Math 135 – Differential Equations

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This is math 135, officially known as Ordinary Differential Equations though we also delve into partial differential equations. It's taught by Professor Hester. We meet weekly on MWF from 12:00 pm to 12:50 pm for lecture. The main textbook used for the class is *Differential Equations with Applications and Historical Notes* 3^{rd} by *Simmons*. Other course notes can be found at my blog site. Please let me know through my email if you spot any concerning typos in the note.

Contents

List of Theorems

List of Definitions

§1 Lec 1: Sep 27, 2021

§1.1 Laplace Transforms

Consider the following questions

1. What is a transform?
2. What is a Laplace transform?
3. What are some examples?
4. What are some general properties?
5. Why are they useful for differential equations?

Let's tackle these questions.

1. Notice that functions: sets $\to$ sets. Transform is in higher hierarchy, i.e.,

$$\text{Transform/Operator: functions} \;\to\; \text{functions}$$

Example 1.1
- differentiation: $\frac{d}{dx}: \ f \mapsto f'$
- integration: $\int^x dx: \ f \mapsto \int^x f'(x)dx$
- multiplication by $g(x)$: $f(x) \to g(x)f(x)$
- shifting: $f(x) \to f(x-a)$

2. Laplace transform $\mathscr{L}$

$$\mathscr{L}: f(t) \mapsto F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

where $f: [0,\infty) \to \mathbb{R}$ and $F: \mathbb{C} \to \mathbb{C}$

3. Examples:

Example 1.2
- $f(t): \ t \mapsto 0 \implies \mathscr{L}[0] = 0$
- $f(t) = 1$

$$\begin{aligned}\mathscr{L}[1] &= \lim_{t\to\infty} \int_0^t e^{-st}\,dt \\ &= \lim_{t\to\infty} \left[\frac{e^{-st}}{-s}\right]_0^t \\ &= \lim_{t\to\infty} \left(\frac{e^{-st}}{-s} + \frac{1}{s}\right) \\ &= \frac{1}{s} \ \text{ if } \ \mathrm{Re}(s) > 0\end{aligned}$$

Example 1.3 • Consider

$$\begin{aligned}\mathscr{L}[t] &= \int_0^\infty te^{-st}\,dt \\ &= \left[\frac{te^{-st}}{-s}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt \\ &= \frac{1}{s^2} \text{ if } \operatorname{Re}(s) > 0\end{aligned}$$

We can generalize this as

$$\mathscr{L}[t^n] = \frac{1}{s^{n+1}}, \qquad \operatorname{Re}(s) > 0, \ \ n \in \mathbb{N}$$

In addition,

$$\begin{aligned}\mathscr{L}[e^{at}] &= \int_0^\infty e^{-(s-a)t}\,dt \\ &= \frac{1}{s-a}, \qquad \operatorname{Re}(s) > a \\ \mathscr{L}\left[\cos\omega t\right] &= \frac{s}{s^2+\omega^2} \\ \mathscr{L}\left[\sin\omega t\right] &= \frac{\omega}{s^2+\omega^2}\end{aligned}$$

4. Properties:

   a) Linear!

$$\begin{aligned}\mathscr{L}\left[f+g\right] &= \mathscr{L}\left[f\right] + \mathscr{L}\left[g\right] \\ \mathscr{L}\left[af\right] &= a\mathscr{L}\left[f\right]\end{aligned}$$

   b) Consider:

$$\begin{aligned}\mathscr{L}\left[e^{at}f(t)\right] &= \int_0^\infty f(t)e^{-(s-a)t}\,dt \\ &= F(s-a) \quad \text{if } \operatorname{Re}(s-a) > 0\end{aligned}$$

   Multiply an exponential in t-space $\xrightarrow{\mathscr{L}}$ shift in s-space.

5. In reverse,

$$\mathscr{L}\left[f(t-a)\right] = \int_0^\infty f(t-a)e^{-st}\,dt = \int_0^\infty f(t')e^{-st'}\,dt' e^{-sa}$$

   where $t' = t - a$. So

$$\mathscr{L}\left[f(t-a)\right] = F(s)e^{-sa}$$

   Thus, a shift in t-space $\xrightarrow{\mathscr{L}}$ multiply an exponential in s-space.

6. Differentiation:

$$\begin{aligned}\mathscr{L}\left[f'\right] &= \int_0^\infty f'(t)e^{-st}\,dt \\ &= \left[fe^{-st}\right]_0^\infty + \int_0^\infty f(t)se^{-st}\,dt \\ &= sF(s) - f(0)\end{aligned}$$

§2 Lec 2: Sep 29, 2021

§2.1 Laplace Transform (Cont'd)

Recap: $\mathscr{L}: f \to F$

$$\mathscr{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$$

where $t > 0$ and $s \in \mathbb{C}$.

Example 2.1
- $\mathscr{L}[t^n] = \frac{1}{s^{n+1}}$, $n \in \mathbb{N}$
- $\mathscr{L}[e^{at}] = \frac{1}{s-a}$

General properties of Laplace transform:

- linear
- shifting $\leftrightarrow$ multiplying by exponential
- $\mathscr{L}[f'] = s\mathscr{L}[f] - f(0)$

Let's now use Laplace transform to solve the following ODE

$$f'' + af' + bf = g(t), \qquad f(0) = f_0,\ \ f'(0) = f_0'$$

Apply $\mathscr{L}$,

$$\begin{aligned}\mathscr{L}[f'' + af' + bf] &= \mathscr{L}[g] \\ \mathscr{L}[f''] + a\mathscr{L}[f'] + b\mathscr{L}[f] &= G(s)\end{aligned}$$

Notice that

$$\mathscr{L}[f''] = s^2F - sf(0) - f'(0)$$

So

$$\begin{aligned}\left(s^2 + as + b\right)F(s) &= G(s) + (s+a)f_0 + f_0' \\ F(s) &= \frac{G(s) + (s+a)f_0 + f_0'}{s^2 + as + b}\end{aligned}$$

To get $f(t)$ we need to invert $\mathscr{L}$.

Example 2.2

Consider:

$$f'' + 4f = 4t, \qquad f(0) = 1, \quad f'(0) = 5$$

Apply $\mathscr{L}$, we get

$$\begin{aligned}(s^2+4)F(s) &= \frac{4}{s^2} + s + 5 \\ F(s) &= \frac{\frac{4}{s^2} + s + 5}{s^2+4} \\ &= \frac{4}{s^2\left(s^2+4\right)} + \frac{s}{s^2+4} + \frac{5}{s^2+4}\end{aligned}$$

Notice that we need to use partial fractions to decompose the first term.

$$\begin{aligned}\frac{4}{s^2(s^2+4)} &= \frac{A}{s^2} + \frac{B}{s^2+4} \\ 4 &= A(s^2+4) + Bs^2 \\ &= (A+B)s^2 + 4A\end{aligned}$$

So, $A = 1,\ B = -1$. Then,

$$\begin{aligned}F(s) &= \frac{1}{s^2} - \frac{1}{s^2+4} + \frac{s}{s^2+4} + \frac{5}{s^2+4} \\ &= \frac{1}{s^2} + \frac{4}{s^2+4} + \frac{s}{s^2+4} \\ \mathscr{L}\left[f\right] &= \mathscr{L}\left[t + 2\sin 2t + \cos 2t\right] \\ \implies f &= t + 2\sin 2t + \cos 2t\end{aligned}$$

§3 Lec 3: Oct 1, 2021

§3.1 Existence of Laplace Transform

Question 3.1. When is Laplace transform is allowed? When does Laplace transform exist?

$$\mathscr{L}[f] = \int_0^\infty f(t)e^{-st}\,dt$$

<u>*Note*</u>: Beware of ∞ – only trust limits.

$$\mathscr{L}[f] = \lim_{\tau\to\infty}\int_0^\tau f(t)e^{-st}\,dt$$

Laplace transform exists when this limit exists?
$\lim_{\tau\to\infty} f^*(\tau)$ converges to $f_\infty \in \mathbb{R}$ if $\forall \varepsilon > 0$, $\exists M > 0$ s.t.

$$|f^*(\tau) - f_\infty| < \varepsilon \quad \text{for all } \tau > M$$

Convergence test for integrals:

$$\lim_{\tau\to\infty}\int_0^\tau f(t)\,dt$$

<u>Comparison Test</u>: If $|f(t)| < g(t)$ and $\int_0^\infty g(t) < \infty$ (converges) then

$$\int_0^\infty f(t)\,dt \le \int_0^\infty |f(t)|\,dt \le \int_0^\infty g(t)\,dt < \infty$$

i.e., $\int_0^\infty f(t)\,dt$ converges. Now, back to the Laplace transform

$$\mathscr{L}[f] = \int_0^\infty f(t)e^{-st}\,dt$$

What could break this integral?

1. fe^{-st} diverges/unbounded ($\lim_{t\to t^*} f(t) = \infty$)
2. fe^{-st} doesn't decay fast enough as $t\to\infty$.

What could prevent these issues?

1. Piecewise continuous: $\lim_{t\to t^-} f(t)$ and $\lim_{t\to t^+} f(t)$ exist.
2. Exponential order
$$|f(t)| < Me^{ct} \text{ for some } M > 0 \;\&\; c$$
Have
$$\begin{aligned} c^{-t} &\le 1\cdot e^{-t} \qquad \forall t > 0\\ 1 &\le 1\cdot e^{0t} \qquad \forall t > 0\\ t &\le 1\cdot e^{t} \qquad \forall t > 0 \end{aligned}$$

Theorem 3.1

If f is piecewise continuous and of exponential order c then $\mathscr{L}[f]$ exists for $s \in \mathbb{C}$ with $\text{Re}(s) > c$.

Proof. Have

$$\begin{aligned}
\mathscr{L}[f](s) &= \int_0^\infty f(t)e^{-st}\,dt \\
\lim_{\tau\to\infty}\int_0^\tau f(t)e^{-st}\,dt &\le \lim_{\tau\to\infty}\int_0^\tau \left|f(t)e^{-st}\right|\,dt \\
&= \lim_{\tau\to\infty}\int_0^\tau |f(t)|\,e^{-s_r t}\,dt \\
&\le \lim_{\tau\to\infty}\int_0^\tau Me^{ct}\cdot e^{-s_r t}\,dt \\
&= \lim_{\tau\to\infty} M\left[\frac{e^{(c-s_r)t}}{-(c-s_r)}\right]_0^\tau \\
&= \frac{1}{s_r-c} \quad \text{if } s_r > c \\
&< \infty
\end{aligned}$$

Thus, $\mathscr{L}[f]$ exists (for $\operatorname{Re}(s) > c$) by comparison test. □

This is a sufficient condition but not necessary.

Example 3.2

Consider the function $f(t) = \frac{1}{\sqrt{t}}$

$$\begin{aligned}
\mathscr{L}\left[\frac{1}{t^{\frac{1}{2}}}\right] &= \int_0^\infty t^{-\frac{1}{2}}e^{-st}\,dt \\
&= s^{-\frac{1}{2}}\int_0^\infty x^{-\frac{1}{2}}e^{-x}\,dx \\
&= s^{-\frac{1}{2}}2\int_0^\infty e^{-z^2}\,dz \\
&= \sqrt{\frac{\pi}{s}}
\end{aligned}$$

However, we can see that $\frac{1}{t^{\frac{1}{2}}}$ isn't continuous on $[0,\infty)$.

§4 Lec 4: Oct 4, 2021

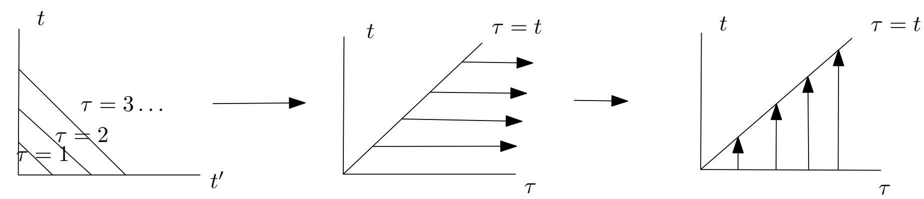
§4.1 Convolution

Question 4.1. Can we invert $\mathscr{L}[f] \cdot \mathscr{L}[g]$?

We have

$$\begin{aligned} F(s)G(s) &= \int_0^\infty f(t)e^{-st}\,dt \int_0^\infty g(t')e^{-st'}\,dt' \\ &= \int_0^\infty \int_0^\infty f(t)g(t')e^{-s(t+t')}\,dt'\,dt \end{aligned}$$

Let's define $\tau = t + t' \implies d\tau = dt'$



$$\begin{aligned} F(s)G(s) &= \int_0^\infty \int_0^\infty f(t)g(t')e^{-s(t+t')}\,dt'\,dt \\ &= \int_0^\infty \int_0^\infty f(t)g(\tau - t)e^{-s\tau}\,d\tau\,dt \\ &= \int_0^\infty \left(\int_0^\tau f(t)g(\tau - t)e^{-s\tau}\,dt \right) d\tau \\ &= \int_0^\infty \left(\int_0^\tau f(t)g(\tau - t)\,dt \right) e^{-s\tau}\,d\tau \\ &= \mathscr{L}\left[\int_0^\tau f(t)g(\tau - t)\,dt \right] \end{aligned}$$

Theorem 4.1 (Convolution)
We have

$$\begin{aligned} (f * g)(\tau) &= \int_0^\tau f(t)g(\tau - t)\,dt \\ \mathscr{L}[f * g] &= \mathscr{L}[f] \cdot \mathscr{L}[g] \end{aligned}$$

§4.2 Application of Laplace Transform – Integral Equation

Consider:

$$f(\tau) = g(\tau) + \int_0^\tau k(\tau - t)f(t)\,dt$$

Notice

$$\begin{gathered}\mathbf{f} = \mathbf{g} + K \cdot \mathbf{f} \\ f(\tau) \approx f_i \\ g(\tau) \approx g_i \\ k(\tau - t) \approx K_{ij}\end{gathered}$$

Have

$$f = g + k * f$$

and we use Laplace

$$\begin{aligned}\mathscr{L}[f] &= \mathscr{L}[g] + \mathscr{L}[k] \cdot \mathscr{L}[f] \\ \mathscr{L}[f] &= \frac{\mathscr{L}[g]}{1 - \mathscr{L}[k]}\end{aligned}$$

Example 4.2

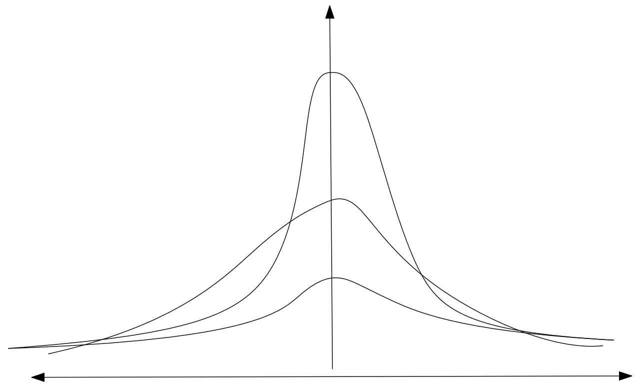
Consider $f(t) = t^3 + \int_0^t \sin(t-\tau) f(\tau) d\tau$.

$$\begin{aligned}F(s) &= \frac{3!}{s^4} + \mathscr{L}[\sin t] F(s) \\ &\vdots \\ F(s) &= 3!(s^{-4} + s^{-6}) \\ f(t) &= t^3 + \frac{t^5}{20}\end{aligned}$$

§5 Lec 5: Oct 6, 2021

§5.1 Dirac Delta "Function"

Visually:



The limit of a function concentrated at zero, with integral

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

Formally:

$$\delta: \quad f(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\, d\tau \implies f = f * \delta$$

δ "picks out" a pointwise value of any function we integrate against/convolve with.
For finite dimension, let $\mathbf{f} \in \mathbb{R}^n$ and $\mathbf{e}_i = [0, \ldots, 0, 1, 0, \ldots]$. So

$$f_i = \mathbf{f} \cdot \mathbf{e}_i$$

For infinite dimension, $f(t): \mathbb{R} \to \mathbb{R}$ for $t \in \mathbb{R}$,

$$f(t) = \int_{\mathbb{R}} f(\tau)\delta(t-\tau)\, d\tau$$

where $\delta(\tau - t) = \delta(t-\tau) = \delta_t(\tau)$. These two notions are analogous, in a sense.
Solving a linear finite dimensional system

$$\mathbf{h} \in \mathbb{R}^n, \quad L \in \mathbb{R}^{n\times n}$$

Solve $L\mathbf{f} = \mathbf{h}$. If we know $L\mathbf{f}_i = \mathbf{e}_i$ where

$$\begin{aligned} \mathbf{e}_i &: \text{ unit vector} \\ \mathbf{f}_i &: \text{ unit response vector} \end{aligned}$$

1. $\mathbf{h} = \sum h_i \mathbf{e}_i$
2. Linear superposition means
$$\mathbf{f} = \sum h_i \mathbf{f}_i$$

and

$$\begin{aligned} L\mathbf{f} &= L\left(\sum_i h_i \mathbf{f}_i\right) \\ &= \sum_i h_i L\mathbf{f}_i \\ &= \sum_i h_i \mathbf{e_i} \\ &= \mathbf{h} \end{aligned}$$

Solving ∞-dim ODE

$$f'' + af' + bf = h(t)(L[f] = h)$$

Let's say we know

$$g_t'' + ag_t' + bg = \delta_t$$

1. $h = h * \delta$
2. Then,

$$\begin{aligned} f &= h * g \\ &= \int_0^t g_t(\tau)h(\tau)\,d\tau \\ &= \int_0^t g(t-\tau)h(\tau)\,d\tau \end{aligned}$$

where g is known as the Green function.

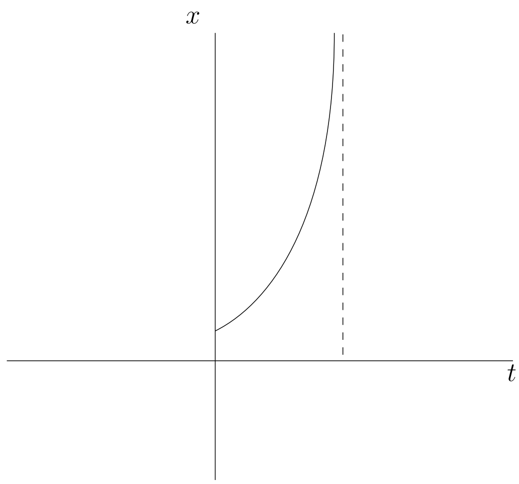
$$e_i \approx \delta_t$$
$$\mathbf{f}_i \approx g_t \mathbf{f} = \sum h_i \mathbf{f}_i \approx f = h * g$$

§6 Lec 6: Oct 08, 2021

§6.1 Existence & Uniqueness of ODE Solutions

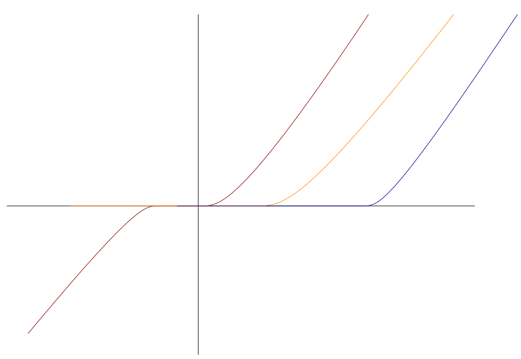
Intuitively, $f(t,x)$ is continuous seems like it guarantees a solution – **this is not true!**

1. Failure of existence over $\mathbb{R}$.
$$\frac{dx}{dt} = x^2, \quad x(0) = 1$$



We can easily solve this and obtain $x(t) = \frac{1}{1-t}$ which blows up in finite time.

2. What about uniqueness?
$$\frac{dx}{dt} = 3x^{\frac{2}{3}}, \quad x(0) = 0$$



This has infinite number of solution through $(0,0)$ – non-unique. Notice that $x' = 3x^{\frac{2}{3}}$ is an autonomous ODE where the solution is $x(t) = t^3$. However, $x(t) = 0$ is also a solution which shows that solutions are not unique.

Question 6.1. What can prove existence and uniqueness?

1. Converting to "nicer" problem, DE $\iff$ integral equation
2. Devise an iterative algorithm to approximate solutions (Picard iteration)
3. Prove the algorithm converges to a unique solution

§7 Lec 7: Oct 11, 2021

§7.1 Picard Iteration

Goal: Find sufficient conditions to prove existence and uniqueness of solution to ODE

$$\dot{x} = f\left(t, x(t)\right), \quad x(t_0) = x_0$$

Idea:

1. Smoother is better (integration is preferred over differentiation). Make things smoother by integrating
$$\dot{x}(t) = f\left(t, x(t)\right), \quad x(t_0) = x_0$$
Then, we can transform it into an integral equation
$$x(t) = x_0 + \int_{t_0}^{t} f\left(t', x(t')\right)\, dt'$$
Notice that f is continuous and x is continuous imply x is differentiable.

2. Iteration: If we can't solve it at first, try again.

> **Example 7.1**
> Newton's root-finding algorithm
> $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Picard Iteration: Iterative approximation to solutions of the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f\left(t', x(t)\right)\, dt'$$

Start with a guess for the function $x_0(t) = x_0$ (can be a constant)

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} f\left(t', x_n(t')\right)\, dt'$$

In general,

$$x_0(t) \xrightarrow{\text{Picard}} x_1(t) \xrightarrow{\text{Picard}} x_2(t) \xrightarrow{\text{Picard}} x_3(t) \underset{\infty}{\longrightarrow} \dots$$

If $x_{n+1}(t) = x_n(t) = \overline{x}(t)$, then $\overline{x}(t)$ has to solve the IE. We want $\lim_{n\to\infty} x_n(t) \to x(t)$ solves IE.

Example 7.2

Consider $\dot{x}(t) = x(t)$, $x(0) = 1$. This is equivalent to the following integral equation

$$x(t) = 1 + \int_0^t x(t')\, dt'$$

<u>Picard</u>:

$$\begin{aligned} x_0(t) &= 1 \\ x_1(t) &= 1 + \int_0^t x_0(t')\, dt' = 1 + \int_0^t 1\, dt' \\ &= 1 + t \\ x_2(t) &= 1 + \int_0^t 1 + t\, dt \\ &= 1 + t + \frac{t^2}{2!} \\ &\vdots \\ x_n(t) &= \sum_{k=0}^{n} \frac{t^k}{k!} \end{aligned}$$

Thus,

$$\lim_{n\to\infty} x_n(t) \to e^t$$

§8 Lec 8: Oct 13, 2021

§8.1 Continuity

Limit of continuous function is not necessarily continuous.

Example 8.1

Consider $x_n(t) = t^n$ on $[0, 1]$

$$\begin{aligned} x_0 &= 1 \\ x_1 &= t \\ x_2 &= t^2 \\ &\vdots \\ \overline{x} = \lim_{n\to\infty} x_n &= \begin{cases} 0, & t < 1 \\ 1, & t = 1 \end{cases} \end{aligned}$$

which is discontinuous.

<u>Idea</u>: We need "more" continuity. Given x, and given any $\varepsilon > 0$, if $|x - x'| < \delta(x, \varepsilon)$ then $|f(x) - f(x')| < \varepsilon$.

Example 8.2

Consider $f(x) = x$ on $\mathbb{R}$. We can see that

$$|x - x'| < \varepsilon \quad \forall\, |x - x'| < \varepsilon$$

in which we pick $\delta(x, \varepsilon) = \varepsilon$.
How about $f(x) = x^2$ on $\mathbb{R}$?

$$\left|x^2 - y^2\right| < \varepsilon$$

If we pick $\delta(x, \varepsilon) = \varepsilon$, then $|x - y| < \delta = \varepsilon$ which does not necessarily imply $\left|x^2 - y^2\right| < \varepsilon$ because

$$\begin{aligned} \left|x^2 - y^2\right| &= |(x + y)(x - y)| \\ &= |x + y|\,|x - y| \\ &\le \varepsilon\,|x + y| \end{aligned}$$

$|f(x) - f(y)| > \varepsilon$. So we need to pick smaller δ as x and y get larger. It would work for $\delta = \frac{\varepsilon}{2\max(|x|,|y|)}$.

Question 8.1. Is $\frac{1}{x}$ continuous?

Ans: It depends on the domain. If we're talking about $\mathbb{R}$, it doesn't work at 0; on $(0, \infty)$, yes it's continuous.

Definition 8.3 (Uniform Continuity) — $\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0$ s.t. $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$.

Remark 8.4. Notice that the definition is similar to continuity except that δ doesn't depend on x.

Example 8.5

x^2 on $\mathbb{R}$ is not uniformly continuous but x^2 on $(a,b) \subseteq \mathbb{R}$ is continuous since

$$\delta = \frac{\varepsilon}{\max(|x|,|y|)} = \frac{\varepsilon}{\max\left(|a|,|b|\right)}$$

Remark 8.6. Uniform continuity also depends on the domain as continuity does.

Exercise 8.1. Is $x^{\frac{1}{2}}$ uniformly continuous on $[0,1]$?

Lipschitz Continuity: "gradient is bounded"

$$\frac{|f(x)-f(y)|}{|x-y|} < L < \infty$$

We can pick $\delta = \frac{\varepsilon}{L}$ everywhere.

Example 8.7

- x^2 on $\mathbb{R}$ is not Lipschitz but it is on a finite interval.
- $x^{\frac{1}{2}}$ is not Lipschitz continuous on $[0,1]$. However, it's uniformly continuous.

§9 Lec 9: Oct 15, 2021

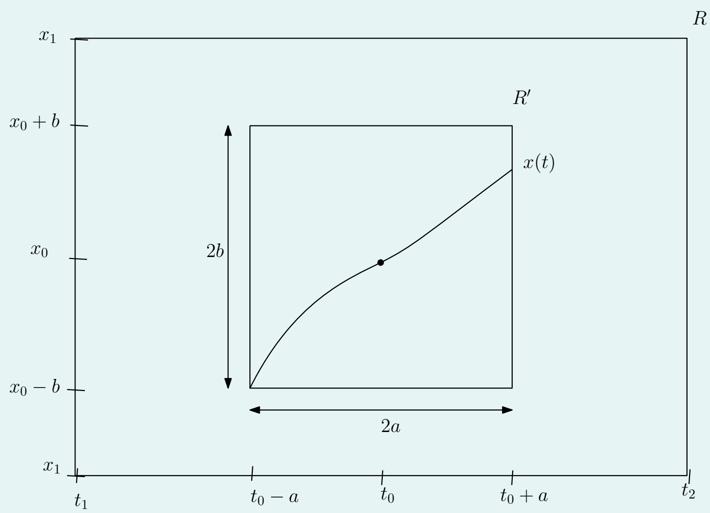
§9.1 Picard's Theorem

Let's prove local existence of the theorem.

Theorem 9.1 (Picard)
If $f(t,x)$ and $\partial_x f(t,x)$ are continuous function on a bounded rectangle $R = [t_1,t_2] \times [x_1,x_2]$ and (t_0,x_0) is in interior of R ($t_1 < t_0 < t_2$, $x_1 < x_0 < x_2$). Then $\exists$ a smaller rectangle $R' = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$ s.t. ODE

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0$$

has a solution in R'.



Note: Since R closed and bounded, then f, $\partial_x f$ are bounded, i.e.,

$$\max_R f(t,x) = M$$
$$\max_R \partial_x f(t,x) = L$$

Thus, f is Lipschitz.
Proof Outline:

1. Solving ODE $\iff$ Soling IE
2. Approximate solutions using Picard iteration
$$x_0(t) = x_0, \quad x_n(t) = x_0 + \int_{t_0}^{t} f(t', x_{n-1}(t'))\, dt'$$
3. Prove Picard iterates converges
$$\lim_{n\to\infty} x_n(t) \to \overline{x}(t)$$
4. Prove limit $\overline{x}(t)$ solves IE.
5. Prove limit $\overline{x}(t)$ is continuous.

6. Prove limit $\overline{x}(t)$ is unique.
7. How big is $R' = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$?

$$\text{Pick } a \ni aL < 1 \text{ \& } b = Ma \leq |x_0 - x_1| \, |x_0 - x_2|$$

Proof. 2. Prove Picard iterates converge

a) We have

$$\lim_{n\to\infty} x_n(t) \iff \lim_{n\to\infty} x_0(t) + \sum_{k=1}^{n} x_k(t) - x_{k-1}(t)$$

telescoping sum!

b) Series $x_0(t) + \sum_{k=1}^{n} x_k(t) - x_{k-1}(t)$ converges by Weierstrass M-test – If $|f_n(x)| < M_n$ $\forall n \in \mathbb{N}$, $x \in D$ and $\sum_{n=0}^{\infty} M_n$ converges, then

$$\sum_{n=0}^{\infty} f_n(x)$$

converges absolutely and uniformly.

i) Show $x_i(t)$ are all in $R' \subseteq R$ so we can use bounds L, M.

$$\begin{aligned}
|x_0(t) - x_0| &= 0 \\
|x_1(t) - x_0| &= \left| \int_{t_0}^{t} f\left(t', x_0(t')\right) \, dt' \right| \\
&\leq \int_{t_0}^{t} \left| f\left(t', x_0(t')\right) \right| dt \\
&\leq \int_{t_0}^{t} M \, dt \\
&\leq Ma = b
\end{aligned}$$

Thus, $x_1(t)$ is in the rectangle. By induction, every $x_n(t)$ in $R' \subseteq R$.

ii) Show $\sum_{i=1}^{\infty} |x_i(t) - x_{i-1}(t)|$ is bounded.
Define $\Delta = \max_{R'} |x_1(t) - x_0|$. Then

$$\begin{aligned}
|x_2(t) - x_1(t)| &= \left| \int_{t_0}^{t} f\left(t', x_1(t')\right) - f\left(t', x_0(t')\right) \, dt' \right| \\
&\leq \int_{t_0}^{t} \left| f\left(t', x_1(t')\right) - f\left(t', x_0(t')\right) \right| dt' \\
&\leq \int_{t_0}^{t} L \left| x_1(t') - x_0(t') \right| dt' \\
&\leq \Delta a L
\end{aligned}$$

and

$$\begin{aligned}
|x_3(t) - x_2(t)| &= \left| \int_{t_0}^{t} f\left(t, x_2(t)\right) - f\left(t, x_1(t)\right) \, dt \right| \\
&\leq \int_{t_0}^{t} \left| f\left(t, x_2(t)\right) - f\left(t, x_1(t)\right) \right| dt \\
&\leq \int_{t_0}^{t} L \left| x_2(t') - x_1(t') \right| dt' \\
&\leq L \left(\Delta a L\right) (t - t_0) \\
&\leq \Delta (aL)^2
\end{aligned}$$

Every $|x_n(t) - x_{n-1}(t)|$ depends on $|x_{n-1}(t) - x_{n-2}(t)|$ recursively. The general pattern is

$$\begin{aligned} |x_n(t) - x_{n-1}(t)| &\le \Delta(aL)^{n-1} \\ \sum_{n=1}^{\infty} |x_n - x_{n-1}| &\le \sum_{n=0}^{\infty} \Delta(aL)^n \\ &= \frac{\Delta}{1 - aL} \\ &< \infty \end{aligned}$$

Thus, $\sum x_n - x_{n-1}$ converges absolutely and uniformly by the Weierstrass M-test. Therefore,

$$\lim_{n\to\infty} x_n(t) = \overline{x}(t) \text{ exists!}$$

3. $\overline{x}$ solves I.E.

   Idea: We know $|\overline{x} - x_n|$ gets small so break $\left|\overline{x} - x_0 - \int_{t_0}^t f(t', \overline{x}(t'))\, dt'\right|$ into pieces like $|\overline{x} - x_n(t)|$.

   $$\text{subtract } x_n(t) - x_0 - \int_{t_0}^t f(t', x_{n-1}(t'))\ dt' = 0$$

   Let $\kappa = \left|\overline{x} - x_0 - \int_{t_0}^t f(t', \overline{x}(t'))\, dt'\right|$.

   $$\begin{aligned} \kappa &= \left| -(x_n - x_0 - \int_{t_0}^t f(t', x_{n-1}(t'))\, dt' \right| \\ &\le |\overline{x} - x_n| + \left| \int_{t_0}^t f(t, \overline{x}) - f(t, x_{n-1})\, dt \right| \\ &\le |\overline{x} - x_n| + \int_{t_0}^t |f(t, \overline{x}) - f(t, x_{n-1})|\, dt \\ &\le |\overline{x} - x_n| + aL\, |\overline{x} - x_{n-1}| \end{aligned}$$

   which approaches 0 as $n \to \infty$ because $\lim_{n\to\infty} x_n = \overline{x}$.

4. $\overline{x} = \lim_{n\to\infty} x_n$ is continuous, i.e., given $\varepsilon > 0$, $\exists \delta > 0$ s.t.

   $$|t - t'| < \delta \implies |\overline{x}(t) - \overline{x}(t')| < \varepsilon$$

   Idea: Split into known things

   $$\begin{aligned} |\overline{x}(t) - \overline{x}(t')| &= |\overline{x}(t) - x_n(t) + x_n(t) - x_n(t') + x_n(t') - \overline{x}(t)| \\ &\le |\overline{x}(t) - x_n(t)| + |x_n(t) - x_n(t')| + |x_n(t') - \overline{x}(t)| \end{aligned}$$

   We pick n s.t. $|\overline{x}(t) - x_n(t)| < \frac{\varepsilon}{3}$ $\forall t$ which is possible because Weierstrass implies uniform convergence. Then pick δ s.t.

   $$|x_n(t) - x_n(t')| < \frac{\varepsilon}{3} \quad \forall\, |t - t'| < \delta$$

   which is possible because x_n is continuous.

5. $\overline{x}$ is unique.

   Idea: Prove $|\overline{x} - \tilde{x}| \le |\overline{x} - \tilde{x}|$.

   - If $\tilde{u}$ is other solution, it also exists in R'.

Proof. (by contradiction) If not, then

$$|\tilde{x}(t_*) - x_0| = b = Ma$$

for some $|t_* - t| < a$. But

$$\begin{aligned} |\tilde{x}(t_*) - x_0| &= \left| \int_{t_0}^{t_*} f\left(t', \tilde{x}(t')\right) dt' \right| \\ &\leq \int_{t_0}^{t_*} |f(t', \tilde{x}(t'))| \, dt' \\ &\leq M\left(t_* - t_0\right) \\ < Ma = b \end{aligned}$$

Contradiction! ☐

- Have

$$\begin{aligned} |\overline{x}(t) - \tilde{x}(t)| &= \left| \int_{t_0}^{t} f\left(t', \overline{x}(t')\right) - f\left(t', \tilde{x}(t')\right) dt' \right| \\ &\leq \int_{t_0}^{t} |f\left(t', \overline{x}(t')\right) - f\left(t', \tilde{x}(t')\right)| \, dt' \\ &\leq \int_{t_0}^{t} L \max |\overline{x}(t') - \tilde{x}(t')| \, dt \\ &\leq La \max |\overline{x}(t') - \tilde{x}(t')| \\ \max |\overline{x}(t) - \tilde{x}(t)| &\leq \max |\overline{x}(t) - \tilde{x}(t)| \end{aligned}$$

which is only possible if $\overline{x}(t) - \tilde{x}(t) = 0$, i.e., solution is unique. ☐

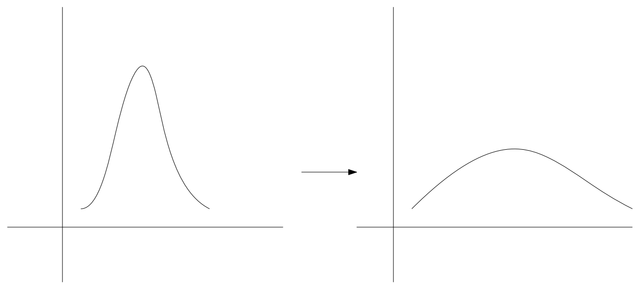
§10 Lec 10: Oct 18, 2021

§10.1 Fourier Series

Goal: Solve linear PDE: 3 canonical examples

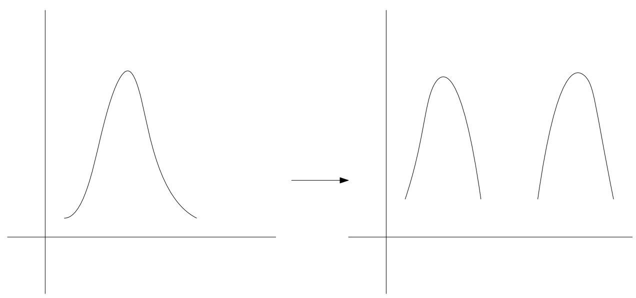
1. Heat/Diffusion equation

$$\partial_t u(t,x) - \partial_x^2 u(t,x) = 0$$



2. Wave equation

$$\partial_t^2 u = \partial_x^2 u$$



3. Laplace equation:

$$\partial_x^2 u + \partial_y^2 u = 0$$

Question 10.1. How do we solve linear PDEs?

Use linearity to split big problems into small ones that you can solve (find the eigenvectors). Then we split 1 PDE $\to \infty$ ODEs. First, let's define Fourier series.

Definition 10.1 (Fourier Series) — Fourier Series is a function written as a sum of sines and cosines

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \sin(nx) + b_n \cos(nx) \\ &= \sum_{-\infty}^{\infty} c_n e^{inx} \end{aligned}$$

where $c_n = c_r + ic_{in}$.

They have amazing properties:

1. They can approximate almost anything
   - analytic function
   - smooth function
   - periodic function
   - differentiable function
   - continuous/discontinuous function
2. They simplify differentiation!

$$\frac{d}{dx}e^{ikx} = ike^{ikx}$$
$$\frac{d^2}{dx^2}\sin kx = -k^2 \sin kx$$
$$\frac{d^2}{dx^2}\cos kx = -k^2 \cos kx$$

   Just like Laplace transform, Fourier series transform differentiation into multiplication problem (easier to deal with).

3. Fourier series are orthogonal

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$$

   or

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0 \quad \text{if } m \neq n$$

   or

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0 \quad \text{if } m \neq n$$

   This gives easy formulas

From these facts follow from linear algebra, because Fourier series are eigenfuncitons of differentiation. They are the correct basis to solve linear PDEs.

§11 Lec 11: Oct 20, 2021

§11.1 Coefficients of Fourier Series

Question 11.1. How do we calculate Fourier Series $a_n, b_n =$?

Consider the domain: $[-\pi, \pi]$, finite dimensions N, vector

$$\mathbf{u} = \sum u_i \mathbf{e}_i$$

How do we calculate u_i?

$$\begin{aligned}\mathbf{u} \cdot \mathbf{e}_i &= \left(\sum_{i=1}^{N} u_i e_i\right) \cdot e_j \\ &= \sum_{i=1}^{N} u_i \left(e_i - e_j\right) \\ &= \sum_{i=1}^{N} \delta_{ij}\end{aligned}$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

We want to do this in ∞ dimensions – inner product

$$N: \ \langle u, v\rangle = u \cdot v = \sum_{i=1}^{N} u_i v_i$$

$$\infty: \ \langle u, v\rangle \propto \int_a^b u(x)v(x)\, dx$$

Inner Product: $\langle u, v\rangle \to \mathbb{R}$ takes in two function & spits out a number. It has to satisfy the following properties

1. Bilinear

$$\langle au + bv, \rangle = a\langle u, vw\rangle + b\langle v, w\rangle$$

2. Symmetric $\langle u, v\rangle = \langle u, v\rangle$.
3. Positivity: $\langle u, u\rangle > 0$ unless $u = 0$.

Inner products are important

- They imply a norm $\|u\| = \sqrt{\langle u, u\rangle}$
- Cauchy-Schwarz Inequality

$$\langle u, v\rangle^2 \le \langle u, u\rangle\langle v, v\rangle$$

- Triangle inequality

$$\|u + v\| \le \|u\| + \|v\|$$

Exercise 11.1. Prove these properties.

Now, we will use inner products to calculate Fourier. Define

$$\langle u, v\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} u(x)v(x)\,dx$$

Under this inner product, $\sin kl$, $\cos kl$ are orthogonal functions, i.e.,

$$\begin{aligned}\langle \sin kx, \cos lx\rangle &= 0 \quad \forall k, l\\ \langle \sin kx, \sin lx\rangle &= 0 \quad \text{if } k \neq l\\ \langle \cos kx, \cos lx\rangle &= 0 \quad \text{if } k \neq l\end{aligned}$$

<u>*Note*</u>: $1 = \cos 0x$

Proof. Left as exercise, but use

$$\begin{aligned}\cos((k+l)x) &= \cos kx \cos lx - \sin kx \sin lx\\ \sin((k+l)x) &= \sin kx \cos lx + \sin lx \cos kx\end{aligned}$$

Also,

$$\begin{aligned}\langle \sin kx, \sin kx\rangle &= 1\\ \langle \cos kx, \cos kx\rangle &= 1 \quad k \neq 0\\ \langle 1, 1\rangle &= 2\end{aligned}$$

□

We have

$$\begin{aligned}f(x) &= \frac{a_0}{2} + \sum a_k \cos kx + b_k \sin kx\\ \langle f, \cos lx\rangle &= \langle \frac{a_0}{2} + \sum a_k \cos kx + b_k \sin kx, \cos lx\rangle\\ &= \frac{a_0}{2}\langle 1, \cos lx\rangle + \sum_{k=1}^{\infty} a_k \langle \cos kx, \cos lx\rangle + \sum_{k=1}^{\infty} b_n \langle \sin kx, \cos lx\rangle\\ \langle f, \cos lx\rangle &= a_l\\ \langle f, \sin lx\rangle &= b_l\end{aligned}$$

So we can write any function $f(x)$

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

where

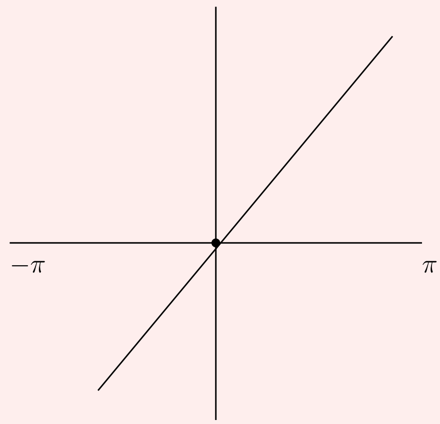
$$\begin{aligned}a_k &= \langle f, \cos kx\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx\\ b_k &= \langle f, \sin kx\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx\end{aligned}$$

Question 11.2. Are these orthogonal functions under $\langle u, v\rangle$?

Question 11.3. Are there any other kind of L^2 inner product?

Example 11.1

Consider $f(x) = x$



We have

$$\begin{aligned} x &= \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \\ a_k &= \langle x, \cos kx \rangle \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos kx \, dx \\ &= 0 - 0 - 0 = 0 \quad \text{(integration by parts)} \\ b_k &= \langle x, \sin kx \rangle \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin kx \, dx \\ &= \frac{1}{\pi} \left[-\pi \frac{\cos k\pi}{k} - (-(-\pi)) \frac{\cos(-k\pi)}{k} \right] \quad \text{(integration by parts)} \\ &= \frac{2(-1)^{k+1}}{k} \end{aligned}$$

Thus,

$$x \sim \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin kx$$

To show that infinite series converges

$$\sum_{k=1}^{\infty} \left| \frac{2(-1)^{k+1}}{k} \right| < 2 \sum_{k=1}^{\infty} \frac{1}{k}$$

which is conclusive (by Weierstrass-M test).

§12 Lec 12: Oct 22, 2021

§12.1 Convergence of Fourier Series

Consider the last example from last lecture

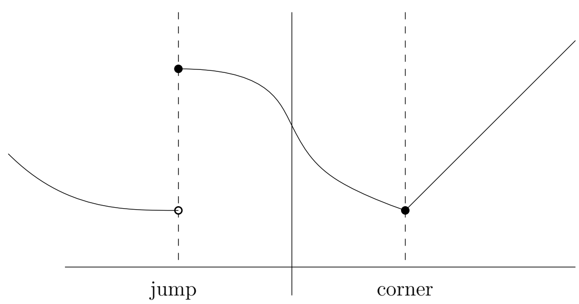
$$f(x) = x \sim \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin kx$$

Question 12.1. In what sense does it converge? (What's happening at $\pm\pi$)

Fourier series must be 2π periodic (because $\cos kx$, $\sin kx$ are 2π-periodic) so the y must converge to a 2π-periodic extension of the function.

$$\tilde{f}(x + 2\pi) = \tilde{f}(x)$$

<u>*Note*</u>: x is C' (derivative continuous) but $\tilde{x}$ is not C'. It is piecewise C' (C': f continuous and $\frac{df}{dx}$ is continuous).
<u>Piecewise C'</u> on $[a, b]$



f is C' except at finitely many points. At any bad point we have

$$\begin{cases} f(x^-) = \lim_{h\to 0} f(x-h) & \text{if } f(x^+) \neq f(x^-) \text{ jump} \\ f(x^+) = \lim_{h\to 0} f(x+h) & \\ f'(x^-) = \lim_{h\to 0} f'(x-h) & \text{if } f(x^+) = f(x^-) \\ f'(x^+) = \lim_{h\to 0} f'(x+h) & \text{but } f'(x^+) \neq f(x^-) \text{ corner} \end{cases}$$

Theorem 12.1 (Fourier Convergence)
If $\tilde{f}(x)$ is 2π-periodic, piecewise C' function, then its Fourier series converges to $\tilde{f}$ everywhere except jump points x where the series converges to $\frac{f(x^+)+f(x^-)}{2}$

Question 12.2. Recall the example at the beginning, why is there no cosines for x?

Odd/even symmetries!

Fact 12.1. We have

$$\text{odd } + \text{odd} = \text{odd}$$
$$\text{even } + \text{even} = \text{even}$$

and

$$\begin{aligned} \text{odd } &\times \text{odd} = \text{even} \\ \text{even } &\times \text{even} = \text{even} \\ \text{odd } &\times \text{even} = \text{odd} \end{aligned}$$

and

$$\begin{aligned} \int_{-a}^{a} \text{odd } dx &= 0 \\ \int_{-a}^{a} \text{even } dx &= 2\int_{0}^{a} \text{even } dx \end{aligned}$$

This implies odd functions f have sine series and even functions have cosine series.

§13 Lec 13: Oct 27, 2021

Recap:

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

where the coefficients are calculated as follows

$$\begin{aligned} a_k &= \langle f, \cos kx \rangle \\ b_k &= \langle f, \sin kx \rangle \\ \langle u, v \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} u(x)v(x)\,dx \end{aligned}$$

Symmetry simplifies a_k, b_k. Fourier series converges for periodic and piecewise C^1 functions.

§13.1 Complex Fourier Series

Recall the Euler's formula

$$e^{ikx} = \cos kx + i \sin kx$$

Also,

$$\begin{aligned} \cos kx &= \frac{e^{ikx} + e^{-ikx}}{2} \\ \sin kx &= \frac{e^{ikx} - e^{-ikx}}{2i} \end{aligned}$$

So,

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \quad \leftrightarrow \quad \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

We want $c_k = \langle f, e^{ikx} \rangle$

$$\langle e^{ikx}, e^{ikx} \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{2ikx}\,dx$$

which is not necessarily positive and we want it to be strictly positive, i.e., norm.

$$\begin{aligned} \int_{-\pi}^{\pi} e^{2ikx}\,dx &= \left[\frac{e^{2ikx}}{2ik} \right]_{-\pi}^{\pi} \\ &= \frac{e^{2\pi ki} - e^{-2\pi ki}}{2ik} \\ &= \frac{\sin 2\pi k}{k} \\ &= 0 \end{aligned}$$

To fix this, let's define Hermitian inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx$$

where $x \in (-\pi, \pi]$ and $f, g : (-\pi, \pi] \to \mathbb{C}$. So

$$\begin{aligned} c_k &= \langle f, e^{ikx} \rangle \\ c_k &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}\,dx \end{aligned}$$

Question 13.1. How do Fourier series work with integration?

Integration makes things smoother. We have

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$
$$\int f(x)\,dx \sim \int \frac{a_0}{2}\,dx + \sum_{k=1}^{\infty} a_k \int \cos kx\,dx + b_k \int \sin kx\,dx$$

Question 13.2. Is this okay?

Notice that

$$\int \cos kx\,dx = \frac{\sin kx}{k} \qquad \int \sin kx\,dx = \frac{-\cos kx}{k}$$

<u>Problem</u>: If $f(x) = 1$, then

$$f \sim 1$$
$$\int_0^x f\,dx \sim 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin kx$$

Constants terms in Fourier series are bad under integration.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

Integration is fine if the function has mean 0

$$\int_{-\pi}^{\pi} f(x)\,dx = 0$$

Compare $f(x) = 1$ and $g(x) = x$.

Remark 13.1. Fourier series need piecewise C^1. To have Fourier of f', it must be C^1 so f must be continuous (can have corners but not jumps).

$$f = a_0 + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$
$$f' = kb_k \cos kx - ka_k \sin kx$$

if f is continuous.

Summary:

- Integrate: divide by k
- Differentiation: multiply by k

§14 Lec 14: Oct 29, 2021

§14.1 Rescaling Intervals of Fourier Series

We know Fourier series on $[-\pi,\pi]$. What about $[-l,l]$? We use coordinate transformation

$$\begin{aligned} y &= \frac{\pi}{l}x \\ F(y) &= f\left(x(y)\right) \\ F\left(y(x)\right) &= f(x) \end{aligned}$$

We have

$$F(y) = f\left(x(y)\right) = f\left(\frac{l}{\pi}y\right)$$

So $F(y) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos ky + b_k \sin ky$

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(y)\cos ky\, dy \\ &= \frac{1}{\pi}\int_{-l}^{l} F\left(y(x)\right)\cos ky(x)\frac{\pi}{l}\, dx \\ &= \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{k\pi}{l}x\right)\, dx \end{aligned}$$

So

$$f(x) = F\left(y(x)\right) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\frac{k\pi}{l}x + b_k \sin\frac{k\pi}{l}x$$

We can find b_k similarly.

§15 Lec 15: Nov 1, 2021

§15.1 The Relationship between Smoothness and Fourier Coefficients

Smoother functions (more differentiable) have faster decaying Fourier coefficients. (infinitely differentiable leads to exponential decay).

Example 15.1
- Discontinuous function $\to c_k \propto \frac{1}{k}$
- $C^0 \to c_k \propto \frac{1}{k^2}$
- $C^1 \to c_k \propto \frac{1}{k^3}$
- $C^2 \to c_k \propto \frac{1}{k^4}$

Why?

Recall these definitions

Definition 15.2 — $\forall \varepsilon, x\ \exists N$ s.t.

$$|f_n(x) - f(x)| < \varepsilon \quad \forall n > N(x)$$

Then, $f_n(x) \to f(x)$ (pointwise convergence).

Definition 15.3 — $\forall \varepsilon$, $\exists N$ s.t.

$$|f_n(x) - f(x)| < \varepsilon \quad \forall n > N, \quad \forall x$$

Then, $f_n(x) \to f(x)$ (uniform convergence).

Series converges $\sum_{k=1}^{\infty} f_k(x) \to g(x)$ if

$$s_n(x) = \sum_{k=1}^{n} f_k(x) \to g(x) \text{ as } n \to \infty$$

Weierstrass M-test: If $|f_n(x)| < M_n$ and $\sum_{n=1}^{\infty} M_n < \infty$, then $\sum_{n=1}^{\infty} f_n(x)$ converges (absolutely/uniformly). So the limit is continuous if f_n are continuous.
Consider a complex Fourier series

$$f \sim \sum_{k=-\infty}^{\infty} c_n e^{ikx}$$

Theorem 15.4
If $\sum_{k=-\infty}^{\infty} |c_k| < \infty$, then the Fourier series is "good", i.e., the limit of the Fourier series is continuous.

Proof. Weierstrass!

$$\left|c_k e^{ikx}\right| \le |c_k| \left|e^{ikx}\right| = |c_k|$$

□

Corollary 15.5

If $|c_k| < \frac{M}{|k|^\alpha}$ where $\alpha > 1$. Then Fourier series is continuous.

Proof. $\sum_{k=1}^{\infty} \frac{M}{k^\alpha} < \infty$ for $\alpha > 1$ by comparison test. □

<u>Note</u>:

$$f \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$
$$f' \sim \sum_{k=-\infty}^{\infty} ikc_k e^{ikx}$$

Differentiation: $c_k \to ikc_k$ or $|c_k| \to k|c_k|$

Theorem 15.6

If $\sum_{k=1}^{\infty} |k|^n |c_k| < \infty$ where $f \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx}$, then $f^{(n)}$ is continuous (f is C^n)

Proof. Have

$$f^{(n)} \sim \sum_{k=-\infty}^{\infty} (ik)^n c_k e^{ikx}$$

then Weierstrass $\left|(ik)^n c_k e^{ikx}\right| \leq |k|^n c_k$ □

Corollary 15.7

If $|c_k| < \frac{M}{|k|^\alpha}$ where $\alpha > n+1$ then f is n times differentiable.

Proof. Comparison test: $c_k = \frac{1}{k^2}$, then

$$|c_k| < \frac{1}{k^{1.5}} \propto \frac{1}{k}$$

So,

$$\frac{1}{k^2} \to C^0$$
$$\frac{1}{k^3} \to C^1$$
$$\frac{1}{k^4} \to C^2$$
$$\vdots$$

□

§16 Lec 16: Nov 3, 2021

§16.1 Hilbert Spaces & Convergence in Norm

Goal: Prove Fourier series converge "in norm". First, we need some definitions.

Definition 16.1 (L^2 integrable) — f is L^2 integrable if $\|f\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 \, dx < \infty$.

Definition 16.2 (Hilbert Space) — Hilbert space is vector space of L^2 integrable function

Proof. Have

- Has a 0 (0 function)
- Closed under addition
$$\|f + g\| \le \|f\| + \|g\| < \infty$$
- Closed under scalar multiplication
$$\|cf\| = |c|\|f\| < \infty \qquad \square$$
- test other axioms ...

Note: L^2 function have Fourier series.

Proof. $|c_k| = \left|\langle f, e^{ikx}\rangle\right| \le \|f\|\|e^{ikx}\| < \infty$ (Cauchy-Schwarz). $\square$

Note: L^2 functions are "abnormal" TBA

Fact 16.1. Hilbert spaces are complete (every "convergent" sequence has a limit that is L^2)

Definition 16.3 — "Convergent" means Cauchy sequence for sequence $a_n \to a$. We need

$$\text{Cauchy} : \forall \varepsilon, \ \exists N \ni \ |a_m - a_n| < \varepsilon \quad \forall m, n > N$$

Aside: Completeness is the difference between rationals $\mathbb{Q}$, and reals $\mathbb{R}$ ($\mathbb{Q}$ isn't complete because π is limit of sequence in $\mathbb{Q}$ but $\pi \notin \mathbb{Q}$). Completeness matters for taking limits.

Definition 16.4 (Convergence in Norm) — $f_n(x) \to f(x)$ if $\|f_n(x) - f(x)\| \to 0$ as $n \to \infty$.

We'll prove Fourier series converge to their function in norm in a general way for a general ∞-dim vector space V with an inner product.

Definition 16.5 (Orthonormal System) — Orthonormal system $\phi_1, \phi_2, \ldots \in V$

$$\langle \phi_i, \phi_j \rangle = \delta_{ij} = \begin{cases} 0, & i \ne j \\ 1, & i = j \end{cases}$$

Definition 16.6 (General Fourier Series) — $f \sim \sum_{k=1}^{\infty} c_k \phi_k$ where $c_l = \langle f, \phi_l \rangle$ and c_k comes from $\langle f, \phi_l \rangle$ on both sides.

Theorem 16.7

The truncated Fourier series

$$s_n = \sum_{k=1}^{\infty} c_k \phi_k$$

is the best approximation to f in least squares sense, that is, consider $V_n = \text{span}\{\phi_1, \ldots, \phi_n\}$ and take any $p_n = \sum_{k=1}^{n} d_k \phi_k \in V_n$ then

$$\|s_n - f\| \le \|p_n - f\| \quad \forall p_n \in V_n$$

Proof. We have

$$\begin{aligned} p_n &= \sum_{k=1}^{n} d_k \phi_k \\ s_n &= \sum_{k=1}^{n} c_k \phi_k \\ c_k &= \langle f, \phi_k \rangle \end{aligned}$$

Then,

$$\begin{aligned} \|p_n\|^2 &= \langle p_n, p_n \rangle \\ &= \langle \sum_{k=1}^{n} d_k \phi_k, \sum_{l=1}^{n} d_l \phi_l \rangle \\ &= \sum_{k=1}^{n} \sum_{l=1}^{n} d_k d_l \langle \phi_k, \phi_l \rangle \\ &= \sum_{k=1}^{n} \sum_{l=1}^{n} d_k d_l \delta_{kl} \\ &= \sum_{k=1}^{n} |d_k^2| \end{aligned}$$

and

$$\begin{aligned} \|p_n - f\|^2 &= \langle p_n - f, p_n - f \rangle \\ &= \langle p_n, p_n \rangle - 2\langle p_n, f \rangle + \langle f, f \rangle \\ &= \sum_{k=1}^{n} |d_k|^2 - 2\left(\sum_{k=1}^{n} d_k \langle \phi_k, f \rangle \right) + \|f\|^2 \\ &= \sum_{k=1}^{n} |d_k - c_k|^2 - \sum_{k=1}^{n} |c_k|^2 + \|f\|^2 \end{aligned}$$

Pick $d_k = c_k$ – norm minimized by s_n. □

§17 Lec 17: Nov 5, 2021

§17.1 Pointwise Convergence of Fourier Series

We know Fourier series converge in norms for continuous, piecewise C^1, periodic functions. But Fourier series seemed to work even for discontinuous functions too (Gibbs phenomenon!). Today we will prove it works pointwise for discontinuous function if $s_n = \sum_{k=-n}^{n} c_k e^{ikx}$. Prove

$$\lim_{n\to\infty} s_n(x) = \frac{1}{2}\left(f(x^+) + f(x^-)\right)$$

1. Use the formulas for c_k

$$\begin{aligned} s_n &= \sum_{k=-n}^{n} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iky}\,dy\right)e^{ikx} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\underbrace{\left(\sum_{k=-n}^{n} e^{ik(x-y)}\right)}\,dy? \end{aligned}$$

Notice that $\sum_{k=-n}^{n} e^{ikx}$ is a geometric series

$$\begin{aligned} \sum_{k=-n}^{n} e^{ikx} &= e^{-inx}\left(\frac{e^{i(2n+1)x}-1}{e^{ix}-1}\right) \\ &= \vdots \\ &= \frac{\sin\left(\left(n+\frac{1}{2}\right)x\right)}{\sin\frac{1}{2}x} \end{aligned}$$

So

$$\begin{aligned} s_n(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\frac{\sin\left(\left(n+\frac{1}{2}\right)(x-y)\right)}{\sin\frac{1}{2}(x-y)}dy \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+y)\frac{\sin\left(n+\frac{1}{2}\right)y}{\sin\frac{1}{2}y}dy \\ &= \frac{1}{2\pi}\int_{0}^{\pi} f(x+y)\frac{\sin\left(n+\frac{1}{2}\right)y}{\sin\frac{1}{2}y}\,dy + \frac{1}{2\pi}\int_{-\pi}^{0} f(x+y)\frac{\sin\left(n+\frac{1}{2}\right)y}{\sin\frac{1}{2}y}\,dy \end{aligned}$$

WTS:

$$\lim_{n\to\infty} \text{top} = f(x^+)$$
$$\lim_{n\to-\infty} \text{bottom} = f(x^-)$$

Note

$$\begin{aligned} \frac{1}{\pi}\int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)y}{\sin y}\,dy &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-n}^{n} e^{iky}\,dy \\ &= 1 \end{aligned}$$

which is valid if only e^{i0} counts.

2. Prove a difference integral $\to 0$ by showing that it's a Fourier coefficient. Prove

$$\frac{1}{2\pi}\int_0^{\pi}\left(f(x+y)-f(x^+)\right)\frac{\sin\left(n+\frac{1}{2}\right)y}{\sin\frac{1}{2}y}\,dy = 0$$

Notice that

$$g(y) \equiv \frac{f(x+y) + f(x^+)}{\sin \frac{1}{2} y}$$

is piecewise continuous $\forall y \in [0, \pi]$. We need $\int_0^\pi g(y) \sin\left(n + \frac{1}{2}\right) y dy = 0$. Note that

$$\sin\left(n + \frac{1}{2}\right) y = \sin \frac{1}{2} y \cos ny + \cos \frac{1}{2} y \sin ny$$

Then,

$$\int_0^\pi \left(g(y) \sin \frac{y}{2}\right) (0) ny + \left(g(y) \cos \frac{y}{2} \sin ny\right)$$

But we know Fourier coefficients decay for all L^2 integrable functions. So these terms $\to 0$ and we prove pointwise convergence.

§18 Lec 18: Nov 8, 2021

§18.1 Heat Equation

We've been learning about Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$
$$= \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

Question 18.1. Why is Fourier series useful?

It's used to solve PDEs; specifically, we want to investigate the heat equation in this lecture. For us,

$$\partial_t u(t,x) - \kappa \partial_x^2 u(t,x) = 0$$

where κ is constant (diffusing constant). Note that

- 1x time derivative $\implies$ 1x initial condition
- 2x space derivative $\implies$ 2x boundary condition

Types of boundary condition

$$u(t,0) = \alpha(t) \quad \text{(Dirichlet boundaries)}$$
$$\partial_x u(t,0) = \mu(t) \quad \text{(Neumann boundaries)}$$
$$\partial_x u + \beta(t) u = \tau(t) \quad \text{(Robin/Mixed boundaries)}$$

Homogeneous $\implies$ RHS $= 0$, i.e., $u = 0$, $\partial_x u = 0$, or $\partial_x u + \beta u = 0$.

Question 18.2. How do we solve this? (infinitely harder than an ODE!)

Assume $u(t,x) = T(t)X(x)$. Substitute into $\partial_t u - \kappa \partial_x^2 u = 0$

$$T'(t)X(x) - \kappa T(t)X''(x) = 0$$
$$\frac{T'(t)}{\kappa T(t)} = \frac{X''(x)}{X(x)}$$
$$\frac{T'(t)}{\kappa T(t)} = \frac{X''(x)}{X(x)}$$

This can only be true if neither depends on t or x, i.e., constant. So

$$\frac{T'}{\kappa T} = \frac{X''}{X} = \lambda$$

What sign is λ?

- If $\lambda > 0$, we get exponential growth which isn't physical.

$$X'' = \lambda X$$
$$X = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$$
$$T' = \kappa \lambda T$$
$$T = Te^{\kappa \lambda t}$$

- $\lambda = 0$,

$$\begin{aligned} X'' &= 0 \\ X &= Ax + B \\ T' &= 0 \\ T &= T_0 \end{aligned}$$

- If $\lambda < 0$, redefine $\lambda \to -\lambda$

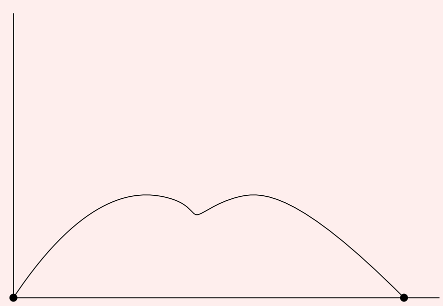
$$\begin{aligned} X'' &= -\lambda X \\ X &= A \sin \sqrt{\lambda} x + B \cos \sqrt{\lambda} x \\ T' &= -\kappa \lambda T \\ T &= T_0 e^{-\kappa \lambda t} \end{aligned}$$

So either $\lambda = 0$, or $\lambda < 0$.

Example 18.1

Consider

$$\begin{aligned} \partial_t u - \partial_x^2 u = 0 \\ u(0, x) = u_0(x) \\ u(t, 0) = 0 \\ u(t, l) = 0 \end{aligned}$$



If we assume $u(t, x) = T(t) X(x)$ we find

$$\frac{T'}{\kappa T} = \frac{X'}{X} = \lambda$$

If $\lambda = 0$, $X = Ax + B$, but $u(t, 0) = u(t, l) = 0 \implies A = B = 0$. So $\lambda < 0$. Let's write $\lambda = -\omega^2$,

$$\begin{aligned} X'' &= -\omega^2 X \\ X &= A \cos \omega x + B \sin \omega x \end{aligned}$$

Use BC's to get A, B.

$$\begin{aligned} u(t, 0) = T(t) X(0) = 0 &\implies X(0) = A = 0 \\ X &= \sin \omega x \end{aligned}$$

Example 18.2 (Cont'd)

And we have

$$u(t,l) = T(t)X(l) = B\sin\omega l = 0$$
$$\implies \omega l = k\pi, \quad k = 1, 2, \ldots$$

Thus, $\omega = \frac{k\pi}{l}$, and it's an eigenvalue and $\sin\frac{n\pi}{l}x$ is an eigenfunction.

The final solution is

$$u(t,x) = \sum_{n=1}^{\infty}\left(\hat{u}_{0,k}e^{-\frac{n^2\pi^2}{l^2}\kappa t}\sin\left(\frac{n\pi}{l}x\right)\right)$$

We get $\hat{u}_{0,k}$ from the Fourier series of the function $u_0(x)$.
Note: The Fourier coefficients decay more quickly as k gets larger so diffusion smooth things out. If we have source term

$$\partial_t u - \partial_x^2 u = f(t,x)$$

We can express f as a Fourier series and solve an ODE for each Fourier coefficient.
Summary:

1. We assume separable solution: $u(t,x) = T(t)X(x)$
2. Substituting gives an eigenvalue problem
$$X'' = \lambda X$$
3. The boundary conditions imply $\lambda = -\omega^2$ where $\omega = \frac{n\pi}{l}$, $x\alpha\sin\left(\frac{n\pi}{l}x\right)$.
4. Linearity mean we sum up all the eigenfunctions
$$u(t,x) = \sum_{n=1}^{\infty}\left(\hat{u}_{0,k}e^{-\kappa\frac{(k-\pi)^2}{2^2}t}\right)\sin k\frac{\pi}{l}x$$
5. We use the initial condition to determine the Fourier series coeff, $\hat{u}_{0,k}$

§19 Lec 19: Nov 10, 2021

§19.1 Wave Equation

Goal: Solve the wave equation

1. Look for separable solutions

$$u(t,x) = T(t)X(x)$$

to

$$\partial_t^2 u(t,x) = c^2 \partial_x^2 u(t,x)$$

c: wave speed $\left(\frac{\text{space}}{\text{time}}\right)$.

$$\begin{aligned}\partial_t^2(TX) &= c^2\partial_x^2(TX)\\ T''X &= c^2TX''\\ \frac{T''(t)}{T(t)} &= \frac{c^2X''(x)}{X(x)} = \lambda\end{aligned}$$

a) $\lambda = \omega^2 > 0$, $T'' = \omega^2 T \implies T = e^{\omega t}$ or $T = e^{-\omega t}$ and $X'' = \frac{\omega^2}{c^2}X$, $X = e^{\frac{\omega x}{c}}$ or $X = e^{-\frac{\omega x}{c}}$

b) $\lambda = 0$:

$$\begin{aligned}T'' = 0 &\implies T = A + Bt\\ X'' = 0 &\implies X = C + Dx\\ TX &= a + bt + cx + d + x\end{aligned}$$

c) $\lambda = -\omega^2 < 0$

$$\begin{aligned}T'' = -\omega^2 T &\implies T = \sin\omega t \text{ or } T = \cos\omega t\\ X'' = \frac{-\omega^2}{c^2}X &\implies X = \frac{\sin\omega x}{c} \text{ or } X = \frac{\cos\omega x}{c}\end{aligned}$$

Next, let's decide on the sign of λ using the boundary conditions, e.g., homogeneous, Dirichlet, boundary conditions

$$u(t,0) = u(t,l) = 0$$

If $\lambda > 0$

$$u = Ae^{\omega t}e^{\frac{\omega x}{c}} + Be^{-\omega t}e^{\omega\frac{x}{c}} + Ce^{\omega t}e^{-\omega\frac{x}{c}} + De^{-\omega t}e^{-\omega\frac{x}{c}}$$

but the boundary condition implies that $A = B = C = D = 0$.

If $\lambda = 0$, similarly $u = 0$ is only possibility.

If $\lambda < 0$, $u = T(t)X(x)$, $T = \sin\omega t, \cos\omega t$, $X = \sin\frac{\omega x}{c}, \cos\frac{\omega x}{c}$

$$\begin{aligned}u(t,0) &= T(t)X(0) = X(0) = 0\\ \implies A\sin\frac{\omega 0}{c} &+ B\cos\frac{\omega 0}{c} = B = 0\end{aligned}$$

So $X = A\sin\frac{\omega x}{c}$.

$$\begin{aligned}u(t,l) &= T(t)X(l) = A\sin\frac{\omega l}{c}\\ \implies \frac{\omega l}{c} &= n\pi, \quad n = 1,\ldots\\ \omega &= \frac{n\pi c}{l}\end{aligned}$$

In general, the solution is

$$u(t,x) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi c}{l} t + B_n \sin \frac{n\pi c}{l} t \right) \sin \frac{n\pi x}{l}$$

TO find A_n, B_n, we use the initial conditions

$$u(0,x) = f(x), \quad \partial_t u(0,x) = g(x)$$

Example 19.1

Consider $\partial_t^2 u = \partial_x^2 u$, $x \in [0,1]$

$$u(t,0) = u(t,1) = 0$$

$$u(0,x) = \begin{cases} x, & 0 \le x \le \frac{1}{2} \\ 1-x, & \frac{1}{2} \le x \le 1 \end{cases}$$

$$\partial_t u(0,x) = 0$$

$$c = 1, \quad l = 1 \implies \omega_n = n\pi$$

Using Dirichlet, the general solution is

$$u(t,x) = \sum_{n=1}^{\infty} (a_n \cos n\pi t + b_n \sin n\pi t) \sin n\pi x$$

We want to get a_n, b_n with ICs. For $t = 0$,

$$\begin{aligned} u(0,x) &= \sum_{n=1}^{\infty} a_n \sin n\pi x \\ &= \begin{cases} x, & 0 \le x \le \frac{1}{2} \\ 1-x, & \frac{1}{2} \le x \le 1 \end{cases} \\ a_n &= \frac{1}{2} \int_0^1 f(x) \sin n\pi x dx \end{aligned}$$

In general,

$$\begin{aligned} \sum_{n=1}^{\infty} a_n \int_0^1 \sin n\pi x \sin m\pi x dx &= \int_0^{\frac{1}{2}} x \sin n\pi x dx + \int_{\frac{1}{2}}^1 (1-x) \sin n\pi x dx \\ &= \begin{cases} 0, & \text{n even} \\ 2\left(- \left[\frac{1}{2} \frac{\cos n\pi/2}{n\pi} \right] + \frac{1}{(n\pi)^2} \left[\sin n\pi \right]_0^{\frac{1}{2}} \right) = \ldots = \frac{4}{\pi^2} \frac{\sum (-1)^k \cos(2k+1)\pi k \sin(2k+1)\pi k}{2k+1} \end{cases} \end{aligned}$$

§20 Lec 20: Nov 12, 2021

§20.1 Midterm 2

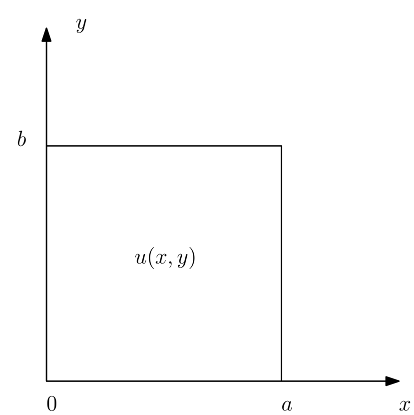
Need to know:

- How to calculate Fourier series
- How to test convergence?
- How does convergence relate to continuity?
- How does symmetry affect Fourier series?
- How does rescaling affect Fourier series?
- How does differentiation/integration affect Fourier series?
- How does smoothness (C^n) relate to Fourier series?
- How to analyze generalized Fourier series in Hilbert spaces?
- How to use Fourier series to solve PDEs?

§20.2 Laplace Equations

Consider

$$\partial_x^2 u + \partial_y^2 u = 0$$



The Laplace equation satisfies the Maximum principle, i.e., the maximum (and minimum) value of a solution to the Laplace equation must occur on the boundary,

Proof. If a function u has a maximum at (x, y) then

1. $\partial_x^2 u < 0$
2. $\partial_y^2 u < 0$

But Laplace says $\partial_x^2 u + \partial_y^2 u = 0$. Thus, we can't have local maximum (or minimum) in the domain. □

Theorem 20.1

Solutions to Laplace equation, with given boundary conditions, are unique.

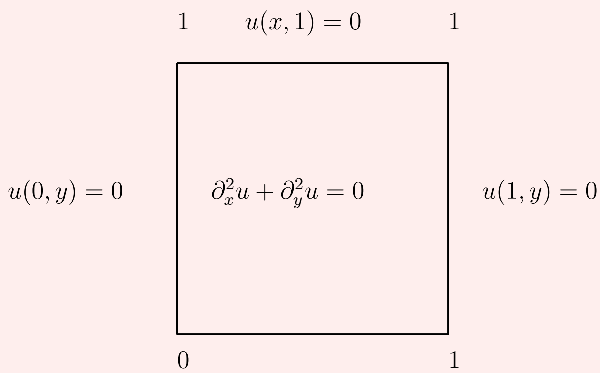
Proof. Suppose there exist u_1 and u_2 where

$$\partial_x^2 u_1 + \partial_y^2 u_1 = 0, \quad u_1 = f \text{ on the boundary}$$
$$\partial_x^2 u_2 + \partial_y^2 u_2 = 0, \quad u_2 = f \text{ on the boundary}$$

Consider $u_1 - u_2 = \Delta u$. Since Laplace equation is linear, Δu solves Laplace. And we know that $\Delta u = 0$ on the boundary. Therefore, $\Delta u = 0$ everywhere (by maximum principle). □

Example 20.2

Consider



and

$$u(0,x) = \begin{cases} x \\ 1-x \end{cases}$$

Have $u = X(x)Y(y)$

$$X''Y + XY'' = 0$$
$$\frac{X''}{X} = \frac{-Y''}{Y} = \lambda$$

- $\lambda = -\omega^2 < 0$

$$X'' = -\omega^2 X \implies X = \cos\omega x,\ \sin\omega x$$
$$Y'' = \omega^2 Y \implies Y = e^{\omega y},\ e^{-\omega y},\ \cosh y,\ \sinh y$$

- $\lambda = 0$,

$$X = 1,\ x$$
$$Y = 1,\ y$$

- $\lambda = \omega^2 > 0$

$$X = e^{\omega x},\ e^{-\omega x}$$
$$y = \cos\omega y,\ \sin\omega y$$

Example 20.3 (Cont'd)

Only $\lambda = -\omega^2 < 0$ works. For $X = \sin n\pi x$,

$$X(0) = X(1) = 0 \implies \omega = n\pi$$

Therefore,

$$\begin{aligned} Y_n &= Ae^{n\pi y} + Be^{-n\pi y} \\ Y(1) = 0 &\implies Ae^{n\pi} + Be^{-n\pi} = 0 \\ &\iff Y(y) \propto \sinh n\pi(1-y) \end{aligned}$$

In general,

$$u = \sum_{n=1}^{\infty} a_n \sinh n\pi(1-y) \sin n\pi x$$

and

$$\begin{aligned} u(x,0) &= \sum_{n=1}^{\infty} a_n \sinh n\pi \sin n\pi x = \begin{cases} x, & x < \frac{1}{2} \\ 1-x, & x > \frac{1}{2} \end{cases} \\ &= \sum_{k=1}^{\infty} (-1)^k \frac{4}{\pi^2 (2k+1)^2} \sin n\pi x \end{aligned}$$

where

$$a_{2k+1} = \frac{(-1)^k 4}{\pi^2 (2k+1)^2}$$

§21 Lec 21: Nov 17, 2021

§21.1 Adjoints

For $u \in U$, $v \in V$ and linear operator L

$$L : U \to V \text{ or } L[u] = v$$

Then,

$$\langle u_1, u_2 \rangle : U \times U \to \mathbb{R}/\mathbb{C}, \qquad \langle\langle v_1, v_2 \rangle\rangle : V \times V \to \mathbb{R}/\mathbb{C}$$
$$\langle u, L^*[v] \rangle = \langle\langle L[u], v \rangle\rangle$$

Example 21.1

$U : \mathbb{R}^m$, $V : \mathbb{R}^n$, $L : A$ matrix with n rows and m columns

$$\langle u_1, u_2 \rangle = u_1 \cdot u_2, \qquad \langle\langle v_1, v_2 \rangle\rangle = v_1 \cdot v_2$$

What is A^*?

$$\begin{aligned} \langle\langle Au, v \rangle\rangle = (Au)^\top v = u^\top A^\top v &= u^\top (A^\top v) \\ &= u^\top \left(A^\top v\right) \\ &= \langle u, A^\top v \rangle \end{aligned}$$

Example 21.2

$\langle u_1, u_2 \rangle = u_1^\top M u_2$, $\langle\langle v_1, v_2 \rangle\rangle = v_1^\top C v_2$

$$\begin{aligned} \langle\langle Au, v \rangle\rangle = (Au)^\top Cv = u^\top A^\top Cv &= u^\top M M^{-1} A^\top Cv \\ &= u^\top M \left(M^{-1} A^\top Cv\right) \\ &= \langle u, M^{-1} A^\top Cv \rangle \end{aligned}$$

$A^* = M^{-1} A^\top C$ – adjoint depends on inner products.

Differential operators as linear operators

$$u \in U = C', \qquad D[u] = \frac{du}{dx}$$

D is linear! $D[u, ru_2] = \frac{du_1}{dx} \cdot r \frac{du_2}{dx} = D[u_1] + D[u_2]$ and $D[cu] = c\frac{du}{dx} = cD[u]$. Say

$$\langle u_1, u_2 \rangle = \int_a^b u_1(x) u_2(x)\, dx$$
$$\langle\langle v_1, v_2 \rangle\rangle = \int_a^b v_1(x) v_2(x)\, dx$$

Question 21.1. What is D^*?

$$\begin{aligned}\langle\langle D[u], v\rangle\rangle &= \int_a^b \frac{du}{dx} v dx = \ldots ? = \langle u, D^*[v]\rangle \\ &= [uv]_a^b - \int_a^b u \frac{dv}{dx} dx \\ &= -\int_a^b u \frac{dv}{dx} dx = \langle u, -D[v]\rangle\end{aligned}$$

If $[uv]_a^b = u(b)v(b) - u(a)v(a) = 0$,

1. Dirichlet BC on u: $u(b) = u(a) = 0$
2. Dirichlet BC on v: $v(b) = v(a) = 0 \iff$ Neumann BC on $u : u'(b) = u'(a) = 0$
3. Periodic BC

Boundary conditions matter. If they're nice then

$$D^* = -D \text{ for standard inner products}$$

Question 21.2. What if

$$\begin{aligned}\langle u_1, u_2\rangle &= \int_a^b u_1(x)u_2(x)k(x)\, dx \\ \langle v_1, v_2\rangle &= \int_a^b v_1(x)v_2(x)p(x)\, dx\end{aligned}$$

Show $D^*[v]$ is $-\frac{1}{p}\frac{d}{dx}(kv)$ if $[uvk]_a^b = 0$.

Fact 21.1. $(L^*)^* = L$

Fact 21.2. $U \underset{L^*}{\overset{L}{\rightleftarrows}} V \underset{M^*}{\overset{M}{\rightleftarrows}} W$

$$\begin{aligned}M \circ L &= M[L[u]] \\ (M \circ L)^* &= L^* \circ M^* = L^*[M^*[w]]\end{aligned}$$

<u>Self-Adjoint</u>: $L^* = L$

$$D^2 = D \circ D, \qquad D^2[u] = \frac{d^2u}{dx^2}$$

Provided correct BCs and inner product

$$D^{2*} = D^* \circ D^* = (-D) \cdot (-D) = D^2$$

<u>Fredholm Alternative</u>:

Question 21.3. When can we solve a linear problem?

$$L[u] = f$$

If $f \perp \text{coker } L \iff \langle\langle f, v\rangle\rangle = 0$ for each $v \in \text{coker } L$.

$$\begin{aligned}\ker L &: \{u | L[u] = 0\} \\ \text{coker } L &: \ker L^* = \{v | L^*[v] = 0\}\end{aligned}$$

Example 21.3

Consider

$$Au = f$$
$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

Way # 1: Direct Gaussian elimination

$$\begin{bmatrix} 1 & 0 & -1 & | & f_1 \\ 0 & 1 & -2 & | & f_2 \\ 1 & -2 & 3 & | & f_3 \end{bmatrix} \dots \sim \begin{bmatrix} 1 & 0 & -1 & | & f_1 \\ 0 & 1 & -2 & | & f_2 \\ 0 & 0 & 0 & | & f_3 - f_1 + 2f_2 \end{bmatrix}$$

Thus, $f_3 - f_1 + 2f_2$ must be 0.
Way # 2: Find kernel of A^*.

$$A^* v = \begin{bmatrix} v_1 + v_3 \\ v_2 - 2v_3 \\ -v_1 - 2v_2 + 3v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$v = t \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}, \quad t \in \mathbb{R}$$

Fredholm $\implies t \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = 0$ which is the same as above.

Example 21.4

What about differential operators?

$$u'' = f, \quad u'(0) = u'(l) = 0$$

Way # 1: Direct

$$\begin{aligned} u'' &= f \\ u' &= a + \int_0^x f(y)\, dy \\ u &= ax + b + \int_0^x \int_0^y f(z)\, dz\, dy \end{aligned}$$

BCs $\implies u'(0) = a = 0$ and $u'(l) = \int_0^l f(x)dx = 0$
Way # 2: Fredholm

Adjoint of $\frac{d^2}{dx^2}$ is $\frac{d^2}{dx^2}$ for standard inner products

$$\begin{aligned} \text{cokernel} = \text{kernel} &= \{g(x) | g'' = 0,\ g'(0) = g'(l) = 0\} \\ &= \{c | c \in \mathbb{R}\} \end{aligned}$$

Thus, $\langle c, f \rangle = c\langle 1, f \rangle = c \int_0^l f(x)dx = 0$.

Proof. If $v \in \operatorname{coker} L$ and $L[u] = f$, then

$$\langle\langle f, v\rangle\rangle = \langle\langle L[u], v\rangle\rangle = \langle u, L^*[v]\rangle = \langle u, 0\rangle = 0$$

□

§22 Lec 22: Nov 19, 2021

§22.1 Self-Adjoint & Positive Definite Linear Operators

Definition 22.1 (Self-Adjoint Opeator) — Self-adjoint linear operator $S : u \to u$ if

$$\langle S[u_1], u_2 \rangle = \langle u_1, S[u_2] \rangle \quad \forall u_1, u_2 \in u$$

Example 22.2

$\frac{d^2}{dx^2}$ with standard $L^2\langle u_1, u_2 \rangle$ and appropriate boundary conditions

$$\begin{aligned} \langle S[u_1], u_2 \rangle &= \int_a^b u_1'' u_2 \, dx \\ &= [u_1' u_2]_a^b - \int_a^b u_1' u_2' \, dx \\ &= [u_1' u_2 - u_1 u_2']_a^b + \int_a^b u_1 u_2'' \, dx \\ &= \langle u_1, S[u_2] \rangle \end{aligned}$$

If $u_1'(b)u_2(b) - u_1(b)u_2'(b) - u_1'(a)u_2(a) + u_1(a)u_2'(a) = 0$ (homogeneous Dirichlet, mixed, and periodic BCs all work)

Definition 22.3 (Positive Definite Opeator) — Positive definite linear operator $S : u \to u$ on inner product space u,

$$\text{positive definite} : s > 0 \text{ if } \langle S[u], u \rangle > 0 \;\; \forall u \neq 0$$
$$\text{positive semi-definite} : s \geq 0 \text{ if } \langle S[u], u \rangle \geq 0 \; \forall u$$

Example 22.4

$S = -\frac{d^2}{dx^2}$ wit standard inner product and appropriate BCs,

$$\begin{aligned} \langle S[u], u \rangle &= -\int_a^b u'' u \, dx \\ &= -[u'u]_a^b + \int_a^b (u')^2 \, dx \\ &= \int_a^b (u')^2 \, dx \end{aligned}$$

for appropriate BCs.

Question 22.1. Is this positive?

1. If homogeneous Dirichlet/Mixed BCs, then no constants solutions with $u' = 0$ and so $S > 0$.
2. If homogeneous Neumann/Periodic BCs, constants allowed where $\langle S[u], u \rangle = 0$ for $u \neq 0$, so $S \geq 0$.

Boundary conditions matter.

Proposition 22.5

Positive definite operators have unique solutions.

Proof. $u \in \ker S \implies S[u] = 0 \implies \langle S[u], u \rangle = 0 \implies u = 0$. So $\ker S = \{0\}$. Then, we get unique solutions (If instead $v \neq 0$, $v \in \ker S$, then $S[u] = f \implies S[u+v] = S[u] + 0 = f$ too!) □

If $L : u \to v$, then $L^* \circ L$ is self-adjoint and positive semi-definite, where $\ker S = \ker L$ and positive definite $\iff \ker L = \ker S = \{0\}$

Proof. 1. $S = L^* \circ L$

$$S^* = (L^* \circ L)^* = L^* \circ L^{**} = L^* \circ L = S$$

2. Have

$$\begin{aligned} \langle u, S[u] \rangle &= \langle u, L^*[L[u]] \rangle \\ &= \langle \langle L[u], L[u] \rangle \rangle \\ &= \| L[u] \|^2 > 0 \text{ unless } L[u] = 0, \text{ i.e., } u \in \ker L = \ker S \end{aligned}$$

□

Question 22.2. Why are $L^* \circ L$ special?

Simple solvability conditions!

Theorem 22.6

If $S = L^* \circ L$, and $S[u] = f$ has solution then $\langle z, f \rangle = 0$ for each $z \in \ker S = \ker S^*$. If multiple solutions $S[u_1] = f, S[u_2] = f$, then $u_2 - u_1 \in \ker S$

Proof. 1. Fredholm alternative for self-adjoint

2. $S[u_1 - u_2] = S[u_1] - S[u_2] = f - f = 0$ so $u_1 - u_2 \in \ker S$. □

Example 22.7

For standard inner products, $S = D^* \circ D = -D \circ D$ is self-adjoint and depending on BCs is positive definite or positive semi-definite.

What about the non-standard inner products?

$$\langle u_1, u_2 \rangle = \int_a^b k u_1 u_2 \, dx$$
$$\langle\langle v_1, v_2 \rangle\rangle = \int_a^b p u_1 u_2 \, dx$$

Let $L[u] = D[u] \implies L^*[v] = -\frac{1}{p}(kv)'$. So

$$S = L^* \circ L = -\frac{1}{p} \frac{d}{dx} \left(k \frac{du}{dx} \right)$$

is self-adjoint and positive semi-definite. And

$$-\frac{1}{p} \frac{d}{dx} \left(k \frac{du}{dx} \right) = f(x) \text{ has unique solutions}$$

§23 Lec 23: Nov 22, 2021

§23.1 Minimization Problem

Theorem 23.1

Consider

$$\mathbf{S}u = f \tag{1}$$

where **S** is positive definite and self-adjoint operator. If u_* solves 1), then u_* minimizes the following optimization problem

$$\frac{1}{2}\langle u, Su\rangle - \langle f, u\rangle = Q(u)$$

Proof. Have

$$\begin{aligned}\langle u, Su\rangle - \langle u, f\rangle &= \langle u, Su\rangle - \langle u, Su_*\rangle \\ &= \langle u_* + u - u_*, S(u_* + u - u_*)\rangle - \langle u, Su_*\rangle \\ &= \frac{1}{2}\langle u_*, Su_*\rangle + \frac{1}{2}\langle u - u_*, S(u - u_*)\rangle + \frac{1}{2}\langle u_*, S(u - u_*)\rangle + \frac{1}{2}\langle (u - u_*), Su_*\rangle - \langle u, Su_*\rangle \\ &= \frac{1}{2}\langle u - u_*, S(u - u_*)\rangle - \frac{1}{2}\langle u_*, Su_*\rangle\end{aligned}$$

where the first term is non-negative and the second term is a constant. Take $u = u_*$ and that's the solution to minimization problem. □

Example 23.2

$-u''(x) = f(x)$ on $u(0) = u(b) = 0$.

$$S = -\frac{\partial^2}{\partial x^2}$$

which is self-adjoint and positive definite operator (under certain BCs) and also

$$S = -D \circ D$$

If u_* solves this, then u_* solves

$$Q(u) = \langle Su, u\rangle - \langle f, u\rangle$$
$$\min_u(\|Du\|^2 - \langle f, u\rangle)$$

§23.2 Sturm-Liouville Problem

Special class of B.V.P. For $x \in [a, b]$, p, q, r real valued function defined on $[a, b]$

1. $(Ly)(x) = p(x)y''(x) + q(x)y'(x) + r(x)y(x)$
2. BC:

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0$$
$$\beta_1 y(b) + \beta_2 y'(b) = 0$$

where

$$(\alpha_1, \alpha_2) \neq (0, 0), \quad (\beta_1, \beta_2) \neq (0, 0)$$

3. BVP/Sturm Liouville problem
$$Ly + \lambda w y = f$$
where w is a known function that is positive and defined on $[a, b]$.

So the S.L. problem is finding solutions to 2.
We need $q = p'$ so that L is self-adjoint and positive definite. Then, 1. becomes

$$(Ly)(x) = (py')' + ry$$

Properties:

1. S.L. is self-adjoint and positive definite.
2. $\lambda > 0$ and eigenvectors are real-valued and orthogonal
3. Eigenvectors will form a complete basis

§24 Lec 24: Nov 24, 2021

§24.1 Review

Motivation: $u(t,x) = v(t)w(x)$

$$\implies \frac{d^2 w}{dx^2} = \lambda w$$

We have countably infinite number of solutions λ_i, w_i

$$u(t,x) = \sum_{i=1}^{\infty} c_i e^{\lambda_i t} w_i(x)$$

and

$$c_i = \frac{\langle u_0(x), w_i \rangle}{\langle w_i, w_i \rangle}$$

Consider the general PDE:

$$\frac{\partial u}{\partial t} = \frac{d}{dx} p(x) \frac{\partial u}{\partial x} + r(x) u(x)$$

where

$$\frac{d}{dx} p(x) \frac{d}{dx} w + r(x) w(x) = \lambda w(x)$$

§24.2 Eigenvalue and Eigenvectors of Self-Adjoint and PD Operator

Consider

$$\langle Su, v \rangle = \langle u, Sv \rangle$$

and $\langle Su, u \rangle > 0 \ \forall u \in \mathcal{H}$ (Hilbert space). Given the operator S, the eigenvalue problem can be described as

$$Su = \lambda u$$

where $\lambda \in \mathbb{C}$ is the eigenvalue and $u \in \mathcal{H}$ is known as the eigenvector.

1. Self-adjoint operators have only purely real set of eigenvalues.
2. If $\lambda_1, \lambda_2 \in \mathbb{C}$ are eigenvalues corresponding to two distinct eigenvectors $u_1, u_2 \in \mathcal{H}$. Then $\langle u_1, u_2 \rangle = 0$.
3. If S is positive (semi)-definite, then $\lambda \geq 0$ or $\lambda > 0$.

Theorem 24.1

If S is self-adjoint, then $\lambda \in \mathbb{R}$.

Proof. Have

$$\begin{aligned} \lambda \langle u, u \rangle &= \langle \lambda u, u \rangle \\ &= \langle Su, u \rangle \\ &= \langle u, Su \rangle \\ &= \langle u, \lambda u \rangle \\ &= \overline{\lambda} \langle u, u \rangle \end{aligned}$$

$\implies \lambda = \overline{\lambda} \implies \lambda \in \mathbb{R}$. □

Theorem 24.2

If S is self-adjoint, $\lambda_1, \lambda_2 \in \mathbb{C}$ are eigenvalues corresponding to two distinct eigenvectors $u_1, u_2 \in \mathcal{H}$. Then $\langle u_1, u_2 \rangle = 0$.

Proof. Have

$$\begin{aligned} \lambda_1 \langle u_1, u_2 \rangle &= \langle \lambda_1 u_1, u_2 \rangle \\ &= \langle S u_1, u_2 \rangle \\ &= \langle u_1, S u_2 \rangle \\ &= \langle u_1, \lambda_2 u_2 \rangle \\ &= \overline{\lambda}_2 \langle u_1, u_2 \rangle \\ &= \lambda_2 \langle u_1, u_2 \rangle \end{aligned}$$

$\implies \langle u_1, u_2 \rangle = 0$. □

Example 24.3

Suppose $A = A^\top \in \mathbb{R}^{n \times n}$. Find an inner product on $\mathbb{R}^n$ s.t. AD is self-adjoint where $D \in \mathbb{R}^{n \times n}$ with positive diagonal elements.
Want

$$\langle ADx, y \rangle_D = \langle x, ADy \rangle_D$$

Have

$$\begin{aligned} \langle x, y \rangle_D &= \langle Dx, y \rangle = \langle x, Dy \rangle \\ \langle ADx, y \rangle_D &= \langle DADx, y \rangle \\ &= \langle ADx, Dy \rangle \\ &= \langle Dx, ADy \rangle \\ &= \langle x, ADy \rangle_D \end{aligned}$$

Theorem 24.4

If S is self-adjoint and positive definite, the smallest eigenvalue λ_1 of S is given by

$$\lambda_1 = \min_{u \in \mathcal{H} \setminus \{0\}} R[u]$$

where

$$R[u] = \frac{\langle u, Su \rangle}{\langle u, u \rangle}$$

§25 Lec 25: Nov 29, 2021

§25.1 Functional Derivatives

Question 25.1. How do we optimize a function f?

1. Calculate the derivative f'/gradient ∇f
2. Find where the derivative/gradient $= 0$

Question 25.2. What about optimizing a function of functions?

In general,

$$I(y) = \int_{x_1}^{x_2} f(x, y, y')\, dx$$

Note: I eats functions y and their derivatives y' and returns numbers (functional).

Example 25.1

Find function y that minimizes path length between (x_1, y_1), (x_2, y_2)

$$I(y) = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx$$

where $y(x_1) = y_1$, $y(x_2) = y2$

$$I = \int_{x_1}^{x_2} f(x, y, y')\, dx$$

$$\frac{\partial f}{\partial y} = 0$$

A derivative is a limit of differences

- 1-dim: $\lim_{\varepsilon \to 0} f(x + \varepsilon) - f(x) \approx \varepsilon f'(x)$ (standard derivative)
- n-dim: $\lim_{\varepsilon \to 0} \mathbf{f}(\mathbf{x} + \varepsilon \mathbf{v}) - \mathbf{f}(\mathbf{x}) \approx \varepsilon \mathbf{v} \cdot \nabla \mathbf{f}$ (partial/directional derivative)
- ∞-dim: $\lim_{\varepsilon \to 0} I(y + \varepsilon \eta) - I(y) \approx \varepsilon \langle \frac{\delta f}{\delta y}, z \rangle$ (functional derivative)

Example 25.2 (Cont'd from above)

We have

$$\begin{aligned} I(y) &= \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx \\ I(y + \varepsilon \eta) &= \int_{x_1}^{x_2} \sqrt{1 + y'^2 + 2\varepsilon y' \eta'} + \varepsilon^2 \eta'^2\, dx \\ &\approx \int_{x_1}^{x_2} \sqrt{1 + y'^2} + \varepsilon(\ldots) + \varepsilon^2(\ldots) + \ldots\, dx \end{aligned}$$

and $\ldots$ is what we want. Differentiate $I(y + \varepsilon \eta)$ with respect to ε and set $\varepsilon = 0$.

$$\frac{dI}{d\varepsilon}(y + \varepsilon \eta) = \int_{x_1}^{x_2} \frac{d}{d\varepsilon} \left(1 + y'^2 + 2\varepsilon y' \eta' + \varepsilon^2 \eta'^2\right)^{\frac{1}{2}}\, dx$$

Example 25.3 (Cont'd)
Then

$$\frac{d}{d\varepsilon}\left(1+y^{'2}+2\varepsilon y'\eta'+\varepsilon^2\eta^{'2}\right)=\frac{1}{2}\left(2y'\eta'+2\varepsilon\eta^{'2}\right)\left(1+y^{'2}+2\varepsilon y'\eta'+\varepsilon^2\eta^{'2}\right)^{-\frac{1}{2}}$$

$$\lim_{\varepsilon\to 0} I(y+\varepsilon\eta)-I(y)=\varepsilon\frac{dI}{d\varepsilon}(y+\varepsilon\eta)\Big|_{\varepsilon=0}=\varepsilon\int_{x_1}^{x_2}\frac{y'\eta'}{\sqrt{1+y'^2}}\,dx$$

Extremes

$$\int_{x_1}^{x_2}\frac{y'\eta'}{\sqrt{1+y'^2}}\,dx=0\quad\forall\eta(x)\text{ where }\eta(x_1)=\eta(x_2)=0$$

So

$$\int_{x_1}^{x_2}(-\eta)\left(y''\left(1+y'^2-y'^2\right)\right)/(1+y'^2)^{\frac{3}{2}}\,dx=0$$

$$\implies\int_{x_1}^{x_2}\frac{y''}{(1+y'^2)^{\frac{3}{2}}}\,\eta(x)\,dx=0$$

This must be true for all η at the optimum. This requires that $\frac{y''}{(1+y'^2)^{\frac{3}{2}}}=0$ everywhere which is solved by $y''=0\implies y=y_1+\frac{x-x_1}{x_2-x_1}y_2$ – straight lines.

Optimizing a functional gave us a differential equation. This DE is called an Euler-Lagrange equation. In general, if

$$I(y)=\int_{x_1}^{x_2}f(x,y,y')\,dx$$

then to optimize we want the functional derivative of f to be zero, which involves IBP.

$$\begin{aligned}\frac{d}{d\varepsilon}I(y+\varepsilon\eta)\Big|_{\varepsilon=0}&=\frac{d}{d\varepsilon}\int_{x_1}^{x_2}f\left(x,y+\varepsilon\eta,y'+\varepsilon\eta'\right)\,dx\\&=\int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\eta+\frac{\partial f}{\partial y'}\eta'\right)dx\\&=\int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)\eta\,dx=0\\&=\langle\frac{\partial f}{\partial y},\eta\rangle\end{aligned}$$

So

$$\frac{\delta f}{\delta y}=\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=\frac{\partial f}{\partial y}-\left(\frac{\partial}{\partial x}\frac{\partial f}{\partial y'}+\frac{\partial}{\partial y}\frac{\partial f}{\partial y'}y'+\frac{\partial f}{\partial y'}\frac{\partial f}{\partial y'}y''\right)=0$$

§26 Lec 26: Dec 1, 2021

§26.1 Calculus of Variations with Constraints

Question 26.1. What if there are more variables?

$$I(y,z)=\int_{x_1}^{x_2} f\left(x,y,z,y',z'\right)\,dx$$

There are more Euler-Lagrange equations

$$\frac{\delta f}{\delta y}=\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0 \quad \& \quad \frac{\delta f}{\delta z}=\frac{\partial f}{\partial z}-\frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right)=0$$

Question 26.2. How do we optimize with constraints?

Example 26.1

$\max z=x^2+y^2$ with constraint $(x-1)^2+y^2=1$.
Obvious way: parametrize the constrained region. For $t\in[0,2\pi)$,

$$\begin{aligned}
x(t)&=1+\cos(t)\\
y(t)&=\sin(t)\\
z(t)&=x(t)^2+y(t)^2\\
&=(1+\cos t)^2+\sin t^2\\
&=1+2\cos t+\cos^2 t+\sin^2 t\\
&=2+2\cos t
\end{aligned}$$

$\max z(t) \implies \frac{dz}{dt}=-2\sin t=0 \implies t=0,\pi$

$$\begin{aligned}
t=0 &\implies z=4\\
t=\pi &\implies z=0
\end{aligned}$$

Thus, maximum at $t=0$, $x=2$, $y=0$.

But parametrizing here isn't always so simple. Let's use a Lagrange multiplier λ. Consider

$$z(x,y,\lambda)=x^2+y^2-\lambda\left((x-1)^2+y^2-1\right)$$

Now, we optimize the easy way. Set $\partial_x z=0$, $\partial_y z=0$, $\partial_\lambda z=0$

$$\begin{aligned}
\partial_x z&=2x-\lambda 2(x-1)=0\\
\partial_y z&=2y-\lambda 2y=0\\
\partial_\lambda z&=-((x-1)^2+y^2-1)=0
\end{aligned}$$

Solve the above system of equations, we obtain

$$\begin{aligned}
x&=2\\
y&=0\\
\lambda&=2
\end{aligned}$$

Remark 26.2. There's no need to PARAMETRIZE!

The Lagrange multiplier allows us to find constrained optima using grad = 0 approach. The optimum along the constraint level set has to happen when moving along the constraint doesn't change the function.

§26.2 Calculus of Variations with Integral Constraints

Let's say we want to maximize

$$I(y) = \int_{x_1}^{x_2} f(x, y, y')\, dx$$

with constraint

$$J(y) = \int_{x_1}^{x_2} g(x, y, y')\, dx = 0$$

Now we want to minimize

$$\begin{aligned}
L(y, \lambda) &= I(y) - \lambda J(y) \\
\frac{\delta L}{\delta y} &= \frac{dL}{d\varepsilon}\left(y + \varepsilon\eta, \lambda\right)\Big|_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon}\int_0^1 f\left(x, y + \varepsilon\eta, y' + \varepsilon\eta'\right) - \lambda g\left(x, y + \varepsilon\eta, y'\varepsilon\eta'\right)\, dx\Big|_{\varepsilon=0} \\
&= \int_0^1 \frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta' - \lambda\left(\frac{\partial g}{\partial y}\eta + \frac{\partial g}{\partial y'}\eta'\right)\, dx \\
&= \int_0^1 \eta\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \lambda\left(\frac{\partial g}{\partial y} - \frac{d}{dx}\left(\frac{\partial g}{\partial y'}\right)\right)\right)\, dx
\end{aligned}$$

and $\partial_\lambda F = J(x, y, y') = 0$. So now

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \lambda\left(\frac{\partial g}{\partial y} - \frac{d}{dx}\left(\frac{\partial g}{\partial y'}\right)\right) = 0$$
$$J(x, y, y') = 0$$

Example 26.3

Minimize arc-length, given fixed area

$$\text{minimize } I(y) = \int_0^1 \sqrt{1 + y'^2}\, dx, \qquad y(0) = 0, \quad y(1) = 0$$

where $J(y) = \int_0^1 y dx = A$ or $J(y) = \int_0^1 y - A\, dx = 0$. So minimize

$$\begin{aligned}
F(y, \lambda) &= I(y) - \lambda J(y) \\
&= \int_0^1 \sqrt{1 + y'^2} - \lambda(y - A)\, dx \\
\frac{\partial F}{\partial y} = \frac{d}{d\varepsilon}F\left(y + \varepsilon\eta, \lambda\right) &= \int_0^1 \frac{d}{d\varepsilon}\left(\sqrt{1 + (y' + \varepsilon\eta')^2} - \lambda(y + \varepsilon\eta - A)\right)\, dx \\
&= \int_0^1 \frac{\eta' y'}{\sqrt{1 + y'^2}} - \lambda\eta\, dx \\
&= \int_0^1 \eta\left(\frac{-y''}{(1 + y')^{\frac{3}{2}}} - \lambda\right)\, dx
\end{aligned}$$

Example 26.4 (Cont'd)
So
$$\frac{y''}{(1+y')^{\frac{3}{2}}} + \lambda = 0 \text{ and } \frac{dF}{d\lambda} = \int_0^1 y - A\,dx$$
Now, $y''\left(1+y'\right)^{-\frac{3}{2}} + \lambda = 0$
$$\begin{aligned}\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}} + \lambda x\right) &= 0\\ \frac{y'}{\sqrt{1+y'^2}} &= -\lambda x + c\\ \frac{y'^2}{1+y'^2} &= (c-\lambda x)^2\\ y' &= \frac{c-\lambda x}{\sqrt{1-(c-\lambda x)^2}}\\ y &= \frac{1}{\lambda}\sqrt{1-(c-\lambda x)^2} + d\\ \lambda^2(y-d)^2 &= 1-(c-\lambda x)^2\end{aligned}$$
This is a circle. It goes through $(0,0)$ and $(0,1)$ and λ is determined by fixing the area.