Theorems about Vector Spaces from Math 115A

Here are some of the main results about vector spaces that you learned and proved in Math 115A.

We use the following notation in the results below:

(1) V is a vector space over an arbitrary field F

(2) S is a non-empty subset of V.

In the following results, V is not assumed to be finite dimensional unless stated otherwise (and you may assume the truth of the generalization). We begin with an

Axiom. Every vector space has a basis.

If you have heard of the Axiom of Choice or Zorn's Lemma, this axiom is equivalent to each of them.

The results from Math 115A show that this axiom is not needed in the case that the vector space V is spanned by a finite number of vectors, i.e., is a *finite dimensional vector space over* F.

Toss Out Theorem. Suppose that $V = \text{Span}(S)$ and s is an element in S. If $s \in \text{Span}(S \setminus \{s\})$, then $V = \text{Span}(S \setminus \{s\})$.

Toss In Theorem. Suppose S is a linearly independent set in V and $\text{Span}(S) < V$, Then $S \cup \{s\}$ is linearly independent for all v in $V \setminus \text{Span}(S)$.

[Here $A < B$ means $A \subset B$ and $A \neq B$.]

Replacement Theorem. Suppose that S is a basis for V and x is an element of V. If $x = \sum_{v \in S} \alpha_v v$ with $\alpha_v \in F$ almost all zero and $\alpha_{v_0} \neq 0$, then $(S \setminus \{v_0\}) \cup \{x\}$ is a basis for V.

Example. If V is finite dimensional, $S = \{v_1, \ldots, v_n\}$ a basis for V, and $x = \alpha_1 v_1 + \cdots \alpha_n v_n$ with $\alpha_1 \neq 0$, then $\{x, v_2, \ldots, v_n\}$ is a basis for V.

Main Theorem. If S is a finite set and $V = \text{Span}(S)$, then any linearly independent subset of V has at most $|S|$ elements.

Application. Let V be a nonzero finite dimensional vector space over F with bases $\mathcal{B}$ and $\mathcal{C}$ (which exists by Toss Out), then

$$|\mathcal{B}| = |\mathcal{C}| < \infty,$$

and this common integer is called the *dimension* of V. It is denoted by $\dim V$ (or $\dim_F V$). We define the dimension of the zero vector space to be 0.

Examples.

(1) $\dim F^{m\times n} = mn$ with

$$\mathcal{S}_{m,n} := \{e_{ij} \in F^{m\times n}) \mid e_{ij} \text{ is the matrix with 1 in the } ij\text{th entry and 0 in all other entries}\}$$

a basis called the *standard basis* for $F^{m\times n}$.

(2) $\dim F[t]_n := \{\text{polynomials of degree at most } n \text{ over } F\} = n + 1$ with $\{1, t, \ldots, t^n\}$ a basis.

(3) $F[t] := \{\text{polynomials over } F\}$ is not a finite dimensional vector space but $\{1, t, \ldots, t^n, \ldots\}$ is a basis.

(4) if $\alpha < \beta$ in $\mathbb{R}$, and $C[\alpha, \beta] := \{f : [\alpha, \beta] \to \mathbb{R} \mid f \text{ continuous}\}$, then $C[\alpha, \beta]$ is not finite dimensional. Do you know a basis?

(5) The complex numbers $\mathbb{C}$ is a finite dimensional real ($= \mathbb{R}$) vector space but an infinite dimensional vector space over the rational numbers $\mathbb{Q}$.

Extension Theorem. Let S be a linearly independent set in V. Then S is part of a basis for V. We say that S *extends* to a basis for V.

Applications.

(1) Let $\mathcal{B} \subset V$ be a subset. Then the following are equivalent:

  (a) $\mathcal{B}$ is a basis for V.

  (b) $\mathcal{B}$ is a maximal linearly independent set in V.

  (c) $\mathcal{B}$ is a minimal spanning set in V.

(2) If V is a finite dimensional vector space over F and $W \subset V$ a subspace, then $\dim W \leq \dim V$ with equality if and only if $W = V$.

Counting Theorem. Let W_1 and W_2 be finite dimensional subspaces of V. Then $W_1 + W_2$ is a finite dimensional vector space over F and

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2).$$