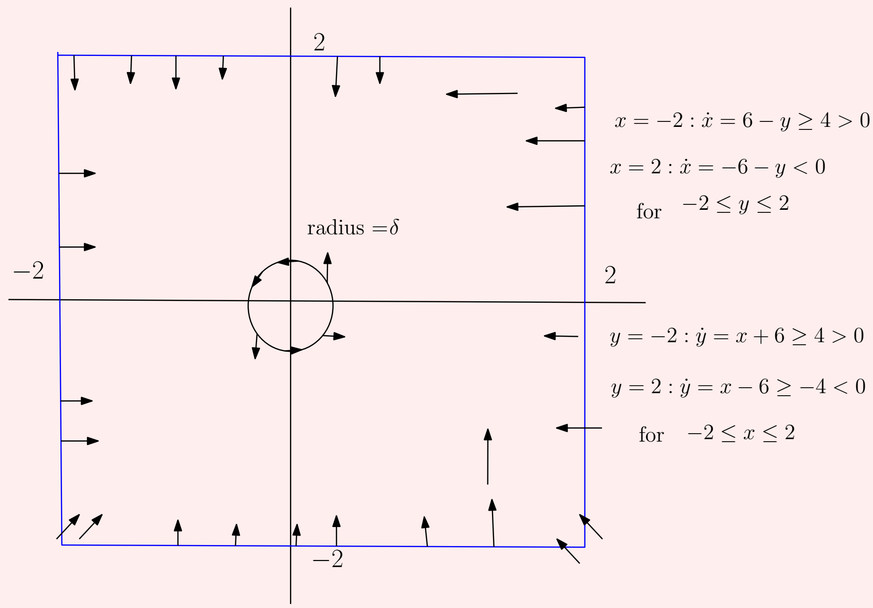
in particular, the nullclines intersect only at (0,0),
so (0,0) is the only fixed point

$$Df(0,0) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \implies \lambda = 1 \pm i$$

$\implies$ $(0,0)$ is an unstable spiral. Let's construct a trapping region

Example 26.5 (Cont'd from the above)

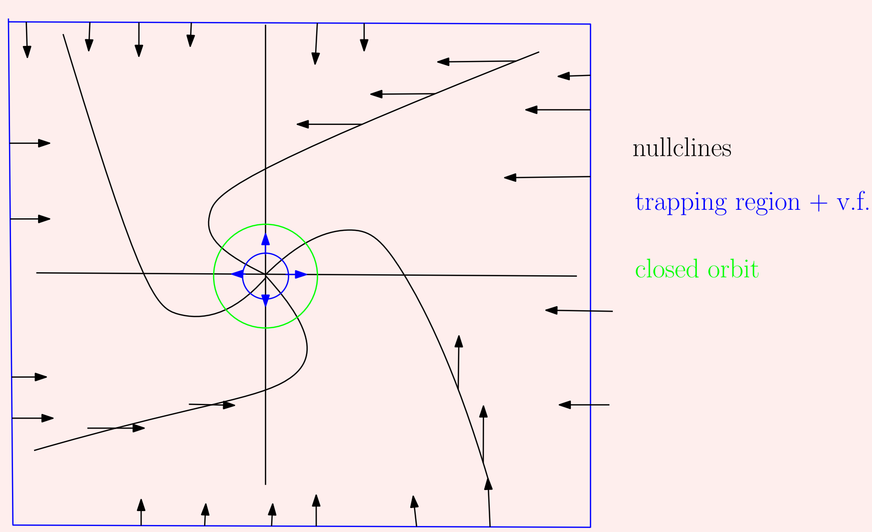
Have:



Thus any trajectory starting in $R = \{(x,y) | -2 \leq x \leq 2, -2 \leq y \leq 2\}$ has to remain in R. Hence

$$R_\delta = R \setminus B_\delta(0) = R \setminus \left\{(x,y) | x^2 + y^2 < \delta^2\right\}$$

for $\delta > 0$ small, is a trapping region, because $(0,0)$ is an unstable spiral, thus all trajectories must leave (and cannot re-enter) $B_\delta(0)$ for some $\delta > 0$ small. Thus, by P-B, R_δ contains a closed orbit.



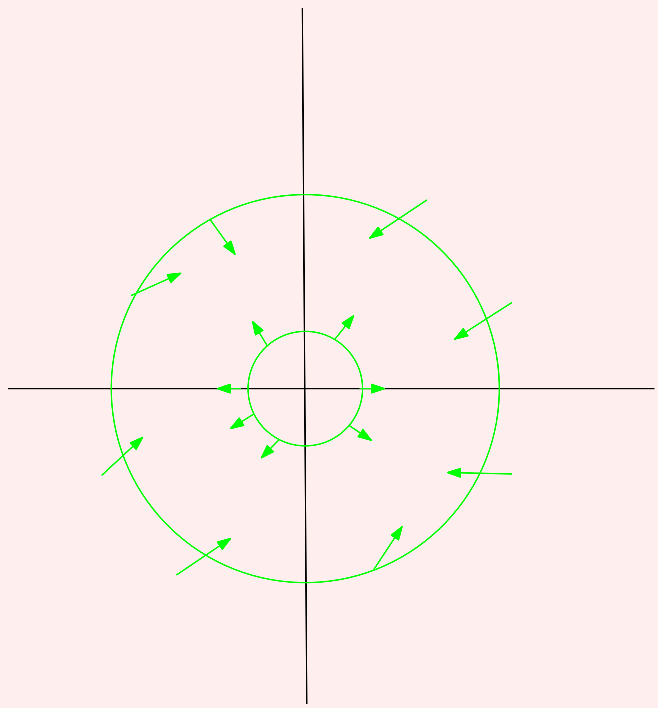
Example 26.6

Consider:

$$\dot{r} = r\left(1 - r^2 + \mu\cos\theta\right)$$
$$\dot{\theta} = 1$$

for a fixed parameter $\mu \in \mathbb{R}$. If $\mu = 0 : \dot{r} = r(1 - r^2)$ and the circle $r = 1$ is a closed orbit.

Claim 26.2. For $\mu \in (0, 1)$ there is a closed orbit.



Proof. Fix $\mu \in (0, 1)$. Then $\dot{r} \geq r(1 - r^2 - \mu) > 0$ if $r^2 < 1 - \mu$ i.e. $0 < r < \sqrt{1-\mu}$.

$$\dot{r} \leq r\left(1 - r^2 + \mu\right) < 0 \quad \text{if } r^2 > 1 + \mu \text{ i.e. } r > \sqrt{1+\mu}$$

Thus, for any $\epsilon > 0$ small, e.g., $\epsilon = \frac{1}{2}$

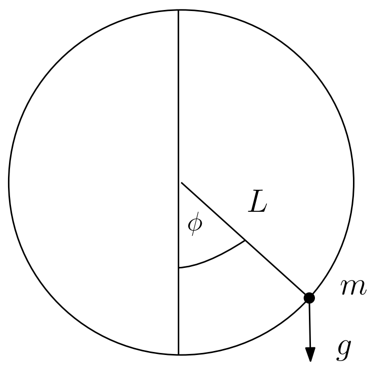
$$R = \left\{(x, y) = r(\cos\theta, \sin\theta) | (1-\epsilon)\cdot\sqrt{1-\mu} \leq r \leq (1+\epsilon)\sqrt{1+\mu}\right\}$$

is a trapping region. Recall R does not contain nay fixed points. Therefore, by P-B, there is a closed orbit in R. □

§27 Lec 25: Mar 10, 2021

§27.1 Pendulum

Consider:



> **Remark 27.1.** For small angles:
> $$\sin\phi \approx \phi, \quad \omega^2 = \frac{g}{L}$$
> $$\ddot{\phi} + \omega^2\phi = 0 \quad \text{(harmonic oscillator)}$$

Normalize $\omega^2 = \frac{g}{L} = 1$. Alternatively, non-dimensional with time scale $T = \frac{1}{\omega}$, $\tau = \frac{t}{T}$. Set $v = \dot{\phi}$. Then

$$\begin{cases} \dot{\phi} = v \\ \dot{v} = -\sin\phi \end{cases}$$

Fixed points, $v_* = 0$, $\phi_* = \pi\mathbb{Z}$ on $\mathbb{R}$, $0, \pi$ on circle.
Linearization at fixed point: $f(\phi, v) = \begin{pmatrix} v \\ -\sin\phi \end{pmatrix}$

$$Df = \begin{pmatrix} 0 & 1 \\ -\cos\phi & 0 \end{pmatrix}$$
$$Df(0,\pi) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \implies \lambda_1 = -1, \lambda_2 = 1$$

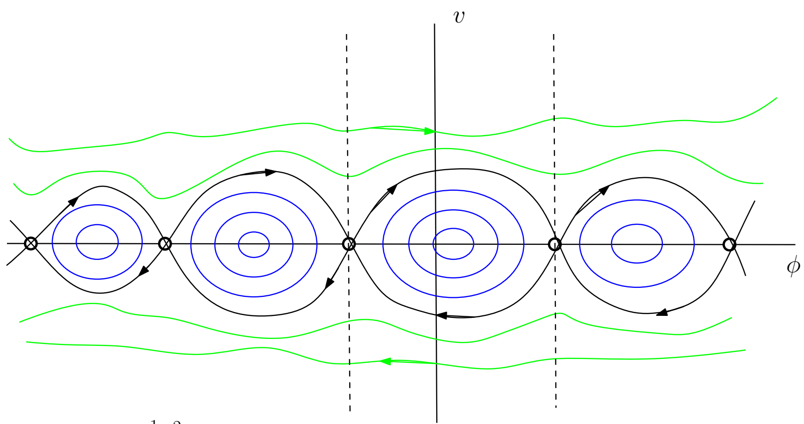
$\implies$ $(0,\pi)$ is a saddle, hyperbolic fixed point, and thus also a saddle for non-linear ODE.

$$Df(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$\implies$ $(0,0)$ is a linear center, in fact, $(0,0)$ is a non-linear center because:

- ODE time-reversible w.r.t $(\phi, v) \mapsto (\phi, -v)$
- $E = \frac{1}{2}v^2 - \cos\phi$ is a conserved quantity (conservation of energy) and $(0,0)$ is an isolated fixed point and local minimum of E.

Phase portrait:



$E = \frac{1}{2}v^2 - \cos\phi$

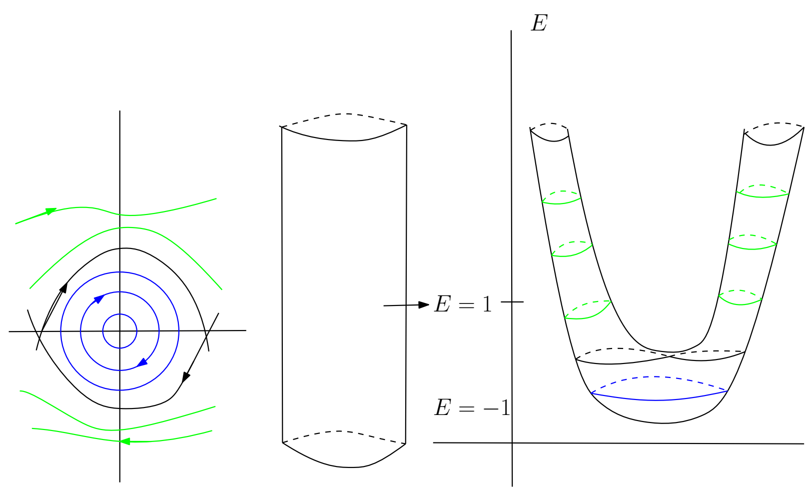
E = -1: fixed points

$E < 1$: closed orbits

E = 1: heteroclinic orbits

$E > 1$: rotations over the top of the pendulum

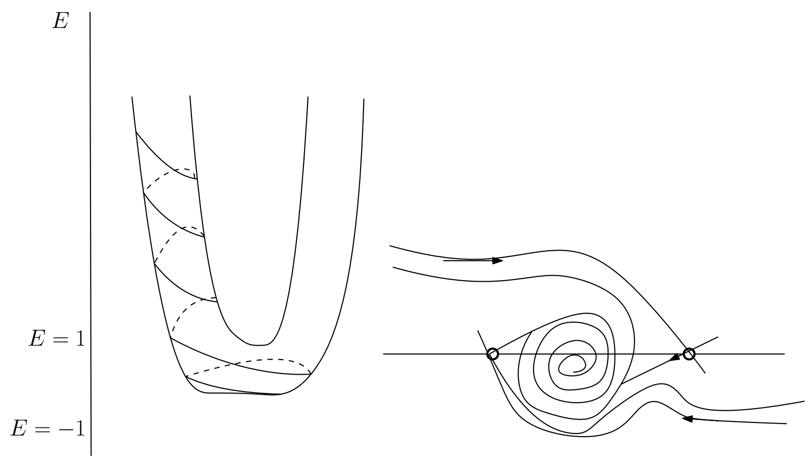
Phase portrait on the cylinder:



Damping: $\ddot{\phi} + b\dot{\phi} + \sin\phi = 0$, $b > 0$ damping constant. Energy is not preserved:

$$\begin{aligned}\frac{d}{dt}E &= \frac{d}{dt}\left(\frac{1}{2}v^2 - \cos\phi\right)\\ &= v\dot{v} + \sin\phi\dot{\phi} = \dot{\phi}\left(\ddot{\phi} + \sin\phi\right) = -b\dot{\phi}^2 \le 0\end{aligned}$$

$\implies$ E non-increasing, decreasing if $\dot{\phi} \neq 0$

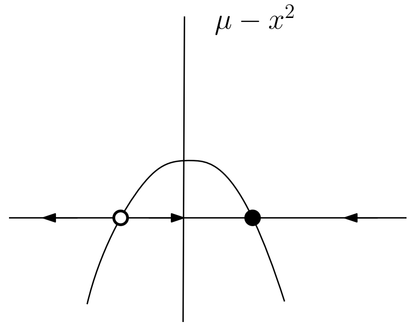


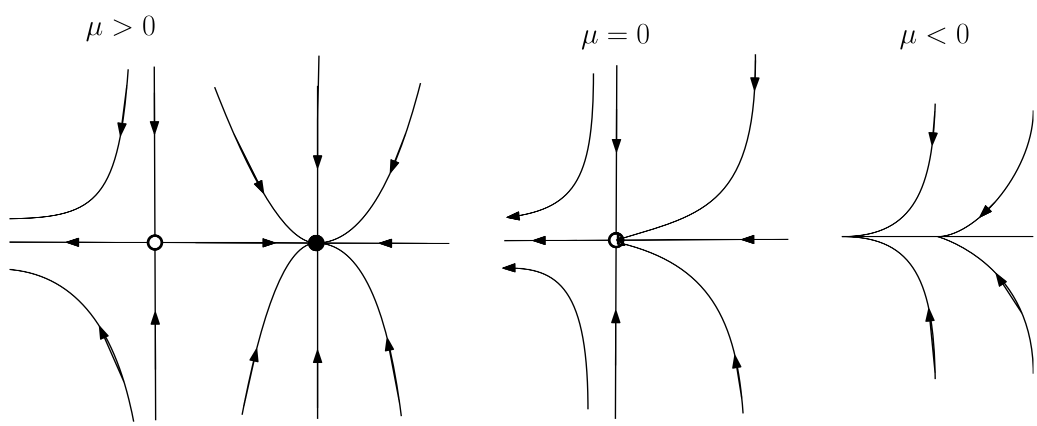
§27.2 Bifurcation in 2D

i) Saddle-node bifurcation

$$\dot{x} = \mu - x^2$$
$$\dot{y} = -y$$

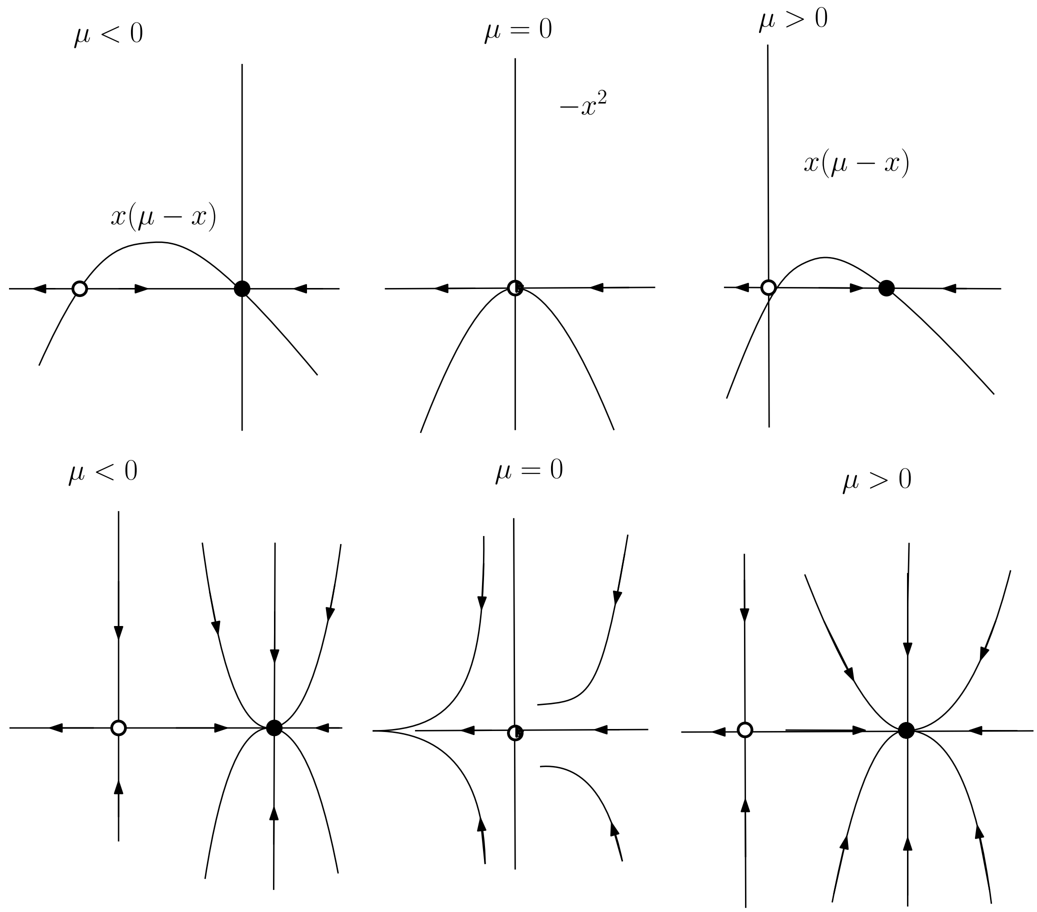
$\mu > 0$:





ii) Transcritical bifurcation

$$\dot{x} = \mu x - x^2$$
$$\dot{y} = -y$$


$\mu < 0$
$\mu = 0$
$\mu > 0$
$-x^2$
$x(\mu - x)$
$x(\mu - x)$
$\mu < 0$
$\mu = 0$
$\mu > 0$

§28 Lec 26: Mar 12, 2021 – Last Lecture :'(

§28.1 Bifurcation in 2D (Cont'd)

Continue from last lecture,

iii) Pitchfork bifurcations:

$$\begin{aligned} \text{subcritical} &: \mu x + x^3, \quad \dot{y} = -y \\ \text{supercritical} &: \mu x - x^3, \quad \dot{y} = -y \end{aligned}$$

Remark 28.1. In all examples, one eigenvalue of $Df(0,0)$ for $\mu = 0$ is equal to zero.

Recall: conditions for bifurcation in 1D

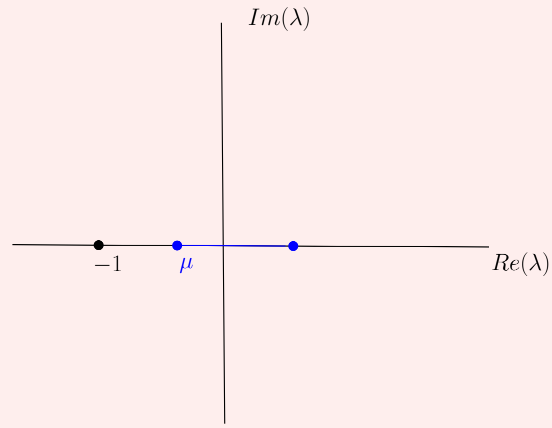
$$\begin{aligned} f &= 0 \\ \frac{\partial f}{\partial x} &= 0 \end{aligned}$$

therefore examples $i) - iii)$ are zero-eigenvalue bifurcations.

Example 28.2

Transcritical bifurcation:

$$\begin{aligned} \dot{x} &= \mu x - x^2 \\ \dot{y} &= -y \\ Df(0,0) &= \begin{pmatrix} \mu & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

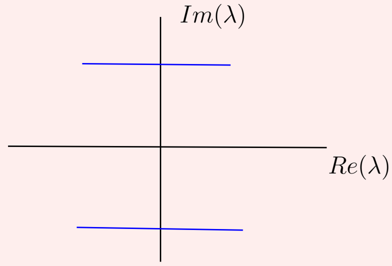


Example 28.3

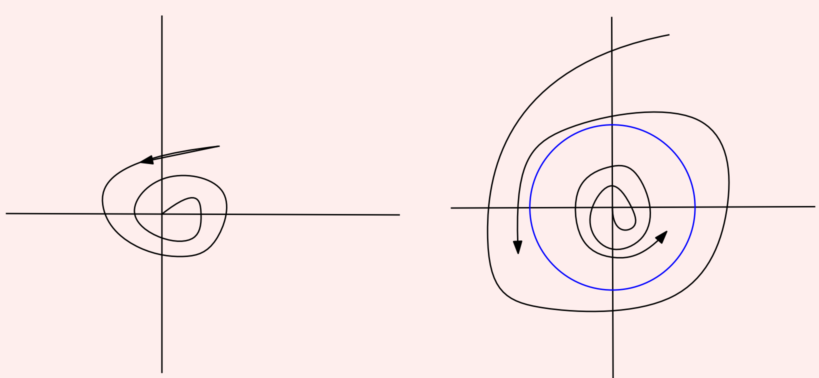
Supercritical Hopf bifurcation:

$$\dot{r} = \mu r - r^3 = r\left(\mu - r^2\right)$$
$$\dot{\theta} = \omega > 0$$

Eigenvalues of linearization at $(x, y) = (0, 0)$: $\lambda_{1,2} = \mu \pm i\omega$



$\mu \leq 0$ $\mu > 0$



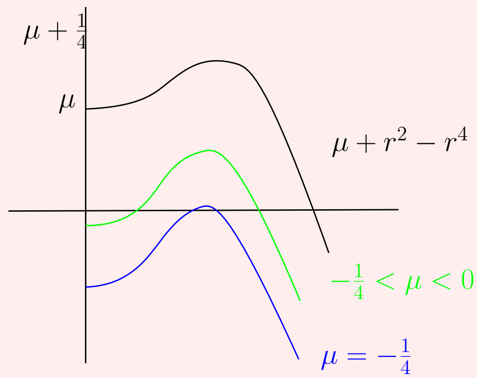
origin: stable spiral origin: unstable spiral

$r = \sqrt{\mu}$ is a closed orbit, in fact, a stable limit cycle.

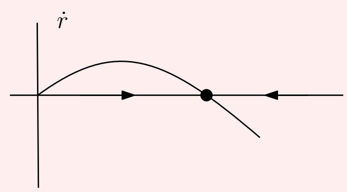
Example 28.4

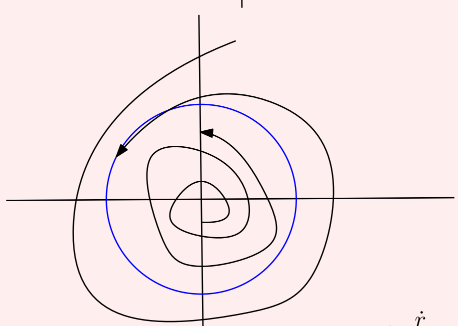
Consider:

$$\dot{r} = \mu r + r^3 - r^5 = r\left(\mu + r^2 - r^4\right)$$
$$\dot{\theta} = \omega > 0$$



$\mu \geq 0$

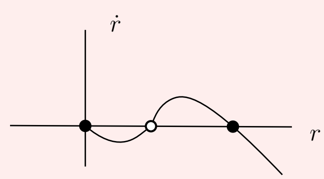


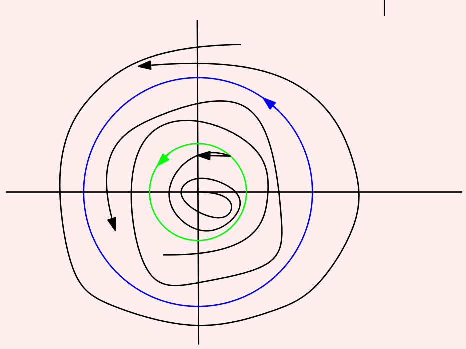


origin: unstable spiral

—— : stable limit cycle

$-\frac{1}{4} < \mu < 0:$





origin: stable spiral

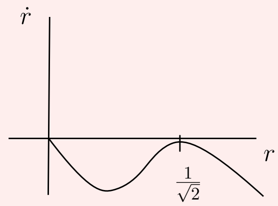
—— : stable limit cycle

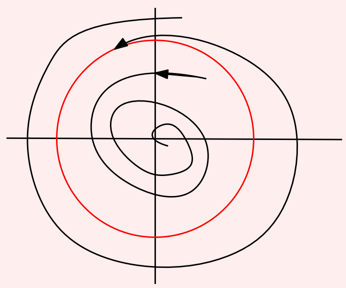
—— : unstable limit cycle

Example 28.5 (Cont'd from above)

We have a subcritical Hopf bifurcation at the origin for $\mu < 0$. (the fixed point (=origin) changes stability and an unstable limit cycle is created)

$\mu = -\frac{1}{4}$:

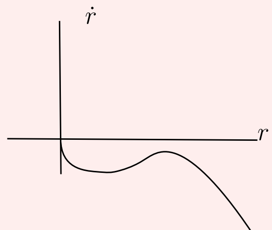


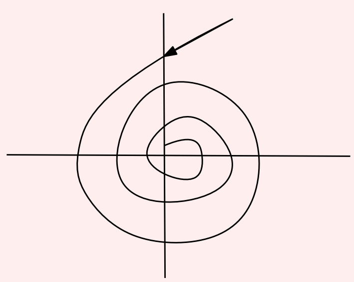


origin: stable spiral

(red line) : semi-stable limit cycle

$\mu < -\frac{1}{4}$:





origin: stable spiral

We have a global bifurcation at the radius $r = \frac{1}{\sqrt{2}}$ (a bifurcation that does take a fixed point), more precisely a "saddle-node bifurcation of limit cycles". A stable and an unstable limit cycles collide and disappear (or appear out of the blue).

Remark 28.6. Degenerate Hopf fibration: center at bifurcation (μ_*, x_*) (recall: sub/supercritical case: spirals)

Example 28.7

Damped pendulum:

$$\ddot{x} + \mu\dot{x} + \sin(x) = 0 \quad \mu \in \mathbb{R}: \text{ damping parameter}$$

Have:

$$\begin{aligned}
\mu > 0 &: \text{ friction: } (x, \dot{x}) = (0,0) \text{ is a stable spiral} \\
\mu = 0 &: \text{ conservative system } (x, \dot{x}) = (0,0) \text{ is a non-linear center} \\
\mu < 0 &: \text{ energy increases: } (x, \dot{x}) = (0,0) \text{ is a stable spiral}
\end{aligned}$$

<u>Recall</u>: $\frac{d}{dt}E = \frac{d}{dt}\left(\frac{1}{2}\dot{x}^2 - \cos(x)\right) = -\mu x^2$.